Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **10**, No. 2, pp. 362-389

DOI: 10.4208/aamm.OA-2017-0067
April 2018

Error Estimate and Superconvergence of a High-Accuracy Difference Scheme for Solving Parabolic Equations with an Integral Two-Space-Variables Condition

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Received 9 March 2017, Accepted (in revised version) 7 July 2017

Abstract. The initial-boundary value problems for parabolic equations with nonlocal conditions have been widely applied in various fields. In this work, we firstly build an implicit Euler scheme for an initial-boundary value problem of one dimensional parabolic equations with an integral two-space-variables condition. Then we prove that this scheme can reach the asymptotic optimal error estimate in the maximum norm. Next, the formulas used to approximate the solution derivatives with respect to time and spatial variables are presented, and it is proved for the first time that they have superconvergence. In the end, numerical experiments demonstrate the theoretical results.

AMS subject classifications: 65M06, 65M12, 65T50

Key words: Parabolic equation, integral condition, finite difference scheme, asymptotic optimal error estimate, superconvergence.

1 Introduction

The initial-boundary value (IBV) problems for parabolic equations with nonlocal conditions (e.g., integral condition) have been widely used in various application fields, such as thermoelasticity, heat conduction, plasma physics, medical science, chemical engineering and population dynamics. The theory and numerical methods of these problems have aroused the concern of many scholars [1–4].

Cannon [5] is the first study of evolution problems involving energy specification (i.e., integral constraint condition) and the existence and uniqueness of the solution are

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proved. Subsequently, the IBV problems for parabolic equations with various nonlocal conditions are investigated and corresponding existence and uniqueness theory are established; see, e.g., [6–9]. As a very important method, finite difference method often has been used to solving these problems and now there is a great deal of research work in this area. To solve the diffusion equation with an integral condition in whole spatial domain, several difference schemes are proposed in [10, 11] and they all reach the asymptotic optimal error estimates in the maximum norm; In [12], two level Crandall's implicit scheme and three level Dufort-Frankel explicit scheme are presented and compared with several usual schemes; In [13, 14], θ method and Laplace transform method are presented respectively. For the diffusion equation with an integral one-space-constant condition, several difference schemes are designed in [15–17]; In [18] a numerical technique based on method of lines and Pade approximants; In [19] a Crank-Nicolson finite-element Galerkin method is presented. To solve the diffusion equation with an integral one-space-variable condition, three implicit schemes are presented in [20] and a discretization scheme based equivalent integral equation is designed in [21]. For the diffusion equation with an integral two-space-variables condition, a numerical method based on the reproducing kernel space theory is presented in [22]. However, at present, not much work has been done in design, calculation, and theoretical analysis of difference schemes which can reach the asymptotic optimal error estimate or the optimal error estimate of the IBV problems for parabolic equations with an integral two-space-variables condition. Besides, in many application problems not only the approximation of the exact solution are concerned but also the approximation of the solution derivatives. Since usual error analysis methods for difference scheme of the IBV problems for parabolic equations with nonlocal condition are difficult to obtain high-accuracy and convergence results for the exact solution and solution derivatives, it is bound to develop some new methods and techniques.

In this work, we first build an implicit Euler difference scheme of a one-dimensional parabolic equation with an integral two-space-variables condition (see [9]). Then, we introduce some new methods and techniques on basis of the discrete Fourier transform (DFT) and prove that under a general condition $\tau \geq Ch^2$ where C is a positive constant independent of mesh size, the errors of the scheme at boundary points and interior points are $\mathcal{O}(\tau|\ln h|)$ and $\mathcal{O}(\tau \ln^2 h)$ respectively. Moreover, we present formulas to approximate partial derivatives of the exact solution and prove that the approximation formula for u_t has the super approximations of

$$\mathcal{O}\Big(\tau\Big(1+\frac{\tau}{h}\Big)|\ln h|\Big) \quad \text{and} \quad \mathcal{O}\Big(\tau\Big(1+\frac{\tau}{h}\Big)\ln^2 h\Big)$$

at the boundary point and the interior point, respectively. Besides, we prove that the approximation formula for u_x has the super approximation of $\mathcal{O}(\tau \ln^2 h)$ at interior points which keep a certain distance with boundaries. It is worthwhile noting that the methods and techniques presented in this work can be extended to other parabolic problems with nonlocal condition for error analysis.

This work is organized as follows. In Section 2, we depict an IBV problem for one dimensional parabolic equation with an integral two-space-variables condition and present