Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **10**, No. 6, pp. 1418-1439

DOI: 10.4208/aamm.OA-2017-0292
December 2018

Improved Symmetry Property of High Order Weighted Essentially Non-Oscillatory Finite Difference Schemes for Hyperbolic Conservation Laws

Wai Sun Don[1], Peng Li[2], Kwun Ying Wong[3] and Zhen Gao[1,*]

[1] *School of Mathematical Sciences, Ocean University of China, Qingdao 266100, Shandong, China*
[2] *Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang 050043, Hebei, China*
[3] *N. T. Heung Yee Kuk Yuen Long District Secondary School, Hong Kong*

Received 9 November 2017; Accepted (in revised version) 8 February 2018

Abstract. This study aims to investigate the rapid loss of numerical symmetry for problems with symmetrical initial conditions and boundary conditions when solved by the seventh and higher order nonlinear characteristic-wise weighted essentially non-oscillatory (WENO) finite difference schemes. Using the one-dimensional double rarefaction wave problem and the Sedov blast-wave problems, and the two-dimensional Rayleigh-Taylor instability (RTI) problem as examples, we illustrate numerically that the sensitive interaction of the round-off error due to the numerical unstable explicit form of the local lower order smoothness indicators in the nonlinear weights definition, which are often given and used in the literature, and the nonlinearity of the WENO scheme are responsible for the rapid growth of asymmetry of an otherwise symmetric problem. An equivalent but compact and numerical stable compact form of the local lower order smoothness indicators is suggested for delaying the onset of and reducing the magnitude of the symmetry error. The benefits of using the compact form of the local lower order smoothness indicators should also be applicable to non-symmetrical strongly non-linear problems in terms of improved numerical stability, reduced rounding errors and increased computational efficiency.

AMS subject classifications: 65M06, 65N06

Key words: Weighted essentially non-oscillatory, symmetry, smoothness indicator, hyperbolic conservation laws.

---

1 Introduction

*Corresponding author.
Emails: donwaisun@outlook.com (W. S. Don), weilailp@163.com (P. Li), yingwky@gmail.com (K. Y. Wong), zhengao@ouc.edu.cn (Z. Gao)

Characteristic-wise Weighted Essentially Non-Oscillation (WENO) conservative finite difference schemes are a class of high order/resolution nonlinear schemes for simulating flows with both shock waves and small scale structures that were initially developed in [11] (for details and history of the WENO scheme, see [19] and references therein). The WENO scheme employs a dynamic set of substencils where a nonlinear convex combination of lower order polynomials *adapts* either to a higher degree polynomial approximation at smooth stencils, or to a lower degree polynomial that avoids interpolation of the function across discontinuities at the cell boundary.

The high (fifth and higher) order WENO schemes with the global Lax-Friedrichs flux splitting (GLF) via the Roe-averaged eigensystem have also been applied to the solution of highly unstable fluid flows with a perturbed interface separating two fluids with different densities. One of the well-known examples is the Richtmyer-Meshkov instability (RMI) (see, e.g., [1, 7, 14, 21] and references therein). The interface is accelerated impulsively by a passing shock wave. The interface perturbation grows linearly initially forming a system of bubbles and spikes, and then nonlinearly after the amplitude of the perturbation becomes sufficiently larger. In the meantime, a large amount of vorticity is created and deposited along the interface where the pressure gradient is perpendicular to the density gradient forming vortical rollup in the shape of the mushroom cap. A similar temporal and spatial evolution can also be found in the Rayleigh-Taylors instability (RTI) (see, e.g., [1,15,22,25] and references therein), where a heavy density fluid rests on top of a lighter density fluid. Instead of an impulsive acceleration of the interface by a passing shock, the perturbed interface is accelerated by a constant bulk force, such as the gravity. This phenomenon can often be observed in a lava lamp.

We are motivated by the recent observations about the fact that these highly unstable fluid systems lose the spatial symmetry of an otherwise symmetrical solution under a symmetrical setup and a sinusoidal perturbation on the interface rapidly when solved by the seventh and ninth order WENO schemes. For example, Fig. 1 shows the density of the RTI problem with a sinusoid perturbation and the temporal history of the symmetry error, which measures the L^2 error in the deviation of the symmetry between the left and right half of the density (see Definition 4.1 below), as computed by the ninth order WENO-JS9 scheme at a very high resolution. Even though the density seems to be symmetric about $x=0.125$ in the eyeball norm, the symmetry error $\mathcal{O}(10^{-11})$ is actually much larger than the machine error $\mathcal{O}(10^{-16})$ at the early time and grows fairly large $\mathcal{O}(10^{-4})$ rapidly at the later time.

In this study, we systematically investigate and identify the source(s) of the symmetry error. It is a surprise to find out that the commonly used explicit form of the local lower order smoothness indicators β_k (smoothness indicators) used in the definition of the nonlinear weights ω_k in a substencil S_k for the seventh and ninth order WENO schemes [5] is the source of rounding off errors, and when interact with the nonlinearity of the WENO scheme, can have a strong influence on the symmetry of the solution. For example, the explicit form of β_k given by Balsara et al. [5] is far less accurate and less numerically stable in term of rounding off errors. A loss of four or more significant digits can be devastating