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A New Finite Difference Well-Balanced Mapped Unequal-Sized WENO Scheme for Solving Shallow Water Equations

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Abstract. In this paper, we propose a newly designed fifth-order finite difference well-balanced mapped unequal-sized weighted essentially non-oscillatory (WBMUS-WENO) scheme for simulating the shallow water systems on multi-dimensional structured meshes. We design new non-linear weights and a new mapping function, so that the WBMUS-WENO scheme can maintain fifth-order accuracy with a small ε even nearby the extreme points in smooth regions. The truncation errors of the scheme is smaller and it has better convergence in simulating some steady-state problems. Unlike the traditional well-balanced WENO-XS scheme [29], this new WBMUS-WENO scheme uses three unequal-sized stencils, denotes the linear weights to be any positive numbers on condition that their summation is one. By incorporating a quartic polynomial on the whole big stencil into WENO reconstruction, the WBMUS-WENO scheme is simple and efficient. Extensive examples are performed to testify the exact C-property, absolute convergence property, and good representations of this new WBMUS-WENO scheme.

AMS subject classifications: 65N06, 65M06 ,65N22

Key words: Shallow water equations, exact C-property, mapping function, well-balanced unequal-sized WENO (WBMUS-WENO) scheme.

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1 Introduction

In this paper, we propose a newly defined fifth-order finite difference well-balanced mapped unequal-sized weighted essentially non-oscillatory (WBMUS-WENO) scheme to simulate discontinuous flows with a non-flat bottom topography. Some advantages of the WBMUS-WENO scheme are briefly described here. The first is that we no longer need to solve the linear weights and the linear weights can be arbitrarily taken as positive numbers with a minor restriction that their summation is one. By incorporating a quartic polynomial on the largest stencil into WENO reconstruction, the dependence on the linear weights is reduced. The second is that a series of central or biased spatial stencils are used in the reconstruction process, which is the same as the number of space stencils in the traditional WENO scheme [29]. The third is that a new nonlinear weights and a new mapping function are designed to keep fifth-order accuracy even nearby the extreme points in smooth regions with a tiny ε and it can pushes the residue of steady-state shallow water problems to settle down close to machine zero. Better convergence and robustness are also the noteworthy advantages of the new WBMUS-WENO scheme.

High-order numerical methods have developed very rapidly in recent years and the WENO scheme is a very popular numerical scheme [4,16]. The high-order WENO scheme is also widely used to simulate the shallow water systems. According to [25], the shallow water system is obtained from the depth averaged compressible NS equations, which are often used to simulate river and coastal flows. This equation can also be viewed as a hyperbolic equation with source term. However, the nature of the equation has changed considerably due to the presence of the source term. Because of this, there is a stationary solution to the system where the nonzero flux gradient could be balanced by the source term. This balance property will be disrupted if the source terms are processed directly. Therefore many numerical methods have been devised to maintain this balance property. For example, Vukovic and Sopta [27] introduced the weighted ENO reconstruction with the exact C-property into solving hyperbolic equations with south term. Nelida and Senka [7] extend this balanced WENO scheme to the hyperbolic conservation laws. Bermúdze and Vazquez [3] generalized the upwind scheme to the discretization of source terms. This property is necessary to make the scheme "exact" for the stationary case. The most contributing work was done by Xing and Shu [29]. They designed a well balanced WENO-XS scheme and successfully solved the shallow water equations. Alcrudo [1] proposed a new Godunov scheme for the free-surface flow equations. The approximate Godunov scheme [24] was introduced into the above framework. LeVeque [15] designed a wave-propagation method by introducing a Riemann solver which can exactly balance the flux difference and source terms. Lu et al. [17] also used this method and combined with the Lax-Wendroff(LW) time discretizations to solve the equations. There are many related works, such as the positivity-preserving limiter [2] on unstructured or adaptive meshes [19], the research of three-dimensional shallow flows [23], and other numerical schemes [6]. More relevant developments on the shallow water wave equation can be found in [8,30].