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Second-Order Schemes for Fokker-Planck Equations with Discontinuous Drift

Boya Zhang[1], Yaming Chen[1] and Songhe Song[1,2,*]

[1] *College of Science, National University of Defense Technology, Changsha, Hunan 410073, China*
[2] *State Key Laboratory of High Performance Computing, National University of Defense Technology, Changsha, Hunan 410073, China*

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Abstract. Second-order finite-difference schemes are developed to solve the corresponding Fokker-Planck equation of Brownian motion with dry friction, which is one of the simplest model of stochastic piecewise-smooth systems. For the Fokker-Planck equation with a discontinuous drift, both explicit and implicit second order schemes are derived by finite volume method. The proposed schemes are proved to be stable both for the one-variable (related to the velocity only) and two-variable (related to the velocity and displacement) cases. Numerical experiments are implemented for both the two cases. Some known analytical results of the considered model are used to confirm the effectiveness and desired accuracy of the schemes.

AMS subject classifications: 65M06, 65M08, 35Q84

Key words: Brownian motion with dry friction, Fokker-Planck equation, discontinuous coefficient, finite volume method, alternating direction implicit method.

1 Introduction

Nonsmooth dynamical systems have received increased attention in recent years, especially in engineering applications [1]. These dynamical systems are often modelled by piecewise-smooth differential equations [2], which may display many unexpected phenomena, such as stick-slip transitions with dry friction forces [3, 4] and bifurcations [1]. For piecewise-smooth dynamical systems perturbed by noise, there are some new features appearing [5, 6]. Nowadays, piecewise-smooth stochastic differential equations are usually used to describe biological and physical systems. For some simple piecewise-linear or piecewise-constant stochastic differential equations, analytical solutions of the

*Corresponding author.
Emails: zhangboya9252@163.com (B. Y. Zhang), chenym-08@163.com (Y. M. Chen), shsong@nudt.edu.cn (S. H. Song)

transition probability distribution can be obtained [6,7]. However, for many other cases, such as a system with inertial term, it is difficult to attain analytical expressions. Hence, some numerical methods should be employed if one wants to know more dynamical behaviours of the systems.

In this paper, we attempt to derive numerical schemes for solving a Fokker-Planck equation with discontinuous drift, which results from a so-called Brownian motion with dry friction [8,9]. This dry friction model describes the motion of a solid object moving over a vibrating plate subjected to the dry friction force and the viscous friction force. We assume the plate is horizontal and vibrated by a Gaussian white noise. Without loss of generality, we can confine our investigation to the following model according to the Newtonian law,

$$\dot{v}(t) = -\mu\sigma(v(t)) - \gamma v(t) + \xi(t), \tag{1.1}$$

where the dot denotes the derivative with respect to time t. Here $\sigma(v)$ denotes the sign of v, representing the dry friction force with coefficient $\mu > 0$, $\gamma \geq 0$ is the viscous coefficient and $\xi(t)$ is the Gaussian white noise with zero mean and the correlation

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{1.2}$$

where $D > 0$. The notation $\langle\cdot\rangle$ stands for the average over all possible realizations of the noise, and δ is the Dirac delta function. Since the motion of solid objects [10] and the interaction of granular gases [11–13] have been widely studied, the dry friction model plays an important role in theory and experiment [14–18].

Introducing a potential

$$\Phi(v) = \mu|v| + \gamma v^2/2 \tag{1.3}$$

and letting $D = 1$, then the transition probability density $p(v,t|v_0,0)$ of Eq. (1.1) satisfies the following Fokker-Planck equation [19,20],

$$\frac{\partial p(v,t|v_0,0)}{\partial t} = \frac{\partial[\dot{\Phi}(v)p(v,t|v_0,0)]}{\partial v} + \frac{\partial^2 p(v,t|v_0,0)}{\partial v^2}, \tag{1.4}$$

where the dot on Φ denotes the derivative with respect to velocity v. In addition, the corresponding initial condition is $p(v,0|v_0,0) = \delta(v-v_0)$ if $v(0) = v_0$ for Eq. (1.1). In this case, analytical expressions of the transition probability distribution are available for the pure dry friction case ($\gamma = 0$) in closed form and the dry and viscous friction case ($\gamma \neq 0$) in series [6]. Hence, it is a good starting benchmark example for us to test the effectiveness of numerical schemes. For Eq. (1.1) with $D = 1$, one can also consider the displacement by adding an equation $\dot{x}(t) = v$. Then we should consider the joint distribution $p(x,v,t|x_0,v_0,0)$ obeying the following two-variable Fokker-Planck equation

$$\begin{aligned}\frac{\partial p(x,v,t|x_0,v_0,0)}{\partial t} = & -v\frac{\partial p(x,v,t|x_0,v_0,0)}{\partial x} + \frac{\partial[\dot{\Phi}(v)p(x,v,t|x_0,v_0,0)]}{\partial v} \\ & + \frac{\partial^2 p(x,v,t|x_0,v_0,0)}{\partial v^2}.\end{aligned} \tag{1.5}$$