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# Analysis and Application of Stochastic Collocation Methods for Maxwell's Equations with Random Inputs

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**Abstract.** In this paper we develop and analyze the stochastic collocation method for solving the time-dependent Maxwell's equations with random coefficients and subject to random initial conditions. We provide a rigorous regularity analysis of the solution with respect to the random variables. To our best knowledge, this is the first theoretical results derived for the standard Maxwell's equations with random inputs. The rate of convergence is proved depending on the regularity of the solution. Numerical results are presented to confirm the theoretical analysis.

**AMS subject classifications**: 65M10, 78A48

**Key words**: Maxwell's equations, random permittivity and permeability, stochastic collocation methods, uncertainty quantification.

## 1 Introduction

Uncertainty is ubiquitous in many complex physical systems, such as wave, sound and heat propagation through random media, and flow and propagation driven by stochastic forces and stochastic initial conditions. Stochastic partial differential equations (SPDEs) have played an important role in the study of uncertainty quantification (UQ) in many branches of science and engineering. In electromagnetics, the fluctuations in the producing process (such as during the lithography) of electromagnetic materials allow us to treat the permittivity and permeability as uncertain parameters (e.g., [4, 7]). Stochastic Maxwell equations with additive noise were investigated in [13, 14]. Due to the high dimensionality of stochastic solutions, it is very challenging to efficiently solve PDEs with uncertain parameters, and has attracted a great attention in recent years (e.g., [8, 18, 23], review articles [12, 20] and books [10, 16, 26, 28]).

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Due to the low convergence rate of the traditional Monte Carlo method, the stochastic Galerkin methods [2, 10, 11] have been developed and show faster convergence rates with increasing order of expansions, provided that the solutions of differential equations are sufficiently smooth in the random space. However, the system of equations resulting from the stochastic Galerkin methods is often coupled and quite expensive to solve especially for problems requiring high-dimensional random spaces. In [26], Xiu and Hesthaven proposed a class of stochastic collocation methods by taking advantage of the strength of Monte Carlo methods and the stochastic Galerkin methods. Their stochastic collocation method achieves fast convergence when the solutions are sufficiently smooth in the random space. More importantly, the stochastic collocation method is simple in implementation and the system of resulting equations is decoupled and hence is efficient to solve. The stochastic collocation methods have been applied to solve various problems. For example, Babuska et al. [1] proposed and analyzed a stochastic collocation method to solve elliptic problems with random coefficients and forcing terms. Zhang and Gunzburger [27] presented a detailed error analysis of a stochastic collocation method for linear parabolic equations with random coefficients, forcing terms, and initial conditions. Motamed et al. [17] proposed and analyzed a stochastic collocation method to solve the elastic wave equation with random coefficients. Tang and Zhou [22] proposed some rigorous regularity analysis for the random transport equation with a random wave speed and demonstrated the convergence of the stochastic collocation methods. As for Maxwell's equations, in 2006, Chauviere et al. [7] solved the time-dependent Maxwell's equations by using both the stochastic Galerkin method and stochastic collocation method. Detailed comparisons of both methods are made for uncertainties caused by physical materials, by the source wave and by the physical domain. In 2015, Benner and Schneider [4] described several techniques for uncertainty quantification for the time-harmonic Maxwell's equations by using stochastic collocation method. In our recent work [15], we analyzed a stochastic collocation method for the metamaterial Maxwell's equations with random input data.

Though stochastic collocation method has been applied to solve Maxwell's equations with uncertain parameters [4, 7], it would be interesting to develop a theory that offers insight to how uncertainty propagates through the dynamical systems and what regularity we can expect as Chauviere et al. pointed out [7, pp. 774]. One of the main purposes of this paper is to fill this gap.

The rest of the paper is organized as follows. In Section 2, we first present detailed regularity analysis of the time-dependent Maxwell's equations with random permittivity and permeability, then we establish the convergence rate for the stochastic collocation method applied to the Maxwell's equations. Numerical results are presented in Section 3 to support our theoretical analysis. Section 4 concludes the paper.

## 2 Maxwell's equations with random inputs

Let $\boldsymbol{x}\in D\subset R^3$ be the spatial coordinate, $t$ be the time variable from set $[0,T]$, and $(\Omega,\mathcal{A},\mathcal{P})$ be a complete probability space, whose event space is $\Omega$ ($\omega\in\Omega$ is the event) and is equipped with $\sigma$-algebra $\mathcal{A}$, and $\mathcal{P}$ is a probability measure. Furthermore, we let $\rho(y):\Gamma\to R^+$ be the probability density function of the random variable $y(\omega)$, $\omega\in\Omega$, whose image $\Gamma:=y(\Omega)\in R$ is an interval in $R$. For simplicity, in the rest of the paper, we omit the symbol $\omega$ and assume that $\Gamma=[-1,1]$.

Consider the stochastic Maxwell's equations: Find the random electric field $\boldsymbol{E}(\boldsymbol{x},t,y)$ and magnetic field $\boldsymbol{H}(\boldsymbol{x},t,y):D\times(0,T)\times\Omega\to R^3$ such that $P$-almost everywhere in $\Omega$, i.e., almost surely (a.s.) satisfy the following equations:

$$\epsilon(y(\omega))\partial_t\boldsymbol{E}=\nabla\times\boldsymbol{H}, \tag{2.1a}$$

$$\mu(y(\omega))\partial_t\boldsymbol{H}=-\nabla\times\boldsymbol{E}, \tag{2.1b}$$

subject to the initial conditions

$$\boldsymbol{E}(\boldsymbol{x},t=0,y(\omega))=\boldsymbol{E}_0(\boldsymbol{x},y(\omega)),\quad \boldsymbol{H}(\boldsymbol{x},t=0,y(\omega))=\boldsymbol{H}_0(\boldsymbol{x},y(\omega)), \tag{2.2}$$

and the perfect conducting (PEC) boundary condition:

$$\boldsymbol{n}\times\boldsymbol{E}=\boldsymbol{0}\quad \text{on } \partial D, \tag{2.3}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are some given functions. To accommodate the uncertainty or randomness of the material, we assume that the permittivity $\epsilon$ and permeability $\mu$ are random. Moreover, $\boldsymbol{n}$ denotes the unit outward normal vector on the boundary $\partial D$, where $D\subset R^3$ is a bounded polyhedral domain with a Lipschitz boundary. Here and below, we denote $\partial_s$ the partial derivative with respect to variable $s$, e.g., $s=t$ and $y$.

To solve problem (2.1a)-(2.3), we use the Lagrange interpolation approach by following [22,26]. We first choose a set of Gauss-Lobatto collocation points $\{y_k\}_{k=0}^N\in\Gamma$, where $N$ denotes the degree of the Lagrange interpolation polynomial. We then solve the problem (2.1a)-(2.3) at each collocation point $y_j, j=0,\cdots,N$, by a Crank-Nicolson scheme described later. To prove the convergence rate of this stochastic collocation method, we need to establish the regularity for the solution of our model problem (2.1a)-(2.3). Detailed proofs are given below.

## 2.1 Regularity analysis

For the solution of problem (2.1a)-(2.3), we have the following energy conservation property.

**Lemma 2.1.** *For the problem* (2.1a)-(2.3), *we have*

$$\begin{aligned}&\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\boldsymbol{E}|^2+\rho(y)\mu(y)|\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(t)\\ =&\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\boldsymbol{E}|^2+\rho(y)\mu(y)|\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(0),\quad \forall t\in[0,T].\end{aligned} \tag{2.4}$$

*Proof.* Multiplying (2.1a) by $2\rho(y)\boldsymbol{E}$ and integrating over $D$, we have

$$\frac{d}{dt}((\rho(y)\epsilon(y)\boldsymbol{E},\boldsymbol{E})_D)=2(\rho(y)\epsilon(y)\partial_t\boldsymbol{E},\boldsymbol{E})_D=2(\rho(y)\nabla\times\boldsymbol{H},\boldsymbol{E})_D, \tag{2.5}$$

here and below, we denote $(\cdot,\cdot)_D$ for the inner product over domain $D$.

Multiplying (2.1b) by $2\rho(y)\boldsymbol{H}$, integrating over $D$ and using the PEC boundary condition (2.3), we obtain

$$\begin{aligned}&\frac{d}{dt}((\rho(y)\mu(y)\boldsymbol{H},\boldsymbol{H})_D)=2(\rho(y)\mu(y)\partial_t\boldsymbol{H},\boldsymbol{H})_D\\ =&-2(\rho(y)\nabla\times\boldsymbol{E},\boldsymbol{H})_D=2(\rho(y)\boldsymbol{E},\nabla\times\boldsymbol{H})_D.\end{aligned} \tag{2.6}$$

Summing up (2.5) and (2.6), and integrating over the space $\Gamma$, we have

$$\frac{d}{dt}\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\boldsymbol{E}|^2+\rho(y)\mu(y)|\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)=0,$$

which concludes the proof. □

Similarly, we have the following energy conservation property for the curl of the solution of problem (2.1a)-(2.3).

**Lemma 2.2.** *For the problem* (2.1a)-(2.3)*, we have*

$$\begin{aligned}&\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(t)\\ =&\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(0),\quad\forall t\in[0,T].\end{aligned} \tag{2.7}$$

*Proof.* Taking $\nabla\times$ of both (2.1a) and (2.1b), we have

$$\epsilon(y(\omega))\partial_t(\nabla\times\boldsymbol{E})=\nabla\times\nabla\times\boldsymbol{H}, \tag{2.8a}$$

$$\mu(y(\omega))\partial_t(\nabla\times\boldsymbol{H})=-\nabla\times\nabla\times\boldsymbol{E}. \tag{2.8b}$$

Multiplying (2.8a) by $2\rho(y)\nabla\times\boldsymbol{E}$ and integrating over $D$, we have

$$\begin{aligned}&\frac{d}{dt}((\rho(y)\epsilon(y)\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E})_D)\\ =&2(\rho(y)\epsilon(y)\partial_t\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E})_D\\ =&2(\rho(y)\nabla\times\nabla\times\boldsymbol{H},\nabla\times\boldsymbol{E})_D.\end{aligned} \tag{2.9}$$

Multiplying (2.8b) by $2\rho(y)\nabla\times\boldsymbol{H}$, and integrating over $D$, we obtain

$$\begin{aligned}&\frac{d}{dt}((\rho(y)\mu(y)\nabla\times\boldsymbol{H},\nabla\times\boldsymbol{H})_D)\\ =&-2(\rho(y)\nabla\times\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{H})_D\\ =&-2\langle\rho(y)\boldsymbol{n}\times\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{H}\rangle_{\partial D}-2(\rho(y)\nabla\times\boldsymbol{E},\nabla\times\nabla\times\boldsymbol{H})_D,\end{aligned} \tag{2.10}$$

where $\langle\cdot,\cdot\rangle_{\partial D}$ denotes the inner product on the surface $\partial D$.

Using (2.1a) and the PEC boundary condition (2.3), we see that the boundary integral is actually zero:

$$\begin{aligned}&-2\langle\rho(y)\boldsymbol{n}\times\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{H}\rangle_{\partial D}\\=&-2\langle\rho(y)\boldsymbol{n}\times\nabla\times\boldsymbol{E},\epsilon(y)\partial_t\boldsymbol{E}\rangle_{\partial D}\\=&-2\langle\rho(y)\epsilon(y)\nabla\times\boldsymbol{E},\partial_t(\boldsymbol{n}\times\boldsymbol{E})\rangle_{\partial D}=0.\end{aligned}$$

Summing up (2.9) and (2.10), and integrating over the space $\Gamma$, we have

$$\frac{d}{dt}\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)=0,$$

which concludes the proof. □

Under some regularity assumptions on the random functions $\epsilon$ and $\mu$, we can prove that the first derivative of the solution of problem (2.1a)-(2.3) with respective to the random variable is bounded in the $L^2$ norm.

**Theorem 2.1.** *Assume that there exist constants $C_\epsilon$ and $C_\mu$ such that*

$$(\ln\epsilon(y))'\Big/\sqrt{\epsilon(y)\mu(y)}\le C_\epsilon,\quad (\ln\mu(y))'\Big/\sqrt{\epsilon(y)\mu(y)}\le C_\mu,\ \textit{almost everywhere in}\ \Gamma,\tag{2.11}$$

*then for any $t\in[0,T]$, we have*

$$\begin{aligned}&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\\\le& e^{t\max(C_\epsilon,C_\mu)}\Big[\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\&+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\Big].\end{aligned}\tag{2.12}$$

*Proof.* Differentiating both sides of (2.1a) with respect to $y$ gives

$$\epsilon(y)\partial_t(\partial_y\boldsymbol{E})=-\epsilon'(y)\partial_t\boldsymbol{E}+\nabla\times\partial_y\boldsymbol{H},\tag{2.13}$$

which, along with (2.1a), leads to

$$\begin{aligned}&\frac{d}{dt}(\rho(y)\epsilon(y)\partial_y\boldsymbol{E},\partial_y\boldsymbol{E})_D\\=&-2\Big(\rho(y)\frac{\epsilon'(y)}{\epsilon(y)}\nabla\times\boldsymbol{H},\partial_y\boldsymbol{E}\Big)_D+2(\rho(y)\nabla\times\partial_y\boldsymbol{H},\partial_y\boldsymbol{E})_D\\=&-2\Big(\frac{(\ln\epsilon(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\mu(y)}\nabla\times\boldsymbol{H},\sqrt{\rho(y)\epsilon(y)}\partial_y\boldsymbol{E}\Big)_D+2(\rho(y)\nabla\times\partial_y\boldsymbol{H},\partial_y\boldsymbol{E})_D\\\le& C_\epsilon\Big(\int_D\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2d\boldsymbol{x}+\int_D\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2d\boldsymbol{x}\Big)+2(\rho(y)\nabla\times\partial_y\boldsymbol{H},\partial_y\boldsymbol{E})_D.\end{aligned}\tag{2.14}$$

Similarly, differentiating both sides of (2.1b) with respect to $y$ gives

$$\mu(y)\partial_t(\partial_y \boldsymbol{H}) = -\mu'(y)\partial_t \boldsymbol{H} - \nabla\times\partial_y \boldsymbol{E}, \tag{2.15}$$

from which and (2.1b), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\rho(y)\mu(y)\partial_y\boldsymbol{H},\partial_y\boldsymbol{H})_D\\
=&2\Big(\rho(y)\frac{\mu'(y)}{\mu(y)}\nabla\times\boldsymbol{E},\partial_y\boldsymbol{H}\Big)_D-2(\rho(y)\nabla\times\partial_y\boldsymbol{E},\partial_y\boldsymbol{H})_D\\
=&2\Big(\frac{(\ln\mu(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\epsilon(y)}\nabla\times\boldsymbol{E},\sqrt{\rho(y)\mu(y)}\partial_y\boldsymbol{H}\Big)_D-2(\rho(y)\nabla\times\partial_y\boldsymbol{E},\partial_y\boldsymbol{H})_D\\
\le&C_\mu\Big(\int_D\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2d\boldsymbol{x}+\int_D\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2d\boldsymbol{x}\Big)-2(\rho(y)\nabla\times\partial_y\boldsymbol{E},\partial_y\boldsymbol{H})_D. 
\end{aligned}\tag{2.16}$$

Adding (2.14) and (2.16) together, and using integration by parts and the PEC boundary condition (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_D(\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
\le&\max(C_\epsilon,C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
&+\max(C_\epsilon,C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}\Big).
\end{aligned}\tag{2.17}$$

Integrating (2.17) over the space $\Gamma$, then using Lemma 2.2 and the Gronwall inequality, we conclude the proof. □

Theorem 2.1 shows that if the following initial conditions are bounded:

$$\Big(\int_\Gamma\int_D\big(\rho(y)\epsilon(y)(|\partial_y\boldsymbol{E}|^2+|\nabla\times\boldsymbol{E}|^2)+\rho(y)\mu(y)(|\partial_y\boldsymbol{H}|^2)+|\nabla\times\boldsymbol{H}|^2)\big)d\boldsymbol{x}dy\Big)(0)\le C,$$

then the derivative of the solution with respect to the random variable $y$ is bounded:

$$\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\le C.$$

To obtain higher order error estimates, we need to prove higher regularity estimates. First, we can prove the following energy conservation property for curl-curl of the solution of problem (2.1a)-(2.3).

**Lemma 2.3.** *For the problem* (2.1a)-(2.3), *we have: for any* $t\in[0,T]$,

$$\begin{aligned}
&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\\
=&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0).
\end{aligned}\tag{2.18}$$

*Proof.* Taking $\nabla\times\nabla\times$ of both (2.1a) and (2.1b), we have

$$\epsilon(y(\omega))\partial_t(\nabla\times\nabla\times\boldsymbol{E})=\nabla\times(\nabla\times\nabla\times\boldsymbol{H}), \tag{2.19a}$$

$$\mu(y(\omega))\partial_t(\nabla\times\nabla\times\boldsymbol{H})=-\nabla\times(\nabla\times\nabla\times\boldsymbol{E}). \tag{2.19b}$$

Multiplying (2.19a) by $2\rho(y)\nabla\times\nabla\times\boldsymbol{E}$ and integrating over $D$, we have

$$\frac{d}{dt}((\rho(y)\epsilon(y)\nabla\times\nabla\times\boldsymbol{E},\nabla\times\nabla\times\boldsymbol{E})_D)=2(\rho(y)\nabla\times(\nabla\times\nabla\times\boldsymbol{H}),\nabla\times\nabla\times\boldsymbol{E})_D. \tag{2.20}$$

Multiplying (2.19b) by $2\rho(y)\nabla\times\nabla\times\boldsymbol{H}$, and integrating over $D$, we obtain

$$\begin{aligned}&\frac{d}{dt}((\rho(y)\mu(y)\nabla\times\nabla\times\boldsymbol{H},\nabla\times\nabla\times\boldsymbol{H})_D)\\ =&-2(\rho(y)\nabla\times(\nabla\times\nabla\times\boldsymbol{E}),\nabla\times\nabla\times\boldsymbol{H})_D\\ =&-2\langle\rho(y)\boldsymbol{n}\times(\nabla\times\nabla\times\boldsymbol{E}),\nabla\times\nabla\times\boldsymbol{H}\rangle_{\partial D}\\ &\quad-2(\rho(y)\nabla\times\nabla\times\boldsymbol{E},\nabla\times(\nabla\times\nabla\times\boldsymbol{H}))_D.\end{aligned} \tag{2.21}$$

Using both (2.1a) and (2.1b), we have

$$\nabla\times\nabla\times\boldsymbol{E}=-\mu(y)\partial_t(\nabla\times\boldsymbol{H})=-\mu(y)\epsilon(y)\partial_{tt}\boldsymbol{E},$$

which leads to

$$\boldsymbol{n}\times(\nabla\times\nabla\times\boldsymbol{E})=-\mu(y)\epsilon(y)\partial_{tt}(\boldsymbol{n}\times\boldsymbol{E})=0\quad\text{on }\partial\Omega, \tag{2.22}$$

where the PEC boundary condition (2.3) was used in the last step.

Summing up (2.20) and (2.21), using (2.22), and then integrating over the space $\Gamma$, we have

$$\frac{d}{dt}\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)=0,$$

which concludes the proof. □

The next theorem proves that the curl of the first derivative of the solution to problem (2.1a)-(2.3) is bounded in $L^2$ norm.

**Theorem 2.2.** *Under the same assumption as Theorem 2.1, for any $t\in[0,T]$, we have*

$$\begin{aligned}&\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(t)\\ \le& e^{t\max(C_\epsilon,C_\mu)}\Big[\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(0)\\ &\quad+\left(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\right)(0)\Big].\end{aligned} \tag{2.23}$$

*Proof.* Taking $\nabla\times$ of both sides of (2.13), we have

$$\epsilon(y)\partial_t(\nabla\times\partial_y\boldsymbol{E})=-\epsilon'(y)\partial_t(\nabla\times\boldsymbol{E})+\nabla\times\nabla\times\partial_y\boldsymbol{H}. \tag{2.24}$$

Multiplying (2.24) by $2\rho(y)\nabla\times\partial_y\boldsymbol{E}$, integrating the resultant over $D$, and using (2.1a), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\rho(y)\epsilon(y)\nabla\times\partial_y\boldsymbol{E},\nabla\times\partial_y\boldsymbol{E})_D\\
=&-2\Big(\rho(y)\frac{\epsilon'(y)}{\epsilon(y)}\nabla\times\nabla\times\boldsymbol{H},\nabla\times\partial_y\boldsymbol{E}\Big)_D+2(\rho(y)\nabla\times\nabla\times\partial_y\boldsymbol{H},\nabla\times\partial_y\boldsymbol{E})_D\\
=&-2\Big(\frac{(\ln\epsilon(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\mu(y)}\nabla\times\nabla\times\boldsymbol{H},\sqrt{\rho(y)\epsilon(y)}\nabla\times\partial_y\boldsymbol{E}\Big)_D\\
&\quad+2(\rho(y)\nabla\times\nabla\times\partial_y\boldsymbol{H},\nabla\times\partial_y\boldsymbol{E})_D\\
\le& C_\epsilon\Big(\int_D\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2d\boldsymbol{x}+\int_D\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2d\boldsymbol{x}\Big)\\
&\quad+2(\rho(y)\nabla\times\nabla\times\partial_y\boldsymbol{H},\nabla\times\partial_y\boldsymbol{E})_D.
\end{aligned}\tag{2.25}$$

Similarly, taking $\nabla\times$ of both sides of (2.15) gives

$$\mu(y)\partial_t(\nabla\times\partial_y\boldsymbol{H})=-\mu'(y)\partial_t(\nabla\times\boldsymbol{H})-\nabla\times\nabla\times\partial_y\boldsymbol{E}, \tag{2.26}$$

from which and (2.1b), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\rho(y)\mu(y)\nabla\times\partial_y\boldsymbol{H},\nabla\times\partial_y\boldsymbol{H})_D\\
=&2\Big(\rho(y)\frac{\mu'(y)}{\mu(y)}\nabla\times\nabla\times\boldsymbol{E},\nabla\times\partial_y\boldsymbol{H}\Big)_D-2(\rho(y)\nabla\times\nabla\times\partial_y\boldsymbol{E},\nabla\times\partial_y\boldsymbol{H})_D\\
=&2\Big(\frac{(\ln\mu(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\epsilon(y)}\nabla\times\nabla\times\boldsymbol{E},\sqrt{\rho(y)\mu(y)}\nabla\times\partial_y\boldsymbol{H}\Big)_D\\
&\quad-2\Big(\rho(y)\nabla\times\nabla\times\partial_y\boldsymbol{E},\nabla\times\partial_y\boldsymbol{H}\Big)_D\\
\le& C_\mu\Big(\int_D\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2d\boldsymbol{x}+\int_D\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2d\boldsymbol{x}\Big)\\
&\quad-2(\rho(y)\nabla\times\nabla\times\partial_y\boldsymbol{E},\nabla\times\partial_y\boldsymbol{H})_D.
\end{aligned}\tag{2.27}$$

From (2.13) and the PEC boundary condition (2.3), we have

$$\boldsymbol{n}\times(\nabla\times\partial_y\boldsymbol{H})=\epsilon'(y)\partial_t(\boldsymbol{n}\times\boldsymbol{E})+\epsilon(y)\partial_{ty}(\boldsymbol{n}\times\boldsymbol{E})=0\quad\text{on }\partial\Omega. \tag{2.28}$$

Summing up (2.25) and (2.27), and using integration by parts and (2.28), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
\le&\max(C_\epsilon,C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
&\quad+\max(C_\epsilon,C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}\Big).
\end{aligned}\tag{2.29}$$

Integrating (2.29) over the space $\Gamma$, then using Lemma 2.3 and the Gronwall inequality, we conclude the proof. $\square$

The following theorem proves that the second derivative of the solution to problem (2.1a)-(2.3) is bounded in $L^2$ norm.

**Theorem 2.3.** *Assume that there exist constants $C_\epsilon^*$ and $C_\mu^*$ such that*

$$\frac{2(\epsilon'(y))^2-\epsilon(y)\epsilon''(y)}{\epsilon^2(y)\sqrt{\epsilon(y)\mu(y)}}\le C_\epsilon^*,\quad \frac{2(\mu'(y))^2-\mu(y)\mu''(y)}{\mu^2(y)\sqrt{\epsilon(y)\mu(y)}}\le C_\mu^*,\quad \textit{almost everywhere in } \Gamma. \tag{2.30}$$

*Then under the same assumption as Theorem 2.1 and for any $t\in[0,T]$, we have*

$$\begin{aligned}
&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\\
\le& e^{t\max(2C_\epsilon+C_\epsilon^*,2C_\mu+C_\mu^*)}\Big[\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\
&+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\
&+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\
&+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\Big].
\end{aligned}\tag{2.31}$$

*Proof.* Differentiating both sides of (2.1a) with respect to $y$ twice gives

$$\epsilon(y)\partial_t(\partial_{y^2}\boldsymbol{E})=-\epsilon''(y)\partial_t\boldsymbol{E}-2\epsilon'(y)\partial_{ty}\boldsymbol{E}+\nabla\times\partial_{y^2}\boldsymbol{H}, \tag{2.32}$$

multiplying which by $2\rho(y)\partial_{y^2}\boldsymbol{E}$, then integrating over $D$ and using (2.1a) to replace $\partial_t\boldsymbol{E}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_D\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2d\boldsymbol{x}\Big)\\
=&-2\Big(\rho(y)\frac{\epsilon''(y)}{\epsilon(y)}\nabla\times\boldsymbol{H},\partial_{y^2}\boldsymbol{E}\Big)_D+4\Big(\rho(y)\epsilon'(y)\Big(\frac{\epsilon'(y)}{\epsilon^2(y)}\nabla\times\boldsymbol{H}-\frac{1}{\epsilon(y)}\nabla\times\partial_y\boldsymbol{H}\Big),\partial_{y^2}\boldsymbol{E}\Big)_D\\
&+2(\rho(y)\nabla\times\partial_{y^2}\boldsymbol{H},\partial_{y^2}\boldsymbol{E})_D\\
=&2\Big(\frac{2(\epsilon'(y))^2-\epsilon(y)\epsilon''(y)}{\epsilon^2(y)\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\mu(y)}\nabla\times\boldsymbol{H},\sqrt{\rho(y)\epsilon(y)}\partial_{y^2}\boldsymbol{E}\Big)_D\\
&-4\Big(\frac{(\ln\epsilon(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\mu(y)}\nabla\times\partial_y\boldsymbol{H},\sqrt{\rho(y)\epsilon(y)}\partial_{y^2}\boldsymbol{E}\Big)_D+2(\rho(y)\nabla\times\partial_{y^2}\boldsymbol{H},\partial_{y^2}\boldsymbol{E})_D\\
\le& C_\epsilon^*\Big(\int_D\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2d\boldsymbol{x}+\int_D\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2d\boldsymbol{x}\Big)+2(\rho(y)\nabla\times\partial_{y^2}\boldsymbol{H},\partial_{y^2}\boldsymbol{E})_D\\
&+2C_\epsilon\Big(\int_D\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2d\boldsymbol{x}+\int_D\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2d\boldsymbol{x}\Big).
\end{aligned}\tag{2.33}$$

Similarly, differentiating both sides of (2.1b) with respect to $y$ twice yields

$$\mu(y)\partial_t(\partial_{y^2}\boldsymbol{H})=-\mu''(y)\partial_t\boldsymbol{H}-2\mu'(y)\partial_{ty}\boldsymbol{H}-\nabla\times\partial_{y^2}\boldsymbol{E}, \tag{2.34}$$

multiplying which by $2\rho(y)\partial_{y^2}\boldsymbol{H}$, then integrating over $D$ and using (2.1b) to replace $\partial_t\boldsymbol{H}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_D\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2d\boldsymbol{x}\Big)\\
=&2\Big(\rho(y)\frac{\mu''(y)}{\mu(y)}\nabla\times\boldsymbol{E},\partial_{y^2}\boldsymbol{H}\Big)_D\\
&-4\Big(\rho(y)\mu'(y)\Big(\frac{\mu'(y)}{\mu^2(y)}\nabla\times\boldsymbol{E}-\frac{1}{\mu(y)}\nabla\times\partial_y\boldsymbol{E}\Big),\partial_{y^2}\boldsymbol{H}\Big)_D-2(\rho(y)\nabla\times\partial_{y^2}\boldsymbol{E},\partial_{y^2}\boldsymbol{H})_D\\
=&-2\Big(\frac{2(\mu'(y))^2-\mu(y)\mu''(y)}{\mu^2(y)\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\epsilon(y)}\nabla\times\boldsymbol{E},\sqrt{\rho(y)\mu(y)}\partial_{y^2}\boldsymbol{H}\Big)_D\\
&+4\Big(\frac{(\ln\mu(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\epsilon(y)}\nabla\times\partial_y\boldsymbol{E},\sqrt{\rho(y)\mu(y)}\partial_{y^2}\boldsymbol{H}\Big)_D-2(\rho(y)\nabla\times\partial_{y^2}\boldsymbol{E},\partial_{y^2}\boldsymbol{H})_D\\
\le& C_\mu^*\Big(\int_D\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2d\boldsymbol{x}+\int_D\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2d\boldsymbol{x}\Big)-2(\rho(y)\nabla\times\partial_{y^2}\boldsymbol{E},\partial_{y^2}\boldsymbol{H})_D\\
&+2C_\mu\Big(\int_D\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2d\boldsymbol{x}+\int_D\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2d\boldsymbol{x}\Big).
\end{aligned}\tag{2.35}$$

Adding (2.33) and (2.35) together, and using integration by parts and the PEC boundary condition (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_D(\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
\le&\max(C_\epsilon^*+2C_\epsilon,C_\mu^*+2C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
&+\max(C_\epsilon^*,C_\mu^*)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
&+2\max(C_\epsilon,C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}\Big).
\end{aligned}\tag{2.36}$$

Integrating (2.36) over the space $\Gamma$, then using Lemma 2.2, Theorem 2.2 and the Gronwall inequality, we conclude the proof. □

Below we show that the curl-curl-curl of the solution to problem (2.1a)-(2.3) is energy conserved.

**Lemma 2.4.** *For the problem* (2.1a)-(2.3)*, we have: for any* $t\in[0,T]$*,*

$$\begin{aligned}
&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\\
=&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0).
\end{aligned}\tag{2.37}$$

*Proof.* Taking $\nabla\times\nabla\times\nabla\times$ of (2.1a), then multiplying the resultant by $2\rho(y)\nabla\times\nabla\times\nabla\times\boldsymbol{E}$ and integrating over $D$, we have

$$\begin{aligned}&\frac{d}{dt}(\rho(y)\epsilon(y)\nabla\times\nabla\times\nabla\times\boldsymbol{E},\nabla\times\nabla\times\nabla\times\boldsymbol{E})_D\\=&2(\rho(y)\nabla\times\nabla\times\nabla\times(\nabla\times\boldsymbol{H}),\nabla\times\nabla\times\nabla\times\boldsymbol{E})_D.\end{aligned} \tag{2.38}$$

Similarly, taking $\nabla\times\nabla\times\nabla\times$ of (2.1b), then multiplying the resultant by $2\rho(y)\nabla\times\nabla\times\nabla\times\boldsymbol{H}$ and integrating over $D$, we have

$$\begin{aligned}&\frac{d}{dt}(\rho(y)\mu(y)\nabla\times\nabla\times\nabla\times\boldsymbol{H},\nabla\times\nabla\times\nabla\times\boldsymbol{H})_D\\=&-2(\rho(y)\nabla\times\nabla\times\nabla\times(\nabla\times\boldsymbol{E}),\nabla\times\nabla\times\nabla\times\boldsymbol{H})_D.\end{aligned} \tag{2.39}$$

Using (2.1a) and (2.1b), we have

$$\begin{aligned}&\boldsymbol{n}\times\nabla\times\nabla\times(\nabla\times\boldsymbol{H})=\boldsymbol{n}\times\epsilon(y)\partial_t(\nabla\times\nabla\times\boldsymbol{E})\\=&\boldsymbol{n}\times\epsilon(y)\partial_t(\nabla\times(-\mu(y)\partial_t\boldsymbol{H}))=-\boldsymbol{n}\times\epsilon(y)\mu(y)\partial_{tt}(\epsilon(y)\partial_t\boldsymbol{E})\\=&-\epsilon^2(y)\mu(y)\partial_{t^3}(\boldsymbol{n}\times\boldsymbol{E})=0\quad\text{on }\partial\Omega,\end{aligned} \tag{2.40}$$

where we used the PEC boundary condition (2.3) in the last step.

Summing up (2.38) and (2.39), using integration by parts and (2.40), then integrating over the space $\Gamma$, we complete the proof. □

Now we can show that the curl-curl of the first random derivative of the solution to problem (2.1a)-(2.3) is bounded in the $L^2$ norm.

**Theorem 2.4.** *Under the same assumption as Theorem 2.1, we have: for any $t\in[0,T]$,*

$$\begin{aligned}&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\\\le&e^{t\max(C_\epsilon,C_\mu)}\Big[\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\&+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\Big].\end{aligned} \tag{2.41}$$

*Proof.* Taking $\nabla\times\nabla\times$ of both sides of (2.1a) gives

$$\epsilon(y)\partial_t(\nabla\times\nabla\times\partial_y\boldsymbol{E})=-\epsilon'(y)\partial_t(\nabla\times\nabla\times\boldsymbol{E})+\nabla\times\nabla\times(\nabla\times\partial_y\boldsymbol{H}). \tag{2.42}$$

Using (2.42) and (2.1a), and following the proof of (2.14), we obtain

$$\begin{aligned}&\frac{d}{dt}(\rho(y)\epsilon(y)\nabla\times\nabla\times\partial_y\boldsymbol{E},\nabla\times\nabla\times\partial_y\boldsymbol{E})_D\\=&-2\Big(\rho(y)\frac{\epsilon'(y)}{\epsilon(y)}\nabla\times\nabla\times\nabla\times\boldsymbol{H},\nabla\times\nabla\times\partial_y\boldsymbol{E}\Big)_D\\&+2(\rho(y)\nabla\times\nabla\times(\nabla\times\partial_y\boldsymbol{H}),\nabla\times\nabla\times\partial_y\boldsymbol{E})_D\end{aligned}$$

$$
\begin{aligned}
&\leq C_\epsilon \Big( \int_D \rho(y)\mu(y)|\nabla\times\nabla\times\nabla\times \boldsymbol{H}|^2 d\boldsymbol{x} + \int_D \rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y \boldsymbol{E}|^2 d\boldsymbol{x} \Big) \\
&\quad + 2(\rho(y)\nabla\times\nabla\times(\nabla\times\partial_y \boldsymbol{H}), \nabla\times\nabla\times\partial_y \boldsymbol{E})_D. \qquad (2.43)
\end{aligned}
$$

Similarly, taking $\nabla\times\nabla\times$ of both sides of (2.1b), we have

$$
\mu(y)\partial_t(\nabla\times\nabla\times\partial_y \boldsymbol{H}) = -\mu'(y)\partial_t(\nabla\times\nabla\times \boldsymbol{H}) - \nabla\times\nabla\times(\nabla\times\partial_y \boldsymbol{E}). \qquad (2.44)
$$

Using (2.44) and (2.1b), and following the proof of (2.16), we can obtain

$$
\begin{aligned}
&\frac{d}{dt}(\rho(y)\mu(y)\nabla\times\nabla\times\partial_y \boldsymbol{H}, \nabla\times\nabla\times\partial_y \boldsymbol{H})_D \\
=&2\Big(\rho(y)\frac{\mu'(y)}{\mu(y)}\nabla\times\nabla\times\nabla\times \boldsymbol{E}, \nabla\times\nabla\times\partial_y \boldsymbol{H}\Big)_D \\
&\quad -2(\rho(y)\nabla\times\nabla\times(\nabla\times\partial_y \boldsymbol{E}), \nabla\times\nabla\times\partial_y \boldsymbol{H})_D \\
\leq& C_\mu \Big( \int_D \rho(y)\epsilon(y)|\nabla\times\nabla\times\nabla\times \boldsymbol{E}|^2 d\boldsymbol{x} + \int_D \rho(y)\mu(y)|\nabla\times\nabla\times\partial_y \boldsymbol{H}|^2 d\boldsymbol{x} \Big) \\
&\quad -2(\rho(y)\nabla\times\nabla\times(\nabla\times\partial_y \boldsymbol{E}), \nabla\times\nabla\times\partial_y \boldsymbol{H})_D. \qquad (2.45)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-\boldsymbol{n}\times\nabla\times(\nabla\times\partial_y \boldsymbol{E}) = \boldsymbol{n}\times\big[\mu'(y)\partial_t(\nabla\times \boldsymbol{H}) + \mu(y)\partial_{ty}(\nabla\times \boldsymbol{H})\big] \\
=&\boldsymbol{n}\times\big[\mu'(y)\epsilon(y)\partial_{tt}\boldsymbol{E} + \mu(y)(\epsilon'(y)\partial_{tt}\boldsymbol{E} + \epsilon(y)\partial_{tty}\boldsymbol{E})\big] \\
=&(\mu(y)\epsilon(y))'\partial_{tt}(\boldsymbol{n}\times \boldsymbol{E}) + \mu(y)\epsilon(y)\partial_{tty}(\boldsymbol{n}\times \boldsymbol{E}) = 0 \quad \text{on } \partial\Omega, \qquad (2.46)
\end{aligned}
$$

where in the first equality we used the $\nabla\times$ of (2.15), in the second equality we used (2.1a) and $\partial_t$ of (2.13), and in the last step we used the PEC boundary condition (2.3).

Adding (2.44) and (2.45) together, and using integration by parts and (2.46), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big( \int_D (\rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y \boldsymbol{E}|^2 + \rho(y)\mu(y)|\nabla\times\nabla\times\partial_y \boldsymbol{H}|^2) d\boldsymbol{x} \Big) \\
\leq& \max(C_\epsilon, C_\mu)\Big( \int_D (\rho(y)\epsilon(y)|\nabla\times\nabla\times\nabla\times \boldsymbol{E}|^2 + \rho(y)\mu(y)|\nabla\times\nabla\times\nabla\times \boldsymbol{H}|^2) d\boldsymbol{x} \Big) \\
&\quad + \max(C_\epsilon, C_\mu)\Big( \int_D (\rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y \boldsymbol{E}|^2 + \rho(y)\mu(y)|\nabla\times\nabla\times\partial_y \boldsymbol{H}|^2) d\boldsymbol{x} \Big). \qquad (2.47)
\end{aligned}
$$

Integrating (2.47) over the space $\Gamma$, then using Lemma 2.4 and the Gronwall inequality, we conclude the proof. □

With the above results, we can show that the curl of the second random derivative of the solution to problem (2.1a)-(2.3) is bounded in the $L^2$ norm.

**Theorem 2.5.** *Under the same assumptions as Theorems 2.1 and 2.3, the following estimate holds for any $t\in[0,T]$:*

$$\begin{aligned}&\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(t)\\ \le& e^{t\max(2C_\epsilon+C_\epsilon^*,2C_\mu+C_\mu^*)}\Big[\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\ &+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\ &+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\\ &+\Big(\int_\Gamma\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}dy\Big)(0)\Big].\end{aligned}\tag{2.48}$$

*Proof.* Taking $\nabla\times$ of both sides of (2.32), we have

$$\epsilon(y)\partial_t(\nabla\times\partial_{y^2}\boldsymbol{E})=-\epsilon''(y)\partial_t(\nabla\times\boldsymbol{E})-2\epsilon'(y)\partial_t(\nabla\times\partial_y\boldsymbol{E})+\nabla\times\nabla\times\partial_{y^2}\boldsymbol{H}.\tag{2.49}$$

Using (2.49), and following the similar proof of (2.33), we can obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\int_D\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2d\boldsymbol{x}\Big)\\ =&2\Big(\frac{2(\epsilon'(y))^2-\epsilon(y)\epsilon''(y)}{\epsilon^2(y)\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\mu(y)}\nabla\times\nabla\times\boldsymbol{H},\sqrt{\rho(y)\epsilon(y)}\nabla\times\partial_{y^2}\boldsymbol{E}\Big)_D\\ &-4\Big(\frac{(\ln\epsilon(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\mu(y)}\nabla\times\nabla\times\partial_y\boldsymbol{H},\sqrt{\rho(y)\epsilon(y)}\nabla\times\partial_{y^2}\boldsymbol{E}\Big)_D\\ &+2(\rho(y)\nabla\times\nabla\times\partial_{y^2}\boldsymbol{H},\nabla\times\partial_{y^2}\boldsymbol{E})_D\\ \le& C_\epsilon^*\Big(\int_D\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2d\boldsymbol{x}+\int_D\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2d\boldsymbol{x}\Big)\\ &+2C_\epsilon\Big(\int_D\rho(y)\mu(y)|\nabla\times\nabla\times\partial_y\boldsymbol{H}|^2d\boldsymbol{x}+\int_D\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2d\boldsymbol{x}\Big)\\ &+2(\rho(y)\nabla\times\nabla\times\partial_{y^2}\boldsymbol{H},\nabla\times\partial_{y^2}\boldsymbol{E})_D.\end{aligned}\tag{2.50}$$

Similarly, taking $\nabla\times$ of both sides of (2.34) yields

$$\mu(y)\partial_t(\nabla\times\partial_{y^2}\boldsymbol{H})=-\mu''(y)\partial_t(\nabla\times\boldsymbol{H})-2\mu'(y)\partial_t(\nabla\times\partial_y\boldsymbol{H})-\nabla\times\nabla\times\partial_{y^2}\boldsymbol{E}.\tag{2.51}$$

Using (2.51), and following the similar proof of (2.35), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\int_D\rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2d\boldsymbol{x}\Big)\\ =&-2\Big(\frac{2(\mu'(y))^2-\mu(y)\mu''(y)}{\mu^2(y)\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\epsilon(y)}\nabla\times\nabla\times\boldsymbol{E},\sqrt{\rho(y)\mu(y)}\nabla\times\partial_{y^2}\boldsymbol{H}\Big)_D\\ &+2\Big(\frac{(\ln\mu(y))'}{\sqrt{\epsilon(y)\mu(y)}}\sqrt{\rho(y)\epsilon(y)}\nabla\times\nabla\times\partial_y\boldsymbol{E},\sqrt{\rho(y)\mu(y)}\nabla\times\partial_{y^2}\boldsymbol{H}\Big)_D\\ &-2(\rho(y)\nabla\times\nabla\times\partial_{y^2}\boldsymbol{E},\nabla\times\partial_{y^2}\boldsymbol{H})_D\end{aligned}$$

$$
\begin{aligned}
&\le C_\mu^*\Big(\int_D \rho(y)\epsilon(y)|\nabla\times\nabla\times \boldsymbol{E}|^2 d\boldsymbol{x}+\int_D \rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2 d\boldsymbol{x}\Big)\\
&\quad +2C_\mu\Big(\int_D \rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y \boldsymbol{E}|^2 d\boldsymbol{x}+\int_D \rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2 d\boldsymbol{x}\Big)\\
&\quad -2(\rho(y)\nabla\times\nabla\times\partial_{y^2}\boldsymbol{E},\nabla\times\partial_{y^2}\boldsymbol{H})_D. \qquad (2.52)
\end{aligned}
$$

Using (2.32) and the PEC boundary condition (2.3), we see that

$$
\begin{aligned}
&\boldsymbol{n}\times(\nabla\times\partial_{y^2}\boldsymbol{H})\\
=&\epsilon''(y)\partial_t(\boldsymbol{n}\times\boldsymbol{E})+2\epsilon'(y)\partial_{ty}(\boldsymbol{n}\times\boldsymbol{E})+\epsilon(y)\partial_{ty^2}(\boldsymbol{n}\times\boldsymbol{E})=0 \quad \text{on } \partial\Omega. \qquad (2.53)
\end{aligned}
$$

Adding (2.50) and (2.52) together, and using integration by parts and (2.53), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
\le&\max(C_\epsilon^*+2C_\epsilon,C_\mu^*+2C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\partial_{y^2}\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\partial_{y^2}\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
&+\max(C_\epsilon^*,C_\mu^*)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\boldsymbol{H}|^2)d\boldsymbol{x}\Big)\\
&+2\max(C_\epsilon,C_\mu)\Big(\int_D(\rho(y)\epsilon(y)|\nabla\times\nabla\times\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\nabla\times\nabla\times\partial_y\boldsymbol{H}|^2)d\boldsymbol{x}\Big). \qquad (2.54)
\end{aligned}
$$

Integrating (2.54) over the space $\Gamma$, then using Lemma 2.4, Theorem 2.4 and the Gronwall inequality, we conclude the proof. □

**Remark 2.1.** We would like to mention that similar regularity analysis can be extended to random vectors (cf. our recent work for more complicated metamaterial Maxwell's equations [15]). If the random parameters are smooth enough, then the boundness of higher derivatives can be proved similarly.

## 2.2 Convergence analysis

To prove the convergence estimate for the collocation method, let us first recall the following interpolation error estimates.

**Lemma 2.5** (see [6]). *Let $I_N u$ denote the polynomial of degree $N$ that interpolates $u$ at the $(N+1)$ Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{y_j\}_{j=0}^N$, i.e., $I_N u(y)=\sum_{j=0}^N u(y_j)\mathcal{L}_j(y)$. Then we have the interpolation error in the $L^2$-norm:*

$$||u-I_N u||_{L^2(-1,1)}\le CN^{-m}|u|_{H^m(-1,1)}, \quad \forall u\in H^m(-1,1) \ \text{with} \ m\ge 1, \qquad (2.55)$$

*and the interpolation error in the $H^l$-norm:*

$$||u-I_N u||_{H^l(-1,1)}\le CN^{2l-\frac{1}{2}-m}|u|_{H^m(-1,1)}, \quad \forall u\in H^m(-1,1) \ \text{with} \ m\ge l\ge 1. \qquad (2.56)$$

*For the Gauss-Lobatto interpolation, we have the following optimal error estimate:*

$$||(u-I_N u)'||_{L^2(-1,1)}\le CN^{1-m}|u|_{H^m(-1,1)}, \quad \forall u\in H^m(-1,1) \ \text{with} \ m\ge 1. \qquad (2.57)$$

To present the error estimate, recall the mean (or expectation) of a function $u$ is defined by

$$E[u]=\int_\Gamma\int_D\rho(y)u(x,y)d\boldsymbol{x}dy, \tag{2.58}$$

and its mean square is defined by

$$M[u]=\left(\int_\Gamma\int_D\rho(y)|u(x,y)|^2d\boldsymbol{x}dy\right)^{1/2}. \tag{2.59}$$

**Theorem 2.6.** *Let* $(\boldsymbol{E},\boldsymbol{H})$ *be the solution of* (2.1a)-(2.3), *and*

$$\boldsymbol{E}^N:=I_N^y\boldsymbol{E}(\boldsymbol{x},t;y)=\sum_{k=0}^N\boldsymbol{E}(\boldsymbol{x},t;y_k)\mathcal{L}_k(y),\quad \boldsymbol{H}^N:=I_N^y\boldsymbol{H}(\boldsymbol{x},t;y)=\sum_{k=0}^N\boldsymbol{H}(\boldsymbol{x},t;y_k)\mathcal{L}_k(y),$$

*be the Gauss-Lobatto interpolation of the solution* $(\boldsymbol{E},\boldsymbol{H})$. *If the assumptions of Theorems 2.1 and 2.2 are satisfied, then the following mean and mean square errors hold: For any* $0<t\le T$,

$$M[\boldsymbol{E}-\boldsymbol{E}^N]+M[\boldsymbol{H}-\boldsymbol{H}^N]+M[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)]+M[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)]\le C_TN^{-1}, \tag{2.60a}$$
$$E[|\boldsymbol{E}-\boldsymbol{E}^N|]+E[|\boldsymbol{H}-\boldsymbol{H}^N|]+E[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)]+E[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)]\le C_TN^{-1}. \tag{2.60b}$$

*Here and below* $C_T$ *is a constant depending on* $T$ *but independent of* $N$. *Furthermore, if the assumptions of Theorems 2.3 and 2.5 are satisfied, then we have the following higher error estimates: For any* $0<t\le T$,

$$M[\boldsymbol{E}-\boldsymbol{E}^N]+M[\boldsymbol{H}-\boldsymbol{H}^N]+M[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)]+M[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)]\le C_TN^{-2}, \tag{2.61a}$$
$$E[|\boldsymbol{E}-\boldsymbol{E}^N|]+E[|\boldsymbol{H}-\boldsymbol{H}^N|]+E[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)]+E[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)]\le C_TN^{-2}. \tag{2.61b}$$

*Finally, if the assumptions of Theorems 2.3 and 2.5 are satisfied, for the Gauss-Lobatto interpolation, we have the error estimate for the derivative of the solution with respect to random variable: For any* $0<t\le T$,

$$M[(\boldsymbol{E}-\boldsymbol{E}^N)']+M[(\boldsymbol{H}-\boldsymbol{H}^N)']+M[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)']+M[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)']\le C_TN^{-1}, \tag{2.62a}$$
$$E[|(\boldsymbol{E}-\boldsymbol{E}^N)'|]+E[|(\boldsymbol{H}-\boldsymbol{H}^N)'|]+E[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)']+E[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)']\le C_TN^{-1}. \tag{2.62b}$$

*Proof.* For any fixed $\boldsymbol{x}$, using (2.55) of Lemma 2.5 for $u=\boldsymbol{E}$ and $u=\boldsymbol{H}$ with $m=1$, respectively, we have

$$\begin{aligned}&\int_\Gamma\Big(\rho(y)\epsilon(y)|\boldsymbol{E}(\boldsymbol{x},t;y)-\boldsymbol{E}^N(\boldsymbol{x},t;y)|^2+\rho(y)\mu(y)|\boldsymbol{H}(\boldsymbol{x},t;y)-\boldsymbol{H}^N(\boldsymbol{x},t;y)|^2\Big)dy\\ \le& CN^{-2}\int_\Gamma\big(\rho(y)\epsilon(y)|\partial_y\boldsymbol{E}|^2+\rho(y)\mu(y)|\partial_y\boldsymbol{H}|^2\big)dy.\end{aligned} \tag{2.63}$$

Similarly, using (2.55) of Lemma 2.5 for $u=\nabla\times\boldsymbol{E}$ and $u=\nabla\times\boldsymbol{H}$ with $m=1$, respectively, we have

$$\begin{aligned}&\int_\Gamma\Big(\rho(y)\epsilon(y)|\nabla\times(\boldsymbol{E}(\boldsymbol{x},t;y)-\boldsymbol{E}^N(\boldsymbol{x},t;y))|^2\\&\quad+\rho(y)\mu(y)|\nabla\times(\boldsymbol{H}(\boldsymbol{x},t;y)-\boldsymbol{H}^N(\boldsymbol{x},t;y))|^2\Big)dy\\\le& CN^{-2}\int_\Gamma\big(\rho(y)\epsilon(y)|\partial_y(\nabla\times\boldsymbol{E})|^2+\rho(y)\mu(y)|\partial_y(\nabla\times\boldsymbol{H})|^2\big)dy.\end{aligned}\tag{2.64}$$

Adding (2.63) and (2.64) together, then integrating the resultant with respect to $\boldsymbol{x}$ over $D$ and using Theorems 2.1 and 2.2, we complete the proof of (2.60a).

The estimates (2.61a) can be proved similarly by using (2.55) of Lemma 2.5 with $m=2$ and the higher regularity proved in Theorems 2.3 and 2.5.

Similarly, using (2.57) of Lemma 2.5 with $m=2$, and the higher regularity proved in Theorems 2.3 and 2.5, we obtain the proof of (2.62a).

Finally, the mean errors (2.60b), (2.61b) and (2.62b) follow from the standard inequality $||u||_{L^1}\le C||u||_{L^2}$ and the estimates (2.60a), (2.61a) and (2.62a). □

With the above interpolation estimate, we can show that the overall errors for solving the Maxwell's equations by the Crank-Nicolson scheme (cf. [24]) can be estimated as follows. Denote $(\boldsymbol{E}^N_{h,\Delta t},\boldsymbol{H}^N_{h,\Delta t})$ for the numerical solution of the fully-discrete solution obtained with the Crank-Nicolson scheme with mesh size $h$, time step size $\Delta t$ and the Gauss-Lobatto interpolation of degree $N$. Using the interpolation, the numerical solution can be written as a function of the random variable, i.e., we treat $\boldsymbol{E}^N_{h,\Delta t}(y)=\sum_{k=0}^N\boldsymbol{E}_{h,\Delta t}(y_k)\mathcal{L}_k(y)$ and $\boldsymbol{H}^N_{h,\Delta t}(y)=\sum_{k=0}^N\boldsymbol{H}_{h,\Delta t}(y_k)\mathcal{L}_k(y)$, where $\boldsymbol{E}_{h,\Delta t}(y_k)$ and $\boldsymbol{H}_{h,\Delta t}(y_k)$ are the numerical solution at each collocation point $y_k$. Denote the discrete $L^2$-norm over the physical space $D$ as $|\cdot|_{l^2(D)}$ (cf. [24]). Then we can obtain the discrete mean square error as following:

$$\begin{aligned}&\Big(\int_\Gamma\rho\epsilon|\boldsymbol{E}-\boldsymbol{E}^N_{h,\Delta t}|^2_{l^2(D)}dy\Big)^{1/2}\\\le&\Big(\int_\Gamma 2\rho\epsilon(|\boldsymbol{E}-\boldsymbol{E}^N|^2_{l^2(D)}+|\boldsymbol{E}^N-\boldsymbol{E}^N_{h,\Delta t}|^2_{l^2(D)})dy\Big)^{1/2}\\\le& C[N^{-m}+(h^2+(\Delta t)^2)],\end{aligned}\tag{2.65}$$

where we used the error estimate of Crank-Nicolson scheme and Theorem 2.6. The error estimate for $\boldsymbol{H}$ can be bounded similarly. Same error bounds can be extened to random vector case (see our recent work [15]).

## 3 Numerical results

To justify our theoretical analysis, here we present some numerical results carried out for the two-dimensional (2D) Maxwell's equations in $TE_z$ mode, whose governing equations are:

$$\epsilon(y)\frac{\partial E_{x_1}}{\partial t}=\frac{\partial H}{\partial x_2}+g_1,\quad \epsilon(y)\frac{\partial E_{x_2}}{\partial t}=-\frac{\partial H}{\partial x_1}+g_2, \tag{3.1a}$$

$$\mu(y)\frac{\partial H}{\partial t}=-\left(\frac{\partial E_{x_2}}{\partial x_1}-\frac{\partial E_{x_1}}{\partial x_2}\right)+g_3, \tag{3.1b}$$

where $g_1$, $g_2$ and $g_3$ are added source terms in order to construct an exact solution to check convergence rate.

First, we would like to mention that the theoretical analysis of Section 2 carries directly to 2D by interpreting the curl operators carefully. For the $TE_z$ mode, the electric field $\boldsymbol{E}=(E_{x_1},E_{x_2})'$ is a 2D vector, the magnetic field $H$ is a scalar, and the 2D curl operators become as

$$\nabla\times\boldsymbol{E}=\frac{\partial E_{x_2}}{\partial x_1}-\frac{\partial E_{x_1}}{\partial x_2},\quad \nabla\times H=\left(\frac{\partial H}{\partial x_2},-\frac{\partial H}{\partial x_1}\right)'.$$

In our numerical tests, we solve the $TE_z$ model (3.1a)-(3.1b) on physical domain $[0,1]^2$ and time domain $[0,1]$ by the following Crank-Nicolson scheme:

$$\begin{aligned}
&\epsilon(y(\omega))\frac{E_{x_1}^{n+1}-E_{x_1}^{n}}{\Delta t}\Big|_{i+1/2,j}\\
&=\frac{1}{\Delta x_2}\left(\frac{H^{n+1}+H^n}{2}\Big|_{i+1/2,j+1/2}-\frac{H^{n+1}+H^n}{2}\Big|_{i+1/2,j-1/2}\right)+g_1\Big|_{i+1/2,j},\\
&\epsilon(y(\omega))\frac{E_{x_2}^{n+1}-E_{x_2}^{n}}{\Delta t}\Big|_{i,j+1/2}=-\frac{1}{\Delta x_1}\left(\frac{H^{n+1}+H^n}{2}\Big|_{i+1/2,j+1/2}-\frac{H^{n+1}+H^n}{2}\right)\Big|_{i-1/2,j+1/2}+g_2\Big|_{i,j+1/2},\\
&\mu(y(\omega))\frac{H^{n+1}-H^{n}}{\Delta t}\Big|_{i+1/2,j+1/2}=\frac{1}{\Delta x_2}\left(\frac{E_{x_1}^{n+1}+E_{x_1}^{n}}{2}\Big|_{i+1/2,j+1}-\frac{E_{x_1}^{n+1}+E_{x_1}^{n}}{2}\Big|_{i+1/2,j}\right)\\
&\qquad\qquad-\frac{1}{\Delta x_1}\left(\frac{E_{x_2}^{n+1}+E_{x_2}^{n}}{2}\Big|_{i+1,j+1/2}-\frac{E_{x_2}^{n+1}+E_{x_2}^{n}}{2}\Big|_{i,j+1/2}\right)+g_3\Big|_{i+1/2,j+1/2},
\end{aligned}$$

where $\Delta t$, $\Delta x_1$, $\Delta x_2$ denote the time step size, and mesh size in $x_1$ and $x_2$ directions, respectively.

**Example 3.1.** Here we choose random parameters: $\mu=U(0.9,1)$ and $\epsilon=U(0.9,1)$, where $U(a,b)$ denotes the uniform distribution on $[a,b]$. To test the convergence rate, we construct the exact solution of (3.1a)-(3.1b) as follows:

$$E_{x_1}=\cos(\pi(x_1+\epsilon(y)))\sin(\pi x_2)e^{-\pi t}, \tag{3.2a}$$

$$E_{x_2}=-\sin(\pi x_1)\cos(\pi(x_2+\epsilon(y)))e^{-\pi t}, \tag{3.2b}$$

$$H=\cos(\pi(x_1+\mu(y)))\cos(\pi(x_2+\mu(y)))e^{-\pi t}, \tag{3.2c}$$

which satisfies the 2D PEC boundary condition and results in the added source terms in

Table 1: Errors of $(E_{x_1}, E_{x_2}, H)$ on the uniform grids.

| Mesh | 1/4 | 1/8 | Rate | 1/16 | Rate | 1/32 | Rate |
|---|---|---|---|---|---|---|---|
| $E[\|E_{x_1}-E^h_{x_1}\|]$ | $8.784989E-03$ | $1.178593E-03$ | 2.8980 | $4.474686E-04$ | 2.1476 | $1.177626E-04$ | 2.0060 |
| $M[\|E_{x_1}-E^h_{x_1}\|]$ | $8.791081E-03$ | $1.205826E-03$ | 2.8660 | $4.507609E-04$ | 2.1428 | $1.184576E-04$ | 2.0060 |
| $E[\|E_{x_2}-E^h_{x_2}\|]$ | $8.784989E-03$ | $1.178593E-03$ | 2.8980 | $4.474686E-04$ | 2.1476 | $1.177626E-04$ | 2.0060 |
| $M[\|E_{x_2}-E^h_{x_2}\|]$ | $8.791081E-03$ | $1.205826E-03$ | 2.8660 | $4.507609E-04$ | 2.1428 | $1.184576E-04$ | 2.0060 |
| $E[\|H-H^h_{x_1}\|]$ | $7.575140E-03$ | $3.164382E-03$ | 1.2593 | $6.988307E-04$ | 1.7191 | $1.687477E-04$ | 1.8644 |
| $M[\|H-H^h_{x_1}\|]$ | $7.602439E-03$ | $3.166105E-03$ | 1.2638 | $7.014300E-04$ | 1.7190 | $1.696368E-04$ | 1.8632 |

Table 2: Errors of $\nabla\times E$ and $\nabla\times H$ on uniform grids.

| Mesh | 1/4 | 1/8 | Rate | 1/16 | Rate | 1/32 | Rate |
|---|---|---|---|---|---|---|---|
| $E[\|\nabla\times(E-E^h)\|]$ | $2.719848E-02$ | $6.000992E-03$ | 2.1803 | $2.529191E-03$ | 1.7134 | $6.990969E-04$ | 1.7092 |
| $M[\|\nabla\times(E-E^h)\|]$ | $6.468009E-02$ | $9.273281E-03$ | 2.8022 | $3.424586E-03$ | 2.1197 | $9.116299E-04$ | 1.9883 |
| $E[\|\nabla\times(H-H^h)\|]$ | $4.240936E-02$ | $2.251843E-02$ | 0.9133 | $4.762980E-03$ | 1.5772 | $1.110742E-03$ | 1.8006 |
| $M[\|\nabla\times(H-H^h)\|]$ | $8.433797E-02$ | $2.392789E-02$ | 1.8175 | $4.549261E-03$ | 2.1062 | $1.027735E-03$ | 2.1471 |
| $CPUtime(s)$ | 0.095223 | 2.865157 | | 128.771754 | | 10128.262903 | |

(3.1a)-(3.1b) as follows:

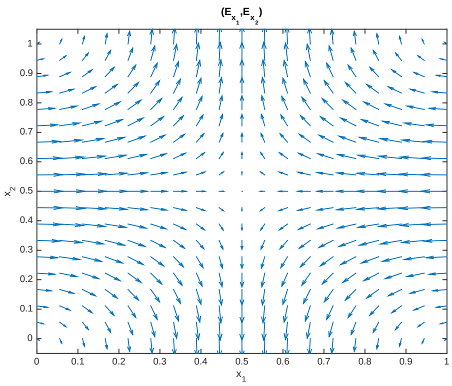
$$
\begin{aligned}
g_1 &= e^{-\pi t}\pi\cos(\pi(x_1+\epsilon(y)))[-\epsilon(y)\sin(\pi x_2)+\sin(\pi(x_2+\mu_0(y)))],\\
g_2 &= -e^{-\pi t}\pi\cos(\pi(x_2+\epsilon(y)))[-\epsilon(y)\sin(\pi x_1)+\sin(\pi(x_1+\epsilon(y)))],\\
g_3 &= -e^{-\pi t}\pi[\cos(\pi(x_2+\mu(y)))\cos(\pi(x_1+\mu(y)))\mu(y)\\
&\quad +\cos(\pi x_1)\cos(\pi(x_2+\mu(y)))+\cos(\pi x_2)\cos(\pi(x_1+\mu(y)))].
\end{aligned}
$$

For simplicity, in our simulation we chose uniform grids with $\Delta t=\Delta x_1=\Delta x_2=1/N$ varying from 1/4 to 1/32. The obtained errors of the solution $(E_{x_1},E_{x_2},H)$ in terms of $E[\cdot]$ and $M[\cdot]$ are presented in Table 1, which shows clearly that the solution converges in the rate of $\mathcal{O}(N^{-2})$ as our theoretical analysis proves.

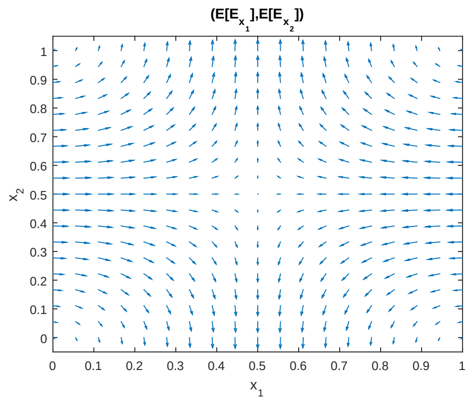
The CPU time (in seconds) and errors of $\nabla\times E$ and $\nabla\times H$ are present in Table 2. The results show that the convergence rate is also $\mathcal{O}(N^{-2})$, which is consistent with the theoretical analysis.

We also presented a sample of the electric field in Fig. 1 to compare with the mean electric field. The sample is obtained by solving the problem on a $20\times 20$ spatial uniform partition of $[0,1]^2$ with random variables $\epsilon=U(0.5,1.5)$ and $\mu=U(0.5,1.5)$. We set the initial value by using (3.2a)-(3.2c) and zero sources $g_1=g_2=g_3=0$. In Fig. 1(a), the electric field $\boldsymbol{E}$ obtained with the random sample $\epsilon=0.8093$ and $\mu=1.2145$ is shown. In Fig. 1(b), the mean value of $\boldsymbol{E}$ is shown. The difference of the mean and the sample $\boldsymbol{E}$ is presented in Fig. 1(c).

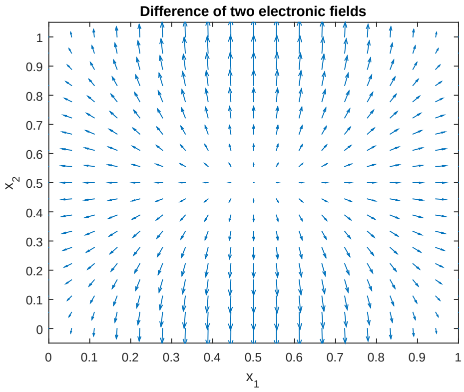
**Example 3.2.** This example is aimed at checking the applicability and efficiency of sparse grid method in solving the stochastic Maxwell's equations. The sparse grid method was introduced in [21] and has been widely studied and used in stochastic collocation methods (e.g., [3,5,19]). We use all the same parameters and algorithm as Example 3.1 but the numerical quadrature. Smolyak sparse grid quadrature has been used in this example.




(a) The sample field $\boldsymbol{E}$: $\epsilon = 0.8093$ and $\mu = 1.2145$




(b) The mean field $\boldsymbol{E}$




(c) The difference of the mean and the sample $\boldsymbol{E}$

Figure 1: Comparison of the mean value and a random sample of electric fields $\boldsymbol{E}$.

To employ the Smolyak sparse quadrature, a Gaussian quadrature formula has been used for each sub-quadrature. The numerical results obtained are presented in Table 3 and Table 4. The errors demonstrate again the second order convergence rates $\mathcal{O}(N^{-2})$ for the solution and curl of the solution by Tables 3 and 4, respectively. In the last row of Table 4, we also presented the CPU time, which shows that there is a big saving of the CPU time

Table 3: Errors of $(E_{x_1}, E_{x_2}, H)$ on sparse grid.

| Mesh | 1/4 | 1/8 | Rate | 1/16 | Rate | 1/32 | Rate |
|---|---|---|---|---|---|---|---|
| $E[\lvert E_{x_1} - E^h_{x_1}\rvert]$ | $2.778348E-02$ | $7.076639E-03$ | 1.9731 | $2.627963E-03$ | 1.7011 | $6.561562E-04$ | 1.7641 |
| $M[\lvert E_{x_1} - E^h_{x_1}\rvert]$ | $1.582512E-02$ | $3.424850E-03$ | 2.2081 | $1.364815E-03$ | 1.7677 | $3.411260E-04$ | 1.7935 |
| $E[\lvert E_{x_2} - E^h_{x_2}\rvert]$ | $2.778346E-02$ | $7.076638E-03$ | 1.9731 | $2.627963E-03$ | 1.7011 | $6.561562E-04$ | 1.7641 |
| $M[\lvert E_{x_2} - E^h_{x_2}\rvert]$ | $1.582513E-02$ | $3.424852E-03$ | 2.2081 | $1.364815E-03$ | 1.7677 | $3.411260E-04$ | 1.7935 |
| $E[\lvert H - H^h_{x_1}\rvert]$ | $2.453646E-02$ | $7.204936E-03$ | 1.7679 | $2.400917E-03$ | 1.6766 | $6.125183E-04$ | 1.7557 |
| $M[\lvert H - H^h_{x_1}\rvert]$ | $1.398775E-02$ | $4.251030E-03$ | 1.7183 | $1.257310E-03$ | 1.7379 | $3.141627E-04$ | 1.8187 |

Table 4: Errors of $\nabla\times E$ and $\nabla\times H$ on sparse grid.

| Mesh | 1/4 | 1/8 | Rate | 1/16 | Rate | 1/32 | Rate |
|---|---|---|---|---|---|---|---|
| $E[\lvert\nabla\times(E-E^h)\rvert]$ | $8.657887E-02$ | $2.169993E-02$ | 1.9963 | $5.064740E-03$ | 2.0477 | $1.165036E-03$ | 2.0746 |
| $M[\lvert\nabla\times(E-E^h)\rvert]$ | $1.104523E-01$ | $2.653268E-02$ | 2.0576 | $5.034825E-03$ | 2.2277 | $1.051978E-03$ | 2.2540 |
| $E[\lvert\nabla\times(H-H^h)\rvert]$ | $1.354196E-01$ | $3.217606E-02$ | 2.0734 | $9.996672E-03$ | 1.8799 | $2.845675E-03$ | 1.8404 |
| $M[\lvert\nabla\times(H-H^h)\rvert]$ | $1.508790E-01$ | $5.451798E-02$ | 1.4686 | $1.114706E-02$ | 1.8793 | $2.824012E-03$ | 1.9509 |
| CPU time (s) | 0.020172 | 0.923809 | | 19.367562 | | 4724.081671 | |

Table 5: Errors of $E$ and $H$.

| Mesh | 1/4 | 1/8 | Rate | 1/16 | Rate | 1/32 | Rate |
|---|---|---|---|---|---|---|---|
| $E[\lvert E-E^h\rvert]$ | $1.224574e-01$ | $3.080085E-02$ | 1.9912 | $7.690579E-03$ | 1.9965 | $1.991338E-03$ | 1.9829 |
| $M[\lvert E-E^h\rvert]$ | $4.076265e-02$ | $1.025696e-02$ | 1.9928 | $2.559606e-03$ | 1.9957 | $6.634349e-04$ | 1.9825 |
| $E[\lvert H-H^h\rvert]$ | $3.497023e-02$ | $8.512993E-03$ | 2.0384 | $2.012699E-03$ | 2.0595 | $5.063236E-04$ | 2.0410 |
| $M[\lvert H-H^h\rvert]$ | $1.167481e-02$ | $2.834366e-03$ | 2.0423 | $6.699305e-04$ | 2.0616 | $1.686740e-04$ | 2.0420 |
| CPU time (s) | 4.986498 | 52.183323 | | 283.131567 | | 6745.458795 | |

by using the sparse grid quadrature compared to the uniform grid quadrature. Note that our simulations are done on a 2017 MacBook Pro laptop with processor of 2.8GHz Intel Core i7, and memory of 16GB 2133MHz LPDDR3.

**Example 3.3.** Upon the suggestion of the reviewer, we added this example for a simple comparison of stochastic collocation method with the generalized polynomial chaos (gPC) method we are working on [9]. We used the 5th order chaotic polynomial to approximate the random $\epsilon$ and $\mu$. We repeated the same numerical example and the numerical result obtained by the gPC method is shown in Table 5 with the CPU time presented in the last row of Table 5. The results shows that the gPC method leads to similar accuracy but costs more CPU time compared to the sparse grid method (cf. Table 3).

# 4 Conclusions

For the first time, we established the rigorous regularity of the solution of the stochastic time-dependent Maxwell's equations with random coefficients and random initial conditions. The stochastic collocation method originally introduced by Xiu and Hesthaven [26] is applied to solve the stochastic Maxwell's equations. The convergence of the method is proved by using the regularity results obtained. Numerical results are presented and justify the theoretical analysis. In the future, we will explore more efficient stochastic methods such as Quasi Monte Carlo method and Mutli-Level Monte Carlo for solving Maxwell's equations.

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