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# Doubling Algorithm for Nonsymmetric Algebraic Riccati Equations Based on a Generalized Transformation

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**Abstract.** We consider computing the minimal nonnegative solution of the nonsymmetric algebraic Riccati equation with $M$-matrix. It is well known that such equations can be efficiently solved via the structure-preserving doubling algorithm (SDA) with the shift-and-shrink transformation or the generalized Cayley transformation. In this paper, we propose a more generalized transformation of which the shift-and-shrink transformation and the generalized Cayley transformation could be viewed as two special cases. Meanwhile, the doubling algorithm based on the proposed generalized transformation is presented and shown to be well-defined. Moreover, the convergence result and the comparison theorem on convergent rate are established. Preliminary numerical experiments show that the doubling algorithm with the generalized transformation is efficient to derive the minimal nonnegative solution of nonsymmetric algebraic Riccati equation with $M$-matrix.

**AMS subject classifications**: 65F50, 15A24

**Key words**: Shift-and-shrink transformation, generalized Cayley transformation, doubling algorithm, nonsymmetric algebraic Riccati equation.

## 1 Introduction

Consider solving the nonsymmetric algebraic Riccati equation

$$XCX-AX-XD+B=0 \tag{1.1}$$

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and its dual form

$$YBY-DY-YA+C=0, \tag{1.2}$$

where $A$, $B$, $C$, $D$ are real matrices of size $m\times m$, $m\times n$, $n\times m$, $n\times n$, respectively. In many real-life applications such as the transport theory related to the transmission of a particle beam [16] and the Markov process [3], the coefficient matrices in (1.1) and (1.2) constitute a block structure of $M$-matrix, that is,

$$K=\begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \tag{1.3}$$

is a nonsingular $M$-matrix or an irreducible singular $M$-matrix. In this sense, we referred such equations as $M$-matrix algebraic Riccati equations (MAREs) [21]. The minimal nonnegative solution of the MARE is of great interest in real applications and its existence has been well studied in [15,16]. Lots of numerical iterative methods including the Newton's method and the fixed-point methods have been extensively studied to find the minimal nonnegative solution of MAREs, see [1,2,5,7,8,10,11,13,16,21,22] and the references therein. Among these methods, the structure-preserving doubling algorithm stands out for its quadratical convergence analogous to Newton's method and the faster convergence than fixed-point methods. With incorporating different matrix transformations, the doubling algorithm has various initial processes and they are respectively referred as the SDA [13], the SDA-ss [6] and the ADDA [21]. Concretely, for a matrix $A$, the SDA employs the Cayley transformation

$$\mathcal{C}(A,\alpha)=(A-\alpha I)(A+\alpha I)^{-1} \tag{1.4}$$

with $\alpha>0$. The SDA-ss admits the shift-and-shrink transformation

$$\mathcal{S}(A,\gamma)=I-A/\gamma \tag{1.5}$$

with $\gamma>0$ and the ADDA exploits the generalized Cayley transformation

$$\mathcal{G}(A,\beta,\alpha)=(A-\beta I)(A+\alpha I)^{-1} \tag{1.6}$$

with $\alpha>0$ and $\beta>0$. By selecting proper parameters $\alpha$, $\beta$ and $\gamma$, each of all above transformations is able to transfer $n$ eigenvalues of Hamiltonian-like matrix

$$H=\begin{pmatrix} D & -C \\ B & -A \end{pmatrix} \tag{1.7}$$

on the left complex semi-plane to ones inside the unit circle and other $n$ eigenvalues of $H$ on the right complex semi-plane to ones outside the unit circle. Then each initial process of these doubling algorithms is well-defined and the whole iteration scheme is feasible.

In this paper, we still aim at the doubling algorithm for solving MAREs, but with a transformation different with (1.4), (1.5) and (1.6). The development story origins from the observation of the shrink-and-shift transformation $I-A/\gamma$ might be seen as one-order Taylor expansion of an exponential transformation $e^{-A/\gamma}$, which has been exploited in [12] for solving a class of differential Riccati equations with $M$-matrix. One may wonder whether a higher-order approximation could be applicable. This helps us develop a new generalized transformation inheriting some characteristics both from the shrink-and-shift transformation (1.5) and the generalized Cayley transformation (1.6). Later we will show in theory that the doubling algorithm based on our new presented transformation could provide a faster convergence rate than the other three doubling algorithms, i.e., the SDA, the SDA-ss and the ADDA.

The rest of this paper is organized as follows. Section 2 presents our generalized transformation as well as several known results about $M$-matrices. Section 3 is devoted to the well-definition of the initial process for the doubling algorithm with our new generalized transformation. The whole algorithm and the corresponding convergence analysis is given in Section 4, also accompanied with the detailed comparison result on convergence rate between the ADDA and our doubling algorithm. Numerical results to demonstrate the efficiency of the doubling algorithm with the new generalized transformation are finally presented in Section 5.

**Notation.** Symbol $\mathbb{R}^{n\times n}$ in this paper stands for the set of $n\times n$ real matrices. For matrices $A$, $B\in\mathbb{R}^{n\times n}$, we write $A\geq B$ $(A>B)$ if $a_{ij}\geq b_{ij}$ $(a_{ij}>b_{ij})$ for all $i$, $j$. A real square matrix $A$ is called a $Z$-matrix if all its off-diagonal elements are nonpositive. It is clear that any $Z$-matrix $A$ can be written as $sI-B$ with $B\geq 0$. A $Z$-matrix $A=sI-B$ with $B\geq 0$ is called an $M$-matrix if $s\geq\rho(B)$, where $\rho(\cdot)$ denotes the spectral radius. It is called a singular $M$-matrix if $s=\rho(B)$ and a nonsingular $M$-matrix if $s>\rho(B)$. Given a square matrix A, we will denote by $\sigma(A)$ the set of eigenvalues of $A$, and $\|A\|$ the Euclid norm of $A$.

## 2 The generalized transformation and some lemmas

It is known that the generalized Cayley transformation (1.6) with two parameters is more sensitive to the magnitude of two matrices than the Cayley transformation (1.4) which uses only one parameter. So the ADDA embedded in transformation (1.6) is superior to the SDA with transformation (1.4). On the other hand, the shrink-and-shift transformation, first given by Ramaswami in [19], was employed to construct the doubling algorithm SDA-ss [6], which had shown dramatic improvements over the SDA in most cases but might still run slower sometimes, although not often happened. Wang, Wang and Li [21] subsequently proved in theory that the ADDA is faster than the SDA-ss and it is the fastest among doubling algorithms with all possible bilinear transformations.

A common feature we note in all three transformations is the injection property, i.e., they are one-to-one maps. This might be unnecessary when used in the doubling algorithms as the core target is mapping the eigenvalues of $H$ in the left complex semi-plane into the inner of the unit circle. On the other hand, an effective combination of advantages from the shift-and-shrink transformation (1.5) and the generalized Cayley transformation

(1.6) might also be feasible. These motivate us to devise the following transformation of a matrix $A$

$$\mathcal{P}(A,\gamma,\beta,\alpha)=(I-A/\gamma)(A-\beta I)(A+\alpha I)^{-1}, \tag{2.1}$$

where $\gamma$, $\beta$, $\alpha$ are some positive constants. Obviously, the new transformation (2.1) is reduced to the generalized Cayley transformation when $\alpha>0$, $\beta>0$, $\gamma\to+\infty$ and to the shift-and-shrink transformation when $\alpha=\beta=0$, $\gamma>0$. Therefore, it is a more generalized transformation. We will show that with some proper positive $\alpha$, $\beta$ and $\gamma$, the doubling algorithm based on (2.1) is well defined and capable of preserving the monotonicity of iteration sequence with the convergence to the required minimal nonnegative solution. More important, the generalized transform (2.1) equipped with proper parameters, when applied to doubling algorithm, is superior to the generalized Cayley transformation (1.6) in terms of the expected convergence rate.

At the end of this section, we present some lemmas associated with $M$-matrix and Symplectic forms. The first is well known (see [20] for example).

**Lemma 2.1.** *For a Z-matrix $A$, the following statements are equivalent:*

*(a) $A$ is a nonsingular M-matrix.*

*(b) $A$ is nonsingular and satisfies $A^{-1}\geq 0$.*

*(c) $Av>0$ for some vector $v>0$.*

*(d) All eigenvalues of $A$ have positive real parts.*

*Moreover, $A$ is an irreducible singular M-matrix if and only if $Av=0$ for some vector $v>0$.*

The next two are also standard (see respectively [4] and [14] for example).

**Lemma 2.2.** *Let $A\in\mathbb{R}^{n\times n}$ be a nonsingular M-matrix, and let $B\in\mathbb{R}^{n\times n}$ be a Z-matrix. If $B\geq A$, then $B$ is also a nonsingular M-matrix.*

**Lemma 2.3.** *Let an $m\times n$ matrix $M$ be partitioned into a block form*

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

*Then*

$$M^{-1}=\begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix},$$

*where inverses when appearing exist.*

The following definition comes from [13].

**Definition 2.1.** A matrix pencil $\lambda L-M$ with $M,L\in\mathbb{R}^{(n+m)\times(n+m)}$ is said to be in *Symplectic* form if it can be written as

$$M=\begin{pmatrix} E & 0 \\ -H & I \end{pmatrix},\quad L=\begin{pmatrix} I & -G \\ 0 & F \end{pmatrix},$$

where the blocks $\pm E\in\mathbb{R}^{n\times n}\geq 0$ and $\pm F\in\mathbb{R}^{m\times m}\geq 0$, $H\in\mathbb{R}^{m\times n}\geq 0$ and $G\in\mathbb{R}^{n\times m}\geq 0$.

A result below on Symeplectic form is useful [18, Theorem 6.19].

**Lemma 2.4.** *Let $\lambda L - M$ be a matrix pencil with $M, L \in \mathbb{R}^{(n+m)\times(n+m)}$, and partition both matrices as*

$$M = (A_1 \ \ A_2), \quad L = (E_1 \ \ E_2),$$

*with $A_1, E_1 \in \mathbb{R}^{(n+m)\times n}$ and $A_2, E_2 \in \mathbb{R}^{(n+m)\times m}$. A Symplectic Form pencil which is right-similar to $\lambda L - M$ exists if and only if*

$$(E_1 \ \ A_2)$$

*is nonsingular; in this case,*

$$\begin{pmatrix} E & -G \\ -H & F \end{pmatrix} = (E_1 \ \ A_2)^{-1}(A_1 \ \ E_2).$$

# 3 Initial process based on the generalized transform

We first review the main framework of doubling algorithm in this section and then show the well-definition of the initial process.

Define $R = D - C\Phi$ and $S = A - B\Psi$, where $\Phi$ and $\Psi$ are respectively the minimal nonnegative solution of the MARE (1.1) and its dual form (1.2). Obviously, (1.1) and (1.2) are equivalent to

$$H\begin{pmatrix} I \\ \Phi \end{pmatrix} = \begin{pmatrix} I \\ \Phi \end{pmatrix} R \quad \text{and} \quad H\begin{pmatrix} \Psi \\ I \end{pmatrix} = \begin{pmatrix} \Psi \\ I \end{pmatrix}(-S) \tag{3.1}$$

with $H$ defined in (1.7). By incorporating the generalized transformation (2.1), we have

$$(I - H/\gamma)(H - \beta I)\begin{pmatrix} I \\ \Phi \end{pmatrix} = (H + \alpha I)\begin{pmatrix} I \\ \Phi \end{pmatrix}\mathcal{P}(R,\gamma,\beta,\alpha), \tag{3.2a}$$

$$(I - H/\gamma)(H - \beta I)\begin{pmatrix} \Psi \\ I \end{pmatrix}\mathcal{Q}(S,\alpha,\beta,\gamma) = (H + \alpha I)\begin{pmatrix} \Psi \\ I \end{pmatrix}, \tag{3.2b}$$

with

$$\mathcal{P}(R,\gamma,\beta,\alpha) = (I - R/\gamma)(R - \beta I)(R + \alpha I)^{-1}$$

and

$$\mathcal{Q}(S,\alpha,\beta,\gamma) = (S - \alpha I)(\beta I + S)^{-1}(I + S/\gamma)^{-1},$$

which could be further rewritten as

$$\mathcal{P}(H,\gamma,\beta,\alpha)\begin{pmatrix} I \\ \Phi \end{pmatrix} = \begin{pmatrix} I \\ \Phi \end{pmatrix}\mathcal{P}(R,\gamma,\beta,\alpha) \tag{3.3}$$

and

$$\mathcal{P}(H,\gamma,\beta,\alpha)\begin{pmatrix} \Psi \\ I \end{pmatrix}\mathcal{Q}(S,\alpha,\beta,\gamma) = \begin{pmatrix} \Psi \\ I \end{pmatrix}, \tag{3.4}$$

given all inverses exist and $\alpha>0$, $\beta>0$, $\gamma>0$.

Let

$$M_0=\begin{pmatrix} E_0 & 0 \\ -H_0 & I \end{pmatrix}, \quad L_0=\begin{pmatrix} I & -G_0 \\ 0 & F_0 \end{pmatrix}. \tag{3.5}$$

Then (3.3) and (3.4), if Lemma 2.4 applicable, could be reduced to

$$M_0\begin{pmatrix} I \\ \Phi \end{pmatrix}=L_0\begin{pmatrix} I \\ \Phi \end{pmatrix}\mathcal{P}(R,\gamma,\beta,\alpha) \tag{3.6}$$

and

$$M_0\begin{pmatrix} \Psi \\ I \end{pmatrix}\mathcal{Q}(S,\alpha,\beta,\gamma)=L_0\begin{pmatrix} \Psi \\ I \end{pmatrix}. \tag{3.7}$$

Also, $\lambda L_0-M_0$ is a Symplectic form. In fact, the whole process is valid thanks to the later Theorem 3.1. Before that, some lemmas will be provided first.

**Lemma 3.1.** *Let*

$$\widetilde{K}=\begin{pmatrix} \alpha^*\gamma I & -\beta^*C-CA+DC \\ 0 & \beta^*\gamma I+\beta^*A+A^2-BC \end{pmatrix}=\begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \\ 0 & \widetilde{K}_{22} \end{pmatrix} \tag{3.8}$$

*and*

$$\widehat{K}=\begin{pmatrix} \gamma\beta^*I-\beta^*D-CB+D^2 & 0 \\ \beta^*B+AB-BD & \gamma\alpha^*I \end{pmatrix}=\begin{pmatrix} \widehat{K}_{11} & 0 \\ \widehat{K}_{21} & \widehat{K}_{22} \end{pmatrix} \tag{3.9}$$

*with*

$$\alpha^*=\max_i\{A_{ii}\}, \quad \beta^*=\max_i\{D_{ii}\}, \quad \gamma=\max\left\{\frac{\alpha^{*2}}{\beta^*}, \frac{\beta^{*2}}{\alpha^*}\right\}. \tag{3.10}$$

*If $K$ in* (1.3) *is a nonsingular M-matrix (or an irreducible singular M-matrix). Then $\widetilde{K}w>0$ and $\widehat{K}w>0$ hold true for some positive vector $w^\top=(u^\top,v^\top)$.*

*Proof.* If $K$ is a nonsingular $M$-matrix, it follows from Lemma 2.1 that there are two positive vectors $u$ and $v$ such that

$$Du>Cv \quad \text{and} \quad Av>Bu.$$

Furthermore, let $D=\beta^*I-D_1$ and $A=\alpha^*I-A_1$ with $D_1$ and $A_1$ nonnegative matrices. It then follows

$$\beta^*u>D_1u+Cv \quad \text{and} \quad \alpha^*v>A_1v+Bu. \tag{3.11}$$

By noting $\alpha^*\gamma/\beta^*\geq\max\{\alpha^*,\beta^*\}$ and $\beta^*\gamma/\alpha^*\geq\max\{\alpha^*,\beta^*\}$, we have

$$\begin{aligned}\widetilde{K}_{11}u+\widetilde{K}_{12}v&=\alpha^*\gamma u-\beta^*Cv+DCv-CAv \\ &=\alpha^*\gamma u-\alpha^*Cv-D_1Cv+CA_1v \\ &\geq\max\{\alpha^*,\beta^*\}(D_1u+Cv)-\alpha^*Cv-D_1Cv+CA_1v \\ &\geq D_1(\max\{\alpha^*,\beta^*\}u-Cv)+CA_1v \\ &>0,\end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
\widetilde{K}_{22}v =& \beta^*\gamma v+\beta^*Av+A^2v-BCv \\
>& \beta^*\gamma v+\beta^*Bu+\alpha^{*2}v-2\alpha^*A_1v+A_1^2v-BCv \qquad (3.13)\\
\geq& \max\{\alpha^*,\beta^*\}(A_1v+Bu)+\alpha^{*2}v-2\alpha^*A_1v+A_1^2v-BCv \qquad (3.14)\\
\geq& \max\{\alpha^*,\beta^*\}A_1v+\alpha^*(A_1v+Bu)-2\alpha^*A_1v+A_1^2v+\max\{\alpha^*,\beta^*\}Bu-BCv \\
>& (\max\{\alpha^*,\beta^*\}-\alpha^*)A_1v+\alpha^*Bu+B(\max\{\alpha^*,\beta^*\}u-Cv) \\
>& 0. \qquad (3.15)
\end{aligned}
$$

So $\widetilde{K}w>0$ with $w^\top=[u^\top,v^\top]^\top$. Analogously, inequalities

$$
\begin{aligned}
\widehat{K}_{11}u =& \beta^*\gamma u-\beta^*Du-CBu+D^2u \\
=& \beta^*\gamma u-\beta^*D_1u-CBu+D_1^2u \\
\geq& (\max\{\alpha^*,\beta^*\}-\beta^*)D_1u+C(\max\{\alpha^*,\beta^*\}v-Bu)+D_1^2u \\
>& 0
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{K}_{21}u+\widehat{K}_{22}v =& \alpha^*\gamma v+\beta^*Bu+ABu-BDu \\
=& \alpha^*\gamma v+\alpha^*Bu-A_1Bu+BD_1u \\
\geq& A_1(\max\{\alpha^*,\beta^*\}v-Bu)+(\alpha^*+\max\{\alpha^*,\beta^*\})Bu+BD_1u \\
>& 0
\end{aligned}
$$

show that $\widehat{K}w>0$ with $w^\top=[u^\top,v^\top]^\top$.

If $K$ is an irreducible singular $M$-matrix, there are also two positive vectors $u$ and $v$, by Lemma 2.1, such that

$$\beta^*u=D_1u+Cv \quad \text{and} \quad \alpha^*v=A_1v+Bu.$$

We can implement the above proof process again, only replacing the inequality with equality in (3.13), to show the conclusion also holds true. □

**Remark 3.1.** The inequality (3.12) in the above proof implies that $\widetilde{K}_{12}-CA_1$ is a non-positive matrix with $\widetilde{K}_{11}u+(\widetilde{K}_{12}-CA_1)v>0$. Also, (3.14) indicates $\widetilde{K}_{22}-A_1^2$ is a $Z$-matrix with $(\widetilde{K}_{22}-A_1^2)v>0$.

To explicitly see the well-definition of the initial process in doubling algorithm, further exploration of block structure for $\gamma K+\widetilde{K}$ and its inversion is required. Here and after, all elements of $K$ in (1.3), without loss of generality, are assumed to be non-zeros. Otherwise, tiny quantities will replace zeros and a continuity argument works. Besides, the division symbol between two nonnegative matrices $A$ and $B$ (i.e., $\frac{A}{B}$) means the element-wise division and $\max_{ij}\{\frac{A}{B}\}$ is the maximal value of the divided matrix $\frac{A}{B}$ excluded infinity.

**Lemma 3.2.** *If*

$$\gamma \geq \max\Big\{\max_{ij}\Big\{\frac{A_1^2-BC}{A_1}\Big\}-\beta^*-2\alpha^*,\ \max_{ij}\Big\{\frac{CA_1-D_1C}{C}\Big\}-\alpha^*\Big\}. \tag{3.16}$$

*Then the matrix $\gamma K+\widetilde{K}$ is a nonsingular M-matrix with*

$$(\gamma K+\widetilde{K})^{-1}=\begin{pmatrix} S_D^{-1} & -S_D^{-1}(\widetilde{K}_{12}-\gamma C)(\gamma A+\widetilde{K}_{22})^{-1} \\ -S_A^{-1}B(D+\alpha^* I)^{-1} & S_A^{-1}\end{pmatrix}, \tag{3.17}$$

*where the Schur complements*

$$S_D=\gamma\Big(D+\alpha^* I+(\widetilde{K}_{12}-\gamma C)(\widetilde{K}_{22}+\gamma A)^{-1}B\Big) \tag{3.18}$$

*and*

$$S_A=\gamma A+\widetilde{K}_{22}+B(D+\alpha^* I)^{-1}(\widetilde{K}_{12}-\gamma C) \tag{3.19}$$

*are nonsingular M-matrices.*

*Proof.* As

$$\gamma A+\widetilde{K}_{22}=\Big(\gamma(\alpha^*+\beta^*)+\alpha^*(\beta^*+1)\Big)I-\Big((\gamma+2\alpha^*+\beta^*)A_1+BC-A_1^2)\Big),$$

so if

$$\gamma\geq\max_{ij}\Big\{\frac{A_1^2-BC}{A_1}\Big\}-\beta^*-2\alpha^*,$$

then $\gamma A+\widetilde{K}_{22}$ is a $Z$-matrix. It from Lemma 3.1 that there is a positive vector $v$ such that $(\gamma A+\widetilde{K}_{22})v>0$. Thus $\gamma A+\widetilde{K}_{22}$ is a nonsingular $M$-matrix by Lemma 2.1.

Since $\widetilde{K}_{12}-\gamma C=CA_1-D_1C-(\gamma+\alpha^*)C$, the choice of

$$\gamma\geq\max_{ij}\Big\{\frac{CA_1-D_1C}{C}\Big\}-\alpha^*,$$

then guarantees its non-positivity. So $S_D$ is a $Z$-matrix. Note that there are two positive vectors $u$ and $v$ in Lemma 3.1 such that $Du>Cv$ and $Av>Bu$, then

$$\begin{aligned}
S_D u=&\gamma Du-\gamma^2C(\gamma A+\widetilde{K}_{22})^{-1}Bu+\alpha^*\gamma u+\gamma\widetilde{K}_{12}(\gamma A+\widetilde{K}_{22})^{-1}Bu\\
>&\gamma C(\gamma A+\widetilde{K}_{22})^{-1}(\gamma Av+\widetilde{K}_{22}v-\gamma Bu)+\alpha^*\gamma u+\gamma\widetilde{K}_{12}(\gamma A+\widetilde{K}_{22})^{-1}Bu\\
=&\gamma C(\gamma A+\widetilde{K}_{22})^{-1}\Big(\widetilde{K}_{22}v+\gamma(Av-Bu)\Big)+\alpha^*\gamma u+\gamma CA_1(\gamma A+\widetilde{K}_{22})^{-1}Bu\\
&+\gamma(\widetilde{K}_{12}-CA_1)(\gamma A+\widetilde{K}_{22})^{-1}Bu\\
>&\gamma C(\gamma A+\widetilde{K}_{22})^{-1}\widetilde{K}_{22}v+\gamma^2C(\gamma A+\widetilde{K}_{22})^{-1}(Av-Bu)\\
&+\gamma CA_1(\gamma A+\widetilde{K}_{22})^{-1}Bu+\alpha^*\gamma u+(\widetilde{K}_{12}-CA_1)v\\
>&0,
\end{aligned} \tag{3.20}$$

where we use the fact $\gamma(Av-Bu)+\widetilde{K}_{22}v>0$ and the non-positivity of $\widetilde{K}_{12}-CA_1$ in Remark 3.1. Similarly,

$$\begin{aligned}S_{_A}v=&\gamma Av-\gamma B(D+\alpha^*I)^{-1}Cv+\widetilde{K}_{22}v+B(D+\alpha^*I)^{-1}\widetilde{K}_{12}v\\>&\gamma B(D+\alpha^*I)^{-1}(Du+\alpha^*u-Cv)+\widetilde{K}_{22}v+B(D+\alpha^*I)^{-1}\widetilde{K}_{12}v\\=&\gamma B(D+\alpha^*I)^{-1}(Du-Cv)+\widetilde{K}_{22}v+B(D+\alpha^*I)^{-1}(\alpha^*\gamma u+\widetilde{K}_{12}v)\\>&0.\end{aligned}$$

So $S_{_D}$ and $S_{_A}$ are nonsingular $M$-matrices.

Finally $\gamma K+\widetilde{K}$ is a $Z$-matrix when $\gamma$ is selected in (3.16). Furthermore, we have $(\gamma K+\widetilde{K})w>0$ with $w^T=(u^T,v^T)$. So $\gamma K+\widetilde{K}$ is a nonsingular $M$-matrix and its inverse form follows readily from Lemma 2.3. □

**Lemma 3.3.** *Let*

$$\gamma\geq\max\Big\{\max_{ij}\Big\{\frac{A_1B-BD_1}{B}\Big\}-\alpha^*,\ \beta^*-\max_{ij}\Big\{\frac{D_1^2-CB}{D_1}\Big\}\Big\}. \tag{3.21}$$

*Then the matrix $\gamma K-\widehat{K}$ is non-positive.*

*Proof.* Since

$$\gamma K-\widehat{K}=\begin{pmatrix}\gamma D-\widehat{K}_{11} & -\gamma C\\ -\gamma B-\widehat{K}_{21} & \gamma A-\alpha^*\gamma I\end{pmatrix}$$

with $\gamma D-\widehat{K}_{11}=CB-D_1^2-(\gamma-\beta^*)D_1$ and $\gamma B+\widehat{K}_{21}=(\gamma+\alpha^*)B+BD_1-A_1B$, then the choice of $\gamma$ in (3.21) guarantees its non-positivity. □

**Remark 3.2.** If the selected $\gamma$ in (3.16) or (3.21) is negative, then the choice of $\gamma$ returns to (3.10), which always preserves the positivity.

**Theorem 3.1.** *Let $\alpha^*=\max_{1\le i\le n}A_{ii}$, $\beta^*=\max_{1\le i\le n}D_{ii}$ and $\gamma^*$ be selected from the maximal of* (3.10), (3.16) *and* (3.21)*. Then* (3.1) *is equivalent to* (3.6) *and* (3.7) *with $E_0\le 0$, $F_0\le 0$, $H_0\ge 0$ and $G_0\ge 0$. Moreover, $I-G_0H_0$ and $I-H_0G_0$ are nonsingular M-matrices.*

*Proof.* According to Lemma 2.4, the Symplectic form in (3.6) and (3.7) satisfies

$$\begin{pmatrix}E_0 & -G_0\\ -H_0 & F_0\end{pmatrix}=(\gamma^*K+\widetilde{K})^{-1}(\gamma^*K-\widehat{K}), \tag{3.22}$$

which together with Lemma 3.2 and Lemma 3.3 yield $E_0\le 0$, $F_0\le 0$, $H_0\ge 0$ and $G_0\ge 0$.

Moreover, if $\beta^*=\max_{1\le i\le n}D_{ii}$ and $\gamma^*$ is selected as the maximal among (3.10), (3.16) and (3.21), then

$$\beta^*I-R=\beta^*I-D+C\Phi\ge 0\quad\text{ and }\quad\gamma^*I-R=\gamma^*I-D+C\Phi\ge 0$$

with their respective diagonal entries nonzero. So

$$\mathcal{P}(R)=-\frac{1}{\gamma^*}(\gamma^* I-R)(\beta^* I-R)(\alpha^* I+R)^{-1}\leq 0$$

with $\mathcal{P}(R)e<0$ ($e=(1,\cdots,1)^T$) by noting the fact $\alpha^* I+R$ is a nonsingular $M$-matrix ($\sigma(R)\subset C^+$). Analogously, it is not difficult to show

$$\mathcal{Q}(S)=-\frac{1}{\gamma^*}(\alpha^* I-S)(\beta^* I+S)^{-1}(\gamma^* I+S)^{-1}\leq 0$$

with $\mathcal{Q}(S)e<0$. Now equating both sides of (3.6) results in

$$E_0=(I-G_0\Phi)\mathcal{P}(R),\quad \Phi-H_0=F_0\Phi\mathcal{P}(R).$$

Since $\mathcal{P}(R)\leq 0$, we can find a positive vector $e$ such that $(I-G_0\Phi)\cdot(-\mathcal{P}(R)e)=-E_0e>0$. Then $I-G_0\Phi$ is a nonsingular $M$-matrix. On the other hand, it follows from $F_0\leq 0$ that $H_0\leq\Phi$. So $I-G_0H_0$ is a nonsingular $M$-matrix by Lemma 2.2. Similarly by equating both sides of (3.7), $I-H_0G_0$ is also not hard to shown be a nonsingular $M$-matrix. $\square$

# 4 Algorithm and convergence

When the initial Symplectic forms in (3.6) and (3.7) are established, the next step is to construct a iteration sequence $\{M_k,L_k\}$ for $k=0$, $1,2,\cdots$, satisfying

$$M_k\begin{pmatrix} I\\ \Phi\end{pmatrix}=L_k\begin{pmatrix} I\\ \Phi\end{pmatrix}(\mathcal{P}(R,\gamma^*,\beta^*,\alpha^*))^{2^k} \tag{4.1}$$

and

$$M_k\begin{pmatrix} \Psi\\ I\end{pmatrix}(\mathcal{Q}(S,\alpha^*,\beta^*,\gamma^*))^{2^k}=L_k\begin{pmatrix} \Psi\\ I\end{pmatrix} \tag{4.2}$$

with

$$M_k=\begin{pmatrix} E_k & 0\\ -H_k & I\end{pmatrix}\quad\text{and}\quad L_k=\begin{pmatrix} I & -G_k\\ 0 & F_k\end{pmatrix}. \tag{4.3}$$

To construct the next iteration, multiply $M_k$ and $L_k$ from the left by

$$\check{M}=\begin{pmatrix} E_k(I-G_kH_k)^{-1} & 0\\ -F_k(I-H_kG_k)^{-1}H_k & I\end{pmatrix}\quad\text{and}\quad \check{L}=\begin{pmatrix} I & -E_k(I-G_kH_k)^{-1}G_k\\ 0 & F_k(I-H_kG_k)^{-1}\end{pmatrix},$$

respectively, and then yield $M_{k+1}=\check{M}M_k$ and $L_{k+1}=\check{L}L_k$ with both of them having the same Symplectic structure in (4.3). This process contributes to the following doubling algorithm with generalized transform (DAGT).

**Algorithm 4.1** (DAGT). (1) Select initial matrices $E_0$, $F_0$, $H_0$ and $G_0$ as in Theorem 3.1.

(2) For $k\geq 0$, do following iterations until the convergence.

$$\begin{aligned}
F_{k+1}&=F_k(I-H_kG_k)^{-1}F_k,\\
E_{k+1}&=E_k(I-G_kH_k)^{-1}E_k,\\
H_{k+1}&=H_k+F_k(I-H_kG_k)^{-1}H_kE_k,\\
G_{k+1}&=G_k+E_k(I-G_kH_k)^{-1}G_kF_k.
\end{aligned}$$

**Theorem 4.1.** *Let $K$ be defined in* (1.3) *and parameters $\alpha$, $\beta$, $\gamma$ be selected in Theorem 3.1. Let $\Phi$, $\Psi$ be the minimal nonnegative solutions of the MARE and its dual equation, respectively. Then the matrix sequences $\{E_k\}$, $\{H_k\}$, $\{G_k\}$ and $\{F_k\}$ generated by the DAGT are well defined, and $E_k\geq 0$, $F_k\geq 0$, $0\leq H_k\leq H_{k+1}\leq\Phi$, $0\leq G_k\leq G_{k+1}\leq\Psi$ and both $I-G_kH_k$ and $I-H_kG_k$ are nonsingular M-matrices. Moreover, we have*

$$\limsup_{k\to\infty}\sqrt[2^k]{||\Phi-H_k||}\leq\rho(\mathcal{P}(R,\gamma^*,\beta^*,\alpha^*))\cdot\rho(\mathcal{Q}(S,\alpha^*,\beta^*,\gamma^*)),\tag{4.4a}$$

$$\limsup_{k\to\infty}\sqrt[2^k]{||\Psi-G_k||}\leq\rho(\mathcal{Q}(S,\alpha^*,\beta^*,\gamma^*))\cdot\rho(\mathcal{P}(R,\gamma^*,\beta^*,\alpha^*)).\tag{4.4b}$$

*Proof.* The proof may be conducted in almost the same way in [13] provided Theorem 3.1 is true. □

We subsequently concentrate on the convergence comparison of existing doubling algorithms. Only comparison of the transformation (2.1) in DAGT with the generalized Cayley transformation (1.6) is considered since ADDA is the most fast in all existing doubling-type algorithms.

**Theorem 4.2.** *Let $\lambda_R$ and $\lambda_S$ be the minimal eigenvalues of two M-matrices $R=D-C\Phi$ and $S=A-\Phi C$, respectively. Define a function*

$$f(\alpha,\beta,\gamma):=\rho(P)\cdot\rho(Q)\tag{4.5}$$

*with $P=(\gamma I-R)(\beta I-R)(\alpha I+R)^{-1}$ and $Q=(\alpha I-S)(\beta I+S)^{-1}(\gamma I+S)^{-1}$.*

*Then for $\alpha>\alpha^*$, $\beta>\beta^*$ and $\gamma>\beta^*$,*

*(a) $f(\alpha,\beta,\gamma)$ equals one when both minimal eigenvalues of $R$ and $S$ are zero (i.e., they are singular M-matrices).*

*(b) $f(\alpha,\beta,\gamma)$ is a strictly increasing function of $\alpha$, $\beta$, $\gamma$ and less than one when at least one of $R$ and $S$ is a nonsingular M-matrix.*

*Proof.* Assume first $R$ and $S$ are irreducible $M$-matrices. Let $\lambda_R$ and $\lambda_S$ be the minimal eigenvalues of $R$ and $S$, respectively. By Perron-Frobenius theorem [20], we have two real positive vectors (unique up to a multiple) $u_R$ and $u_S$ such that

$$Ru_R=\lambda_R u_R,\quad Su_S=\lambda_S u_S.$$

It is clear that $P$ and $Q$ are also nonnegative irreducible matrices if $\alpha>\alpha^*$, $\beta>\beta^*$ and $\gamma>\beta^*$. Using Perron-Frobenius theorem [20] again we have

$$\rho(P)=(\gamma-\lambda_R)(\beta-\lambda_R)(\alpha+\lambda_R)^{-1},$$
$$\rho(Q)=(\alpha-\lambda_S)(\beta+\lambda_S)^{-1}(\gamma+\lambda_S)^{-1}.$$

Thus

$$\begin{aligned} f(\alpha,\beta,\gamma)=&\rho(P)\cdot\rho(Q)\\ =&\frac{\alpha-\lambda_S}{\alpha+\lambda_R}\cdot\frac{\beta-\lambda_R}{\beta+\lambda_S}\cdot\frac{\gamma-\lambda_R}{\gamma+\lambda_S}\\ :=&g_1(\alpha)\cdot g_2(\beta)\cdot g_3(\gamma). \end{aligned}$$

If $R$ and $S$ are both singular, $\lambda_R=\lambda_S=0$ and (a) holds true. If one of $R$ and $S$ is nonsingular, $f(\alpha,\beta,\gamma)$ is a strictly increasing function since $g_1$, $g_2$ and $g_3$ are strictly increasing in $\alpha$, $\beta$, $\gamma$.

For $R$ and $S$ are reducible matrices, we can using a similar way in [21] to reduce such a case to a irreducible one. $\square$

The function (4.5) attains the minimal value at $\alpha=\alpha^*$, $\beta=\beta^*$ and $\gamma=\beta^*$, which implies the best convergence of doubling algorithm based on generalized transformation (2.1). Unlike the exact choice of $\alpha$ and $\beta$, the tight choice of $\gamma=\beta^*$ seems hard to guarantees the well-definition under the frame of $M$-matrix theory. As a substitution, the choice of $\gamma=\gamma^*$ in Theorem 3.1 is employed. The practical numerical experiments in next section show that such a choice in (2.1) could also help DAGT performing no worse than ADDA with generalized Cayley transformation. The following result further interprets such numerical behaviors in terms of comparison of convergence rate.

**Corollary 4.1.** *The convergence rate of DAGT based on the generalized transformation* (2.1) *with* $\gamma=\gamma^*$ *is faster than the ADDA with transformation* (1.6).

*Proof.* Let $\lambda_R$ and $\lambda_S$ be the minimal eigenvalue of two $M$-matrices $R$ and $S$, respectively. Then for $\gamma=\gamma^*$, the convergence rate of DAGT based on the generalized transformation (2.1) is

$$r_{dagt}=\frac{\alpha^*-\lambda_S}{\alpha^*+\lambda_R}\cdot\frac{\beta^*-\lambda_R}{\beta^*+\lambda_S}\cdot\frac{\gamma^*-\lambda_R}{\gamma^*+\lambda_S}. \tag{4.6}$$

On the other hand, we have known the convergence rate of the ADDA is

$$r_{adda}=\frac{\alpha^*-\lambda_S}{\alpha^*+\lambda_R}\cdot\frac{\beta^*-\lambda_R}{\beta^*+\lambda_S}. \tag{4.7}$$

Then we have

$$\frac{r_{dagt}}{r_{adda}}=\frac{\gamma^*-\lambda_R}{\gamma^*+\lambda_S}=1-\frac{\lambda_R+\lambda_S}{\gamma^*+\lambda_S}<1,$$

which shows the convergence rate of doubling algorithm with transformation (2.1) is always less than that of ADDA with transformation (1.6). $\square$

## 5 Numerical experiments

We compare performances of the new algorithm DAGT with those of ADDA in this section. It is seen that the DAGT shares the same complexity of ADDA at each iterative step. Thus we only need to compare the decrease of the equation residual as the iterations increase. Our computations is implemented in MATLAB with a machine error $error \approx 1.11\times 10^{-16}$. We take $\alpha=\alpha^*$, $\beta=\beta^*$ and $\gamma=\gamma^*$ The stop criterion is NRes$<tol$ with

$$\text{NRes}=\frac{||\hat{\Phi}C\hat{\Phi}-A\hat{\Phi}-\hat{\Phi}D+B||_1}{||\hat{\Phi}||_1(||\hat{\Phi}||_1||C||_1+||A||_1+||D||_1)+||B||_1},$$

$tol=10^{-14}$, and $\hat{\Phi}$ the approximation of $\Phi$, derived by the new method or the ADDA when termination occurs.

**Example 5.1.** Our first example is taken from Examples 1 in [21]. Here coefficient matrices are

$$D=\begin{pmatrix} 3 & -1\\ -1 & 3\end{pmatrix},\quad C=\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix},\quad A=\xi*D,\quad B=\xi*C,$$

with $\xi$ a positive constant. The MARE lies in the non-critical case when $\xi=1.5$ and close to the critical case when $\xi=1+10^{-6}$. The corresponding minimal nonnegative solution is

$$X^*=\frac{1}{2}\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

The convergence histories for ADDA and our DAGT are plotted in Fig. 1 the left for $\xi=1.5$ and the right for $\xi=1+10^{-6}$. Obviously, both two algorithms converged quadratically when MARE was far from the critical case and ours with the selected $\gamma^*\approx 6.75$ was faster than the ADDA. However, their numerical performances became linearly as the right plot shown where the MARE was very close to the critical case. Again, our DAGT with the selected $\gamma^*\approx 3.00$ converged always faster than ADDA did. Especially, our method seemed attaining the smaller residual level than the ADDA did in the last step, although both of them retained almost the same linear convergence rate.

**Example 5.2.** Consider the MARE arises from a problem in neutron transport theory [16]. Matrices $B$ and $C$ are of rank-one forms

$$B=ee^T,\quad C=qq^T,$$

with

$$e=(1,1,\cdots,1)^T,\quad q=(q_1,q_2,\cdots,q_n)^T,\quad q_i=\frac{c_i}{2\omega_i},\quad i=1,2,\cdots,n,$$

and $A$ and $D$ are of diagonal-plus-rank-one forms

$$A=\Delta-eq^T,\quad D=\Gamma-qe^T,$$

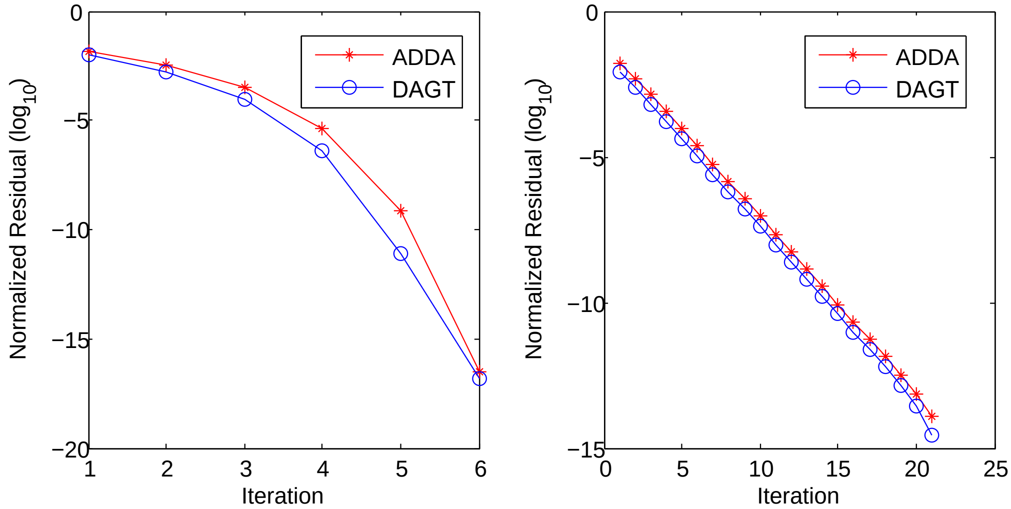


Figure 1: Normalized residual histories for $\xi=1.5$ (left) and $\xi=1+10^{-6}$ (right) in Example 5.1.

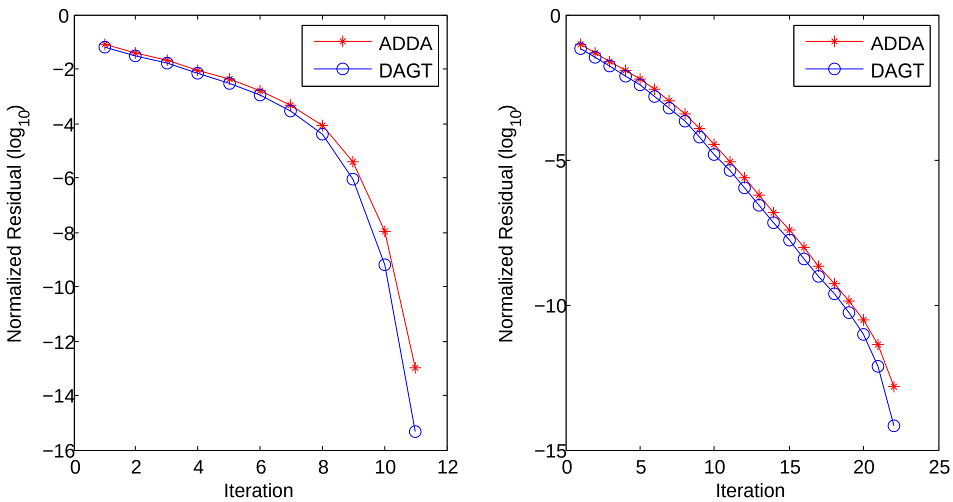


Figure 2: Normalized residual histories for $c=0.5$, $\alpha=0.5$ (left) and $c=1-10^{-8}$, $\alpha=10^{-8}$ (right) in Example 5.2.

with

$$\Delta=\mathrm{Diag}(\delta_1,\delta_2,\cdots,\delta_n),\quad \Gamma=\mathrm{Diag}(\gamma_1,\gamma_2,\cdots,\gamma_n),$$
$$\delta_i=\frac{1}{c\omega_i(1+\alpha)},\quad \gamma_i=\frac{1}{c\omega_i(1-\alpha)},\quad i=1,2,\cdots,n,$$

and $\alpha\in[0,1)$, $c\in(0,1]$. The two parameter sets $\{\omega_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$ denote the nodes and weights, respectively, of the Gauss-Legendre formula satisfying $0<\omega_n<\cdots<\omega_1<1$, and

$$\sum_{i=1}^{n} c_i = 1$$

with $c_i > 0$.

We set $n=2^7=128$ and take the non-critical case with ($c=0.5$, $\alpha=0.5$) and the approaching critical case with ($c=1-10^{-8}$,$\alpha=10^{-8}$) to test numerical performances of both algorithms. The corresponding normalized residual histories are plotted in Fig. 2. Clearly, the left in Fig. 2 showed that both ADDA and DAGT with $\gamma^*\approx 14413.7$ possessed quadratically convergent behaviors and DAGT attained less residual than ADDA did. Similarly, they degenerated to be linearly convergent when MARE became close to the critical case in which the DAGT with selected $\gamma^*\approx 15796.3$ was still superior to the ADDA as the right plot indicated.

**Example 5.3.** This example comes from a positive recurrent Markov chain with non-square coefficient matrices in fluid model, originally from [3], see also in [17]. Here

$$D=\begin{pmatrix} 28 & -22 \\ -21 & 27 \end{pmatrix}, \quad A=\begin{pmatrix} 26 & -22 & -2 \\ -21 & 24 & -1 \\ -21 & -1 & 24 \end{pmatrix}, \quad B=\mathbf{1}_{3,2}, \quad C=2B^T,$$

with $\mathbf{1}$ a matrix of all elements ones. The minimal nonnegative solution in this example is

$$X^*=\begin{pmatrix} 0.163265306122449 & 0.170068027210884 \\ 0.163265306122449 & 0.170068027210884 \\ 0.163265306122449 & 0.170068027210884 \end{pmatrix}$$

and eigenvalues of $R=D-CX^*$ are 4 and 49. Normalized residual histories for both algorithm are plotted in Fig. 3, which showed that the DAGT with selected $\gamma^*\approx 30.15$ converged faster than ADDA did.

## 6 Conclusions

We have devised a new transformation in this paper, generalizing the shift-and-shrink transformation and the generalized Cayley transformation. The doubling algorithm with generalized transform, referred as DAGT, was then presented for computing the minimal nonnegative solution of the nonsymmetric algebraic Riccati equation with $M$-matrix, showing theoretically and numerically faster convergence than the existing doubling algorithms.

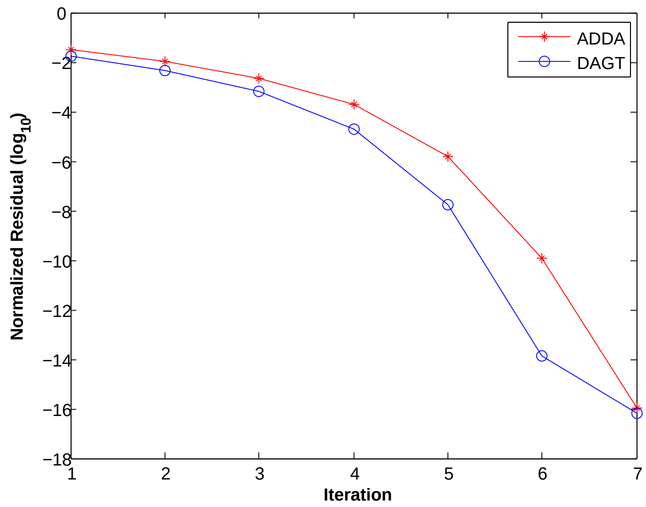


Figure 3: Normalized residual history in Example 5.3.


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