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A Priori Error Analysis of an Euler Implicit, Finite Element Approximation of the Unsteady Darcy Problem in an Axisymmetric Domain

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Abstract. We consider the time dependent Darcy problem in a three-dimensional axisymmetric domain and, by writing the Fourier expansion of its solution with respect to the angular variable, we observe that each Fourier coefficient satisfies a system of equations on the meridian domain. We propose a discretization of these equations in the case of general solution. This discretization relies on a backward Euler's scheme for the time variable and finite elements for the space variables. We prove a priori error estimates both for the time steps and the meshes.

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Key words: Darcy's equations, axisymmetric domain, Fourier truncation, finite element discretization.

1 Introduction

Let $\breve{\Omega}$ be a bounded three-dimensional domain which is invariant by rotation around an axis. The boundary $\breve{\Gamma}$ of this domain is divided into two parts $\breve{\Gamma}_p$ and $\breve{\Gamma}_u$. We are interested in the following model, suggested by Rajagobal [10],

$$\begin{cases} \partial_t \breve{\boldsymbol{u}} + \alpha \breve{\boldsymbol{u}} + \mathbf{grad}\, \breve{p} = \breve{\boldsymbol{f}} & \text{in } \breve{\Omega} \times [0,T], \\ \operatorname{div} \breve{\boldsymbol{u}} = 0 & \text{in } \breve{\Omega} \times [0,T], \\ \breve{p} = \breve{p}_b & \text{on } \breve{\Gamma}_p \times [0,T], \\ \breve{\boldsymbol{u}} \cdot \breve{\boldsymbol{n}} = \breve{g} & \text{on } \breve{\Gamma}_u \times [0,T], \\ \breve{\boldsymbol{u}} = \breve{\boldsymbol{u}}_0 & \text{in } \breve{\Omega} \text{ at time } t = 0, \end{cases} \tag{1.1}$$

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where the unknowns are the velocity $\breve{\boldsymbol{u}}$ and the pressure $\breve{p}$ of the fluid. The data are the quantities $\breve{\boldsymbol{f}}$, $\breve{g}$, the pressure on the boundary $\breve{p}_b$ and the initial value of the velocity $\breve{\boldsymbol{u}}_0$. The parameter α is a positive constant representing the drag coefficient. If the problem is set in a domain which is symmetric by rotation around an axis, it is proved in [3] that, when using the Fourier expansion with respect to the angular variable, a three-dimensional problem is equivalent to a system of two-dimensional problems on the meridian domain, each problem being satisfied by a Fourier coefficient of the solution. Here we are going to present the unsteady Darcy equations in three-dimensional axisymmetric geometries, and we propose a discretization of this problem in the case of a general solution, i.e., for the Fourier coefficient of order k, $k\in\mathbb{Z}$. We recall that the same problem are considered in [6] but in the case of an axisymmetric solution, i.e., only for the Fourier coefficient of order $k=0$.

In this work, we assume that the boundary conditions and the external forces are in general case. Axisymmetric problems without any assumption on the data can be transformed into problems which are invariant by rotation see [2, Chap I, prop 1.2.8]. A natural way for reducing axisymmetric problems on $\breve{\Omega}$ to a family of problems on the meridian domain Ω, which we will make precise later, relies on the use of Fourier expansions with respect to the angular variable θ. Then, by using cylindrical coordinates, we can write a variational formulation of this problem in Ω. We prove the well-posedness and some regularity properties of the solution for such a system in the appropriate weighted Sobolev spaces. Next, we propose a time semi-discrete problem that relies on the backward Euler's scheme. We prove that this problem has a unique solution and derive error estimates. Concerning the space discretization, we consider a conforming finite element method which leads to a well-posed discrete problem for which we prove a priori error estimates.

An outline of the paper is as follows:

- In Section 2, we write a variational formulation of problem (1.1) in the case of an axisymmetric domain, we prove its well-posedness and the error issued from Fourier truncation.
- Section 3 is devoted to the description and a priori analysis of the discrete problem in the meridian domain Ω.
- In Section 4, we present some numerical experiments.

2 The two-dimensional problems

Let (x,y,z) denotes a set of Cartesian coordinates in $\mathbb{R}^3$ such that $\breve{\Omega}$ is invariant by rotation around the axis $x=y=0$. We introduce the system of cylindrical coordinates (r,θ,z), with $r\geq 0$ and $-\pi\leq\theta<\pi$, defined by $x=r\cos\theta$ and $y=r\sin\theta$. If Γ_0 denotes the intersection between $\breve{\Omega}$ and axis $r=0$, then there exists a meridian domain Ω in $\mathbb{R}_+\times\mathbb{R}$ such that

$$\breve{\Omega}=\{(r,\theta,z);\ (r,z)\in\Omega\cup\Gamma_0 \ \text{ and } \ -\pi<\theta\leq\pi\}.$$