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Superconvergence Analysis of Bilinear Finite Element for the Nonlinear Schrödinger Equation on the Rectangular Mesh

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Abstract. In this paper, we investigate the superconvergence property of a time-dependent nonlinear Schrödinger equation with the bilinear finite element method on the rectangular mesh. We prove the superclose error estimate in H^1-norm with order $\mathcal{O}(h^2)$ between the approximated solution and the elliptic projection of the exact solution. Moreover, we obtain the global superconvergence result in H^1-norm with order $\mathcal{O}(h^2)$ by the interpolation post-processing operator.

AMS subject classifications: 65M60, 65M15

Key words: Finite element method, nonlinear Schrödinger equation, superconvergence, interpolation post-processing.

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1 Introduction

In this paper, we consider an initial boundary value problem of the two-dimensional time-dependent nonlinear Schrödinger equation. Let Ω be a bounded rectangular-type domain in R^2 with boundary $\partial\Omega$, and let $0<T<\infty$ be given. We seek a complex-valued

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function $u=u(x,y,t)$, defined on $\Omega\times[0,T]$ and satisfying

$$\begin{cases} iu_t=-\Delta u+Vu+|u|^2u+f, & \forall(x,y,t)\in\Omega\times[0,T],\\ u(x,y,t)=0, & \forall(x,y,t)\in\partial\Omega\times[0,T],\\ u(x,y,0)=u_0(x,y), & \forall(x,y)\in\Omega, \end{cases} \tag{1.1}$$

where $i=\sqrt{-1}$ is the complex unit, functions $u_0(x,y)$ and $f(x,y,t)$ are complex-valued, the trapping potential function $V(x,y)$ is real-valued and non-negative bounded for all $(x,y)\in\Omega$, and

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

The Schrödinger equation may describe many physical phenomena in quantum mechanics, optics, seismology, and plasma physics. Numerical methods and analysis for the Schrödinger equation have been investigated extensively. For instance, the finite difference method [1–3], the spectral method [4, 5], the finite element method [6–14], the discontinuous Galerkin method [15,16], the mixed finite element method [17,18], and the two-grid method [19, 20]. In [1], Han et al. proposed a finite-difference scheme for the one-dimensional time-dependent Schrödinger equation. In [4], Bao et al. proposed an exponential wave integrator sine pseudospectral method for the nonlinear Schrödinger equation with wave operator and carry out rigorous error analysis. In [6], Akrivis et al. presented several implicit Crank-Nicolson Galerkin FEMs for the nonlinear Schrödinger equation. The authors obtained the optimal error estimate with time-step condition $\tau=o(h^{\frac{d}{4}})$, $(d=2,3)$. In [16], Karakashian et al. analyzed the convergence of the discontinuous Galerkin method for the nonlinear Schrödinger equation. In [18], Zhao et al. established a new mixed finite element approximate formulation with less degree of freedoms for nonlinear Schrödinger equation based on spaces of bilinear finite element and its gradient. In [20], Jin et al. solved the time-dependent Schrödinger equation by the finite element two-grid method and analyzed the convergence. The semi-discrete schemes are proved to be convergent with an optimal convergence order and the full-discrete schemes are verified by a numerical example.

Superconvergence analysis is a powerful tool to improve the approximation accuracy and the efficiency of the finite element method. It is well known that superconvergence results often happen when the underlying differential equations have smooth solutions and are solved on very structured meshes such as rectangular grids or strongly regular triangular grids. Many excellent superconvergence results have been obtained for elliptic and parabolic problems [21–30]. However, there exist not many superconvergence results for the Schrödinger equations. In [10], Lin et al. considered a kind of initial boundary value problem of the Schrödinger equation and presented superconvergence estimates in semi-discrete and and fully discrete schemes by the interpolation error theory. In [11], Shi et al. applied the simplest anisotropic linear triangular finite element to solve the nonlinear Schrödinger equation, and provided the superconvergence error estimate in the