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Optimal Error Estimate of the Penalty FEM for the Stationary Conduction-Convection Problems

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Abstract. In this paper, a penalty finite element method is presented for the two dimensional stationary conduction-convection problems. The existence and the convergence of the penalty stationary conduction-convection formulation are shown. An optimal error estimate of the numerical velocity, pressure and temperature is provided for the penalty finite element method when the parameters ϵ and h are sufficiently small. Our numerical experiments show that our method is effective and our analysis is right.

AMS subject classifications: 76M10, 65N12, 65N30, 35K61

Key words: Conduction-convection problems, penalty finite element method, existence and convergence, error estimates.

1 Introduction

Conduction-convection system is one of the most important model in the atmosphere dynamics, which has a forced nonlinear term. Not only the velocity and the pressure are contained in this problems, but also the temperature is coupling. A numerical simulation of the conduction-convection problems is a fundamental problem both of numerical analysis and fluid dynamics. In this article, we consider the stationary conduction-convection problems

$$\begin{cases} -\nu\Delta u+(u\cdot\nabla)u+\nabla p=\lambda jT, & x\in\Omega,\\ \text{div } u=0, & x\in\Omega,\\ -\Delta T+\lambda u\cdot\nabla T=0, & x\in\Omega,\\ u=0,\quad T=T_0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega\subset\mathbb{R}^2$ is a suitable smooth bounded convex polygon, $u=u(x,t)=(u_1(x,t),u_2(x,t))$ represents the velocity vector of a viscous incompressible fluid. $p=p(x,t)$ represents the pressure, $T=T(x,t)$ represents the temperature, $\nu>0$ is the viscosity coefficient, $\lambda>0$ is the Groshoff number, $j=(0,1)$ is the two-dimensional vector, and T_0 is the boundary condition. There are some numerical results about this system, see [6,12,32]. Many works [13–16] have developed for mixed finite element solution for conduction-convection problems by Luo and his collaborators. Newton type iterative methods [21, 23, 25] and defect-correction methods [22, 24, 26] for the conduction-convection equations were presented. In [27], a gauge-Uzawa projection method was presented. A modified characteristics projection finite element method for time-dependent conduction-convection problems was shown in [28].

We note that the velocity u, the pressure p and the temperature T in (1.1) are coupled together by the incompressibility constraint div $u=0$, which makes the system difficult to solve numerically. On the basis of solving Navier-Stokes equations experience, a popular strategy to overcome this kind difficulty is to relax the incompressibility constraint in an appropriate way, resulting in a class of pseudo-compressibility methods, among which are the penalty method [17], the artificial compressibility method and the projection method (see [2–4, 7, 11, 18–20] for details). In this paper, we give the penalty finite element method to solve the conduction-convection problems. The penalty stationary conduction-convection problems is as follows

$$\begin{cases} -\nu\Delta u_\epsilon+(u_\epsilon\cdot\nabla)u_\epsilon+\nabla p_\epsilon=\lambda jT_\epsilon, & x\in\Omega,\\ \text{div } u_\epsilon+\dfrac{\epsilon}{\nu}p_\epsilon=0, & x\in\Omega,\\ -\Delta T_\epsilon+\lambda u_\epsilon\cdot\nabla T_\epsilon=0, & x\in\Omega,\\ u_\epsilon=0,\quad T_\epsilon=T_0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and preliminary results for the stationary conduction-convection problems (1.1). In Section 3, we prove the existence, uniqueness and the convergence of the penalty solution. In Section 4, we give the optimal error estimate for the penalty steady conduction-convection problems. From our analysis, the method that we used in this paper has an optimal error estimate in H_1-norm. Finally, we use Taylor Hood finite element space in our numerical experiments. Through a series of numerical experiments, it is verified that our method is effective and our analysis is right.

2 Preliminaries

In this section, we aim to describe some of the notations and results which will be frequently used in this paper. For the mathematical setting of the conduction-convection problems (1.1) and the penalty conduction-convection problems (1.2), we introduce the