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New Residual Based Stabilization Method for the Elasticity Problem

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Abstract. In this paper, we consider the mixed finite element method (MFEM) of the elasticity problem in two and three dimensions (2D and 3D). We develop a new residual based stabilization method to overcome the inf-sup difficulty, and use Langrange elements to approximate the stress and displacement. The new method is unconditionally stable, and its stability can be obtained directly from Céa's lemma. Optimal error estimates for the H^1-norm of the displacement and $H(\mathrm{div})$-norm of the stress can be obtained at the same time. Numerical results show the excellent stability and accuracy of the new method.

AMS subject classifications: 65N15, 65N30

Key words: Elasticity, MFEM, residuals, stabilization.

1 Introduction

In this paper, we consider the MFEM of the elasticity problem based on the Hellinger-Reissner variational principle. As is known to all, this method requires the pairs of the finite element space satisfy the so-called inf-sup condition. Since the stress tensor requires symmetry, it is difficult to construct the stable mixed finite elements (MFEs). Some early works employed composite elements [1], or imposed the symmetry of stress tensor weakly [2–6]. Until 2002, Arnold and Winther proposed the first family of stable MFEs with respect to triangular meshes which used polynomial shape functions to approximate the stress and displacement [7], of which the simplified lowest order element has 21 degrees of freedom for the stress and 3 for the displacement (21 plus 3 DOFs), and

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optimal order error estimates were obtained for all variables. An analogous family of conforming MFEs based on rectangular meshes were proposed in [8], involving 36 plus 3 DOFs for the simplified lowest order element. In [11], using the similar method of [7], Arnold et al. presented some stable elements in 3D with respect to simplicial meshes, and even the simplified lowest element has 156 plus 6 DOFs. In addition, some efficient nonconforming MFEs for this problem also have been proposed. For example, two triangular elements were presented in [9], and the simplified element has 12 plus 3 DOFs; and a group of rectangular elements were introduced in [10], with the $\mathcal{O}(h)$ convergence order in L^2-norm for both the stress and the displacement, and the simplest element employed 12 plus 4 DOFs. Although many other stable elements were also constructed based on the ideas of [7], (see [12–17]), these elements still have too many DOFs, and the implementations are expensive [18], especially for the 3D case.

Recently, some new methods were proposed to construct stable elements for elasticity problem. In [19], a family of nonconforming rectangular and cubic elements were constructed, and an explicit constructional proof of the discrete inf-sup condition was given. The DOFs are 2 plus 1 in 1D, 7 plus 2 in 2D, and 15 plus 3 in 3D, and the error estimates for all variables are optimal. In [20–22], some conforming rectangular and cubic elements were presented, of which the lowest order elements have 8 plus 2 DOFs in 2D, and 18 plus 3 DOFs in 3D. In [23–25], some conforming elements on simplicial meshes were developed, and the lowest order elements only involve 18 plus 3 DOFs in 2D and 48 plus 6 in 3D. Compared with Arnold-Winther elements, these elements are more compact, and have less DOFs.

On the other hand, some stabilized methods were also studied for the elasticity problem to overcome the inf-sup difficulty, such as Galerkin least-squares method [26], Brezzi-Pitkäranta stabilization [27], variational multiscale method [28], projection stabilization method [29, 30], edge stabilization method [31, 32], and least-squares method [33–36]. In this paper, we propose a new residual based stabilization method for the elasticity problem. The equilibrium term is used to augment the coercivity, and the term derived from the pure displacement equation is used to control the H^1-norm of the displacement. The method is consistent and unconditionally stable. The bilinear form is strongly coercive, and its stability can be obtained directly from Céa's lemma. The Language elements of any order can be used to approximate stress and displacement, so the lowest elements on simplicial meshes have 9 plus 6 DOFs in 2D, and 24 plus 12 DOFs in 3D, and the numerical implementations are more easily. In addition, Optimal error estimates for the H^1-norm of the displacement and $H(\mathrm{div})$-norm of the stress can be obtained at the same time.

The rest of this paper is organized as follows. In Section 2, we introduce the mixed form of the elasticity problem and some notations used throughout the paper. In Section 3, we present the new stabilization scheme, prove the stability, and give the error analysis. In Section 4, we implement two numerical examples to test the stability and convergence rate of the new method. Throughout the paper we use C to denote a generic positive constant whose value may change from place to place but that remains independent of