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# Finite Element Scheme with H2N2 Interpolation for Multi-Term Time-Fractional Mixed Sub-Diffusion and Diffusion-Wave Equation

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**Abstract.** In this paper, two numerical schemes for the multi-term fractional mixed diffusion and diffusion-wave equation (of order $\alpha$, with $0<\alpha<2$) are developed to solve the initial value problem. Firstly, we study a direct numerical scheme that uses quadratic Charles Hermite and Newton (H2N2) interpolation polynomials approximations in the temporal direction and finite element discretization in the spatial direction. We prove the stability of the direct numerical scheme by the energy method and obtain a priori error estimate of the scheme with an accuracy of order $3-\alpha$. In order to improve computational efficiency, a new fast numerical scheme based on H2N2 interpolation and an efficient sum-of-exponentials approximation for the kernels is proposed. Numerical examples confirm the error estimation results and the validity of the fast scheme.

**AMS subject classifications**: 65M10, 78A48

**Key words**: The multi-term fractional mixed diffusion and diffusion-wave equation, finite element method, H2N2 interpolation, fast algorithm, stability and convergence.

## 1 Introduction

Inspired by the nonlocality of fractional derivatives, the study of fractional partial differential equations (FPDEs) has attracted much attention in recent years, see [3,10,23,24]. To our knowledge, although many experts and scholars have made many meaningful contributions in their theoretical analysis, the exact solutions of most fractional equations are still difficult to obtain [8, 11]. Even if an exact solution is obtained, it is often too

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complexity to apply in real life. Due to the difficulties in obtaining accurate solutions for FPDEs, there is an urgent need to study effective numerical methods for FPDEs. In recent decades, the numerical methods for FPDEs has been well applied, such as the finite element methods [12,29], finite difference methods [3,10,26,27] and spectral methods [5,17] et al..

With the rapid development of science and technology, FPDEs have been widely studied in rheology, wave propagation, fluid flows, etc. as seen in [6,13,16,21]. However, compared to classical diffusion equations or wave equations, multi-term time fractional mixed sub-diffusion and diffusion wave equations may be more consistent and accurate in modeling some processes of disturbance. Zhang et al. [32] established a compact ADI scheme for the two-dimensional time fractional diffusion-wave equation. Jin et al. [13] derived a fully discrete scheme for the multi-term time fractional diffusion equation with bounded convex polyhedral domain conditions. Using the L2-$1_\sigma$ formula, Gao et al. [10] constructed a temporal second order difference scheme for the time multi-term and distributed-order fractional sub-diffusion equations. In [6], the authors adopted a high order difference scheme for the multi-term time fractional partial differential equations applied the energy method and the Galerkin spectral method. By an effective asymptotic expansion of the error equation of the shifted Grünwald–Letnikov formula, Zeng et al. [30] investigated a new modified weighted shifted Grünwald–Letnikov formula for multi-term fractional ordinary and partial differential equations. Sun [20] et al. proved two temporal second-order difference schemes for the multi-term time fractional diffusion-wave equation based on the order reduction technique. Feng et al. [9] presented a mixes L scheme to approximate the novel two-dimensional multi-term time-fractional mixed sub-diffusion and diffusion-wave equation on convex domains. A fully discrete approximate scheme for the 2D multi-term time-fractional mixed diffusion and diffusion-wave equations were established by using linear triangle finite element method in space and classical L1 time-stepping method combined with Crank-Nicolson scheme in time [31]. Jiang et al. [11] proposed the multi-term space-time Caputo-Riesz fractional advective-diffusion equation with Dirichlet inhomogeneous boundary conditions. Bhrawy and Zaky [2] considered an efficient operational formulation of spectral tau method for multi-term time-space fractional differential equation under Dirichlet boundary conditions. Based on a mixed difference scheme in time and an unstructured mesh finite element method in space, Fan et al. [7] studied a fully discrete scheme for two-dimensional multi-term time-space fractional diffusion-wave equations, which defined on an irregular convex domain with the time fractional orders belonging to the whole interval $(0,2)$.

However, most of the above results only considered the L1 approximation in the time direction, or the numerical methods were used in the relevant equations with high computational costs and large storage problems. To overcome these difficulties, several techniques were employed to reduce the cost of calculating and storing derivation methods. Ke et al. [14] attempted to deal with the block lower triangular Toeplitz-like with tri-diagonal blocks system which arises from the time-fractional partial differential equa-