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Nonconforming FEMs for the p-Laplace Problem

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Abstract. The p-Laplace problems in topology optimization eventually lead to a degenerate convex minimization problem $E(v) := \int_\Omega W(\nabla v)dx - \int_\Omega f v dx$ for $v \in W_0^{1,p}(\Omega)$ with unique minimizer u and stress $\sigma := DW(\nabla u)$. This paper proposes the discrete Raviart-Thomas mixed finite element method (dRT-MFEM) and establishes its equivalence with the Crouzeix-Raviart nonconforming finite element method (CR-NCFEM). The sharper quasi-norm a priori and a posteriori error estimates of this two methods are presented. Numerical experiments are provided to verify the analysis.

AMS subject classifications: 65N12, 65N30, 65Y20

Key words: Adaptive finite element methods, nonconforming, p-Laplace problem, dual energy.

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1 Introduction

We consider the following nonlinear p-Laplace problem $(2 \le p < \infty)$ in the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with the given $f \in L^q(\Omega)$ (q conjugate of p),

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

This type of equation appears in many mathematical models of physical process, nonlinear diffusion and filtration, power-law materials, and viscoelastic materials, see [18, 27] for example. Most of these mathematical modeling are equivalent to the convex minimization problem [15] with energy

$$E(v) := \int_\Omega W(\nabla v)dx - F(v) \quad \text{for } v \in V := W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = 0\}. \tag{1.2}$$

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Here and throughout this paper, $F(v):=\int_\Omega fv dx$ and the energy density function $W:\mathbb{R}^2\to\mathbb{R}$ reads $W(A):=|A|^p/p$ with the derivative $DW(A)=|A|^{p-2}A$ for all $A\in\mathbb{R}^2\setminus\{0\}$ and the dual function

$$W^*(A):=\frac{|A|^q}{q}\quad\Big(\frac{1}{p}+\frac{1}{q}=1\Big). \tag{1.3}$$

Finite element approximation for (1.1) has been extensively studied by many authors, the previous works on a priori and a posteriori error estimations in the conventional $W^{1,p}(\Omega)$-norm can be found, for example, in [15, 16, 18, 25, 28]. Sharper a priori error estimates were derived in [4, 17, 20] by developing the quasi-norm techniques, and these techniques were extended to establish improved a posteriori error estimators of residual type for the $\mathcal{P}_1$ conforming finite element methods (CFEM) and nonconforming finite element methods (NCFEM) [12, 14, 21, 22]. In [19], Kim applied quasinorm techniques to a mixed finite volume method. Nevertheless, the NCFEM analysis of flux $\sigma:=DW(\nabla u)$, which is important in physical process and also the topic here, is almost not covered in the above references.

This paper focuses on (1.2) and the analysis of flux σ, proposes some simplified mixed finite element method (MFEM) with one-point numerical quadrature and explores some surprising advantages of the novel discrete Raviart-Thomas mixed finite element method (dRT-MFEM). First, the dRT-MFEM is equivalent to the Crouzeix-Raviart nonconforming first-order finite element method (CR-NCFEM). This generalizes the Marini representation [3, 24] and Arbogast [2] from linear and general variable coefficients elliptic PDEs to nonlinear p-Laplace problems. Second, the quasi-norm convergence analysis of dRT-MFEM (CR-NCFEM) leads to some optimal convergence rates with effective a posteriori error control.

The remaining parts of this paper are organized as follows. Section 2 introduces the precise notation and states the CR-NCFEM and dRT-MFEM for the p-Laplace problem. Section 3 establishes the equivalence result of dRT-MFEM and CR-NCFEM. The quasi-norm a priori and a posteriori error estimates of CR-NCFEM and dRT-MFEM follow in Section 4 and Section 5. Some numerical experiments conclude the paper in Section 6 with empirical evidence of the superiority of the new NCFEM also for adaptive mesh-refinement.

Standard notation applies throughout this paper to Lebesgue and Sobolev spaces $L^p(\Omega)$, $H^s(\Omega)$, and $H(\text{div},\Omega)$, as well as to the associated norms $\|\cdot\|_{p,\Omega}:=\|\cdot\|_{L^p(\Omega)}$, $|||\cdot|||_{p,\Omega}:=\|\nabla\cdot\|_{L^p(\Omega)}$, and $|||\cdot|||_{NC,p,\Omega}:=\|\nabla_{NC}\cdot\|_{L^p(\Omega)}$ with the piecewise gradient $\nabla_{NC}\cdot|_T:=\nabla(\cdot|_T)$ for all T in a regular triangulation $\mathcal{T}$ of the polygonal Lipschitz domain Ω. Here and throughout, ":" denotes the scalar product in $\mathbb{R}^{m\times n}$ and the expression "$\lesssim$" abbreviates an inequality up to some multiplicative generic constant, i.e., $A\lesssim B$ means $A\le CB$ with some generic constant $0\le C<\infty$, which depends on the interior angles of the triangles but not their sizes.