Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **10**, No. 6, pp. 1440-1458

DOI: 10.4208/aamm.OA-2018-0038
December 2018

Jacobi Spectral Collocation Method Based on Lagrange Interpolation Polynomials for Solving Nonlinear Fractional Integro-Differential Equations

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Received 31 January 2018; Accepted (in revised version) 13 May 2018

Abstract. In this paper, we study a class of nonlinear fractional integro-differential equations, the fractional derivative is described in the Caputo sense. Using the properties of the Caputo derivative, we convert the fractional integro-differential equations into equivalent integral-differential equations of Volterra type with singular kernel, then we propose and analyze a spectral Jacobi-collocation approximation for nonlinear integro-differential equations of Volterra type. We provide a rigorous error analysis for the spectral methods, which shows that both the errors of approximate solutions and the errors of approximate fractional derivatives of the solutions decay exponentially in L^∞-norm and weighted L^2-norm.

AMS subject classifications: 65R20, 45J05, 65N12

Key words: Spectral method, nonlinear, fractional derivative, Volterra integro-differential equations, Caputo derivative.

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1 Introduction

During the past three decades, the subject of fractional calculus (that is, calculus of integrals and derivatives of arbitrary order) has gained considerable popularity and importance, mainly due to its demonstrated applications in numerous diverse and widespread fields in science and engineering. For example, fractional calculus has been successfully applied to problems in system biology, physics, chemistry and biochemistry, hydrology, medicine and finance. In many cases these new fractional-order models are more adequate than the previously used integer-order models, because fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes that are governed by anomalous diffusion. Hence, there is a growing need to find the solution behaviour of these fractional differential equations. However, the analytic solutions of most fractional differential equations generally cannot be obtained. As a consequence, approximate and numerical techniques are playing an important role in identifying the solution behaviour of such fractional equations and exploring their applications.

In this article, we are concerned with the numerical study of the following nonlinear fractional integro-differential equation:

$$D^{\gamma}y(t)=\widehat{f}(t,y(t))+\int_0^t \widehat{K}(t,\tau,y(\tau))d\tau+\widehat{g}(t), \quad 0<\gamma<1, \quad t\in[0,T], \tag{1.1a}$$

$$y(0)=y_0, \tag{1.1b}$$

where $0<\gamma<1$, $\widehat{f}:[0,T]\times\mathbb{R}\to\mathbb{R}$, kernel function $\widehat{K}:S\times\mathbb{R}\to\mathbb{R}$ (where $S:=\{(t,\tau):0\le\tau\le t\le T\}$) and $\widehat{g}(t):[0,T]\to\mathbb{R}$ are known, $y(t)$ is the unknown function to be determined. D^{γ} denotes fractional derivative of fractional order γ defined as Caputo derivative.

It will always be assumed that problem (1.1) possesses a unique solution, namely, the given functions $\widehat{f}(t,y)$, $\widehat{K}(t,\tau,y)$ and $\widehat{g}(t)$ will be subject to the conditions that $\widehat{g}\in C[0,T]$, $\widehat{f}$ is continuous for all x and all u and satisfies the (uniform) Lipschitz conditions:

$$|\widehat{f}(t,y_1)-\widehat{f}(t,y_2)|\le M|y_1-y_2|, \tag{1.2}$$

$\widehat{K}$ is continuous for all S, $\widehat{K}\in H^m$ for y and satisfies the Lipschitz conditions:

$$|\widehat{K}(t,\tau,y_1)-\widehat{K}(t,\tau,y_2)|\le L_0|y_1-y_2|, \tag{1.3a}$$

$$\Big|\frac{\partial^i\widehat{K}(t,\tau,y_1)}{\partial y^i}-\frac{\partial^i\widehat{K}(t,\tau,y_2)}{\partial y^i}\Big|\le L_i|y_1-y_2|, \quad i=1,\cdots,m, \tag{1.3b}$$

for all $t\in[0,T]$, $(t,\tau)\in S$ and y_1, $y_2\in\mathbb{R}$, with Lipschitz constants M and L_i being independent of y_1 and y_2.

Let $\Gamma(\cdot)$ denote the Gamma function. For any positive integer n and $n-1<\gamma<n$, the Caputo derivative, Riemann-Liouville derivative and fractional integral of order γ are respectively defined as:

left Caputo derivative:

$$ {}_{a}^{C}D_t^{\gamma}u(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t\frac{u^{(n)}(s)}{(t-s)^{(\gamma-n+1)}}ds,\quad t\in[a,b], \tag{1.4}$$

right Caputo derivative:

$$ {}_{t}^{C}D_b^{\gamma}u(t)=\frac{(-1)^{-n}}{\Gamma(n-\gamma)}\int_t^b\frac{u^{(n)}(s)}{(t-s)^{(\gamma-n+1)}}ds,\quad t\in[a,b]. \tag{1.5}$$

I^{γ} denotes the Riemann-Liouville fractional integral of order γ and is defined as

$$I^{\gamma}u(t)=\frac{1}{\Gamma(\gamma)}\int_a^t(t-s)^{\gamma-1}u(s)ds, \tag{1.6}$$

and we have

$$I^{\gamma}({}_{a}^{C}D_t^{\gamma}u(t))=u(t)-\sum_{k=0}^{n-1}u^{(k)}(a)\frac{t^k}{k!}. \tag{1.7}$$

The numerical solution of fractional differential equations has attached considerable attention from many researchers and some reliable and efficient techniques are developed to solve fractional differential equations [1,2], such as generalized differential transform method [3], variational iteration method [4]. Moreover,orthogonal functions also play an important role in finding numerical solutions for fractional differential equations [5], such as Block pulse functions [6], Bernstein polynomials [7], shifted Legendre polynomials [8], Chebyshev wavelets [9], Legendre wavelets [10], etc. However, many of these methods cannot be applied to nonlinear FDEs. Further, to the authors knowledge, a systematic convergence analysis of these methods does not appear in the literature. It is well known that the spectral methods offer exponential rates of convergence/spectral accuracy for smooth problems in simple geometries.

This paper presents a numerical solution scheme for a class of fractional integro-differential equations (FIDEs). In this approach, the FIDEs are expressed in terms of Caputo type fractional derivative. Properties of the Caputo derivative allow one to reduce the FIDEs into a Volterra type integral-differential equations with singular kernel. Once this is done, a number of numerical schemes developed for Volterra type integral equation can be applied to find numerical solution of FIDEs. Recently, we provided Jacobi spectral-collocation method or spectral Petrov-Galerkin method [12–14] and convergence analysis for integro-differential equations. The main objective of this paper is to convert the nonlinear FIDEs into equivalent nonlinear integral-differential equations of Volterra type with singular kernel and develop new effective numerical methods and supporting analysis, based on the spectral Jacobi-collocation methods. The convergence of our proposed numerical methods are also investigated. Numerical experiments are carried out in support of our theoretical analysis. We also emphasise that the numerical methods we develop are applicable for many other types of fractional partial differential equations.

This paper is organized as follows. In Section 2, We convert the fractional integro-differential equations into equivalent integral-differential equations of Volterra type with singular kernel using the properties of the Caputo derivative and propose a spectral Jacobi-collocation approximation for nonlinear integro-differential equations of Volterra type. Some lemmas useful for establishing the convergence results will be provided in Section 3. The convergence analysis will be carried out in Section 4 and Section 5 contains numerical results, which will be used to verify the theoretical results obtained in Section 4.

2 Jacobi-collocation method

Let $\omega^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ be a weight function in the usual sense, for $\alpha,\beta>-1$. As defined in [15–17], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$ -orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1)=\{v: v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta}}<\infty\},$$

equipped with the norm

$$\|v\|_{\omega^{\alpha,\beta}}=\left(\int_{-1}^{1}|v(x)|^2\omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

and the inner product

$$(u,v)_{\omega^{\alpha,\beta}}=\int_{-1}^{1}u(x)v(x)\omega^{\alpha,\beta}(x)dx,\quad \forall u,v\in L^2_{\omega^{\alpha,\beta}}(-1,1).$$

For a given $N\geq 0$, we denote by $\{\theta_k\}_{k=0}^{N}$ the Legendre Gauss points and by $\{\omega_k\}_{k=0}^{N}$ the corresponding Legendre weights (i.e., Jacobi weights $\{\omega_k^{0,0}\}_{k=0}^{N}$). Then, the Legendre Gauss integration formula is

$$\int_{-1}^{1}f(x)dx\approx\sum_{k=0}^{N}f(\theta_k)\omega_k. \tag{2.1}$$

Similarly, we denote by $\{\tilde{\theta}_k\}_{k=0}^{N}$ the Jacobi Gauss points and by $\{\omega_k^{\alpha,\beta}\}_{k=0}^{N}$ the corresponding Jacobi weights. Then, the Jacobi Gauss integration formula is

$$\int_{-1}^{1}f(x)\omega^{\alpha,\beta}(x)dx\approx\sum_{k=0}^{N}f(\tilde{\theta}_k)\omega_k^{\alpha,\beta}. \tag{2.2}$$

For a given positive integer N, we denote the collocation points by $\{x_i^{\alpha,\beta}\}_{i=0}^{N}$, which is the set of $(N+1)$ Jacobi Gauss points, corresponding to the weight $\omega^{\alpha,\beta}(x)$. Let $\mathcal{P}_N$

denote the space of all polynomials of degree not exceeding N. For any $v\in C[-1,1]$, we can define the Lagrange interpolating polynomial $I_N^{\alpha,\beta}v\in\mathcal{P}_N$, satisfying

$$I_N^{\alpha,\beta}v(x_i)=v(x_i),\quad 0\le i\le N.$$

The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta}v(x)=\sum_{i=0}^{N}v(x_i)F_i(x),\quad 0\le i\le N,$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^N$.

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t=\frac{1}{2}T(1+x),\quad x=\frac{2t}{T}-1,$$
$$\tau=\frac{1}{2}T(1+s),\quad s=\frac{2\tau}{T}-1,$$

and let

$$u(x)=y\Big(\frac{1}{2}T(1+x)\Big),\qquad g(x)=\widehat{g}\Big(\frac{1}{2}T(1+x)\Big),$$
$$f(x)=\widehat{f}\Big(\frac{1}{2}T(1+x)\Big),\qquad K(x,s)=\frac{T}{2}\widehat{K}\Big(\frac{1}{2}T(1+x),\frac{1}{2}T(1+s)\Big).$$

The nonlinear fractional integro-differential equation in one dimension (1.1) is of the form

$$D^\gamma u(x)=f(x,u(x))+\int_{-1}^{x}K(x,s,u(s))ds+g(x),\quad 0<\gamma<1,\quad x\in I=[-1,1], \tag{2.3a}$$
$$u(-1)=u_{-1}=y_0. \tag{2.3b}$$

In order that the Jacobi collocation methods are carried out naturally, using (1.7), we restate (2.3) as

$$D^\gamma u(x)=f(x,u(x))+\int_{-1}^{x}K(x,s,u(s))ds+g(x),\quad 0<\gamma<1,\quad x\in I, \tag{2.4a}$$
$$\frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{\gamma-1}D^\gamma u(s)ds=u(x)-u(-1). \tag{2.4b}$$

Let $-\mu=\gamma-1$. Set the collocation points as the set of $(N+1)$ Jacobi Gauss points, $\{x_i^{-\mu,-\mu}\}_{i=0}^N$ associated with $\omega^{-\mu,-\mu}$. Assume that Eq. (2.4) holds at $x_i^{-\mu,-\mu}$:

$$D^\gamma u(x_i^{-\mu,-\mu})=f(x_i^{-\mu,-\mu},u(x_i^{-\mu,-\mu}))+\int_{-1}^{x_i^{-\mu,-\mu}}K(x_i^{-\mu,-\mu},s,u(s))ds+g(x_i^{-\mu,-\mu}), \tag{2.5a}$$
$$u(x_i^{-\mu,-\mu})=u(-1)+\frac{1}{\Gamma(\gamma)}\int_{-1}^{x_i^{-\mu,-\mu}}(x_i^{-\mu,-\mu}-s)^{-\mu}D^\gamma u(s)ds. \tag{2.5b}$$

The main difficulty in obtaining high order of accuracy is to compute the integral term in (2.5). In particular, for small values of $x_i^{-\mu,-\mu}$, there is little information available for $u(s)$. To overcome this difficulty, we will transfer the integral interval $[-1,x_i^{-\mu,-\mu}]$ to a fixed interval $[-1,1]$ then make use some appropriate quadrature rule. More precisely, we first make a simple linear transformation:

$$s(x,\theta)=\frac{1+x}{2}\theta+\frac{x-1}{2},\quad -1\le\theta\le 1. \tag{2.6}$$

Then (2.5) becomes

$$\begin{aligned} D^\gamma u(x_i^{-\mu,-\mu}) &= f(x_i^{-\mu,-\mu},u(x_i^{-\mu,-\mu}))+\int_{-1}^{1}\tilde{K}(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta),u(s(x_i^{-\mu,-\mu},\theta)))d\theta \\ &\quad +g(x_i^{-\mu,-\mu}), \end{aligned} \tag{2.7a}$$

$$u(x_i^{-\mu,-\mu})=u(-1)+\frac{1}{\Gamma(\gamma)}\left(\frac{1+x_i^{-\mu,-\mu}}{2}\right)^{\gamma}\int_{-1}^{1}(1-\theta)^{-\mu}D^\gamma u(s(x_i^{-\mu,-\mu},\theta))ds, \tag{2.7b}$$

where

$$\begin{aligned} &\tilde{K}(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta),u(s(x_i^{-\mu,-\mu},\theta))) \\ =&\frac{1+x_i^{-\mu,-\mu}}{2}K(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta),u(s(x_i^{-\mu,-\mu},\theta))). \end{aligned} \tag{2.8}$$

Next, using a $(N+1)$-point Gauss quadrature formula relative to Legendre weights $\{\omega_k\}_{k=0}^N$ (i.e., Jacobi weights $\{\omega_k^{0,0}\}_{k=0}^N$), the integration term in (2.7a) can be approximated by

$$\begin{aligned} &\int_{-1}^{1}\tilde{K}(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta),u(s(x_i^{-\mu,-\mu},\theta)))d\theta \\ =&\sum_{k=0}^{N}\tilde{K}(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta_k),u(s(x_i^{-\mu,-\mu},\theta_k))),\omega_k^{0,0}, \end{aligned} \tag{2.9}$$

where the set $\{\theta_k\}_{k=0}^N$ is the Legendre Gauss points corresponding Legendre weight $\{\omega_k^{0,0}\}_{i=0}^N$. Similarly, the integration term in (2.7b) can be approximated by

$$\int_{-1}^{1}(1-\theta)^{-\mu}D^\gamma u(s(x_i^{-\mu,-\mu},\theta))d\theta=\sum_{k=0}^{N}D^\gamma u(s(x_i^{-\mu,-\mu},\tilde{\theta}_k))\omega_k^{-\mu,0}, \tag{2.10}$$

where $\{\tilde{\theta}_k\}_{k=0}^N$ is the set of Jacobi Gauss points corresponding to the weight $\{\omega_k^{-\mu,0}\}_{k=0}^N$.

We use u_i, u_i^γ, $0\le i\le N$ to approximate the function value $u(x_i^{-\mu,-\mu})$, $D^\gamma u(x_i^{-\mu,-\mu})$, $0\le i\le N$ and use

$$U(x)=\sum_{j=0}^{N}u_jF_j(x),\quad U^\gamma(x)=\sum_{j=0}^{N}u_j^\gamma F_j(x), \tag{2.11}$$

where $F_j(x)$ is the Lagrange interpolation basis function associated with $\{x_i^{-\mu,-\mu}\}_{i=0}^N$ which is the set of $(N+1)$ Jacobi Gauss points. Combining the above equation and (2.7) yields

$$\begin{aligned} u_i^\gamma = & f(x_i^{-\mu,-\mu},u_i)+\left(\sum_{k=0}^N \tilde{K}(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta_k),\sum_{j=0}^N u_jF_j(s(x_i^{-\mu,-\mu},\theta_k)))\omega_k^{0,0}\right)\\ & +g(x_i^{-\mu,-\mu}), \end{aligned} \tag{2.12a}$$

$$u_i = u(-1)+\frac{1}{\Gamma(\gamma)}\left(\frac{1+x_i^{-\mu,-\mu}}{2}\right)^\gamma \sum_{j=0}^N u_j^\gamma \left(\sum_{k=0}^N F_j(s(x_i^{-\mu,-\mu},\tilde{\theta}_k))\omega_k^{-\mu,0}\right). \tag{2.12b}$$

The numerical scheme (2.12) leads to a nonlinear system for $\{u_i\}_{i=0}^N$ and $\{u_i^\gamma\}_{i=0}^N$, we can get the values of $\{u_i\}_{i=0}^N$ and $\{u_i^\gamma\}_{i=0}^N$ by solving the system of nonlinear equations using a proper solver (e.g., Newton method).

3 Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section. Let $I:=(-1,1)$.

Lemma 3.1 (see [18]). *Assume that an $(N+1)$-point Gauss quadrature formula relative to the Jacobi weight is used to integrate the product $u\varphi$, where $u\in H^m(I)$ with I for some $m\geq 1$ and $\varphi\in\mathcal{P}_N$. Then there exists a constant C independent of N such that*

$$\left|\int_{-1}^1 u(x)\varphi(x)dx-(u,\varphi)_N\right| \leq CN^{-m}|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)}\|\varphi\|_{L^2_{\omega^{\alpha,\beta}}(I)}, \tag{3.1}$$

where

$$|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{j=\min(m,N+1)}^m \|u^{(j)}\|^2_{L^2_{\omega^{\alpha,\beta}}(I)}\right)^{1/2},$$

and

$$(u,\varphi)_N = \sum_{j=0}^N u(x_j)\varphi(x_j)\omega_j. \tag{3.2}$$

Lemma 3.2 (see [18,19]). *Assume that $u\in H^{m,N}_{\omega^{-\mu,-\mu}}(I)$ and denote by $I_N^{-\mu,-\mu}u$ its interpolation polynomial associated with the $(N+1)$ Jacobi-Gauss points $\{x_j\}_{j=0}^N$, namely,*

$$I_N^{-\mu,-\mu}u = \sum_{i=0}^N u(x_i)F(x_i).$$

Then the following estimates hold:

$$\|u-I_N^{-\mu,-\mu}u\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\le CN^{-m}|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)}, \tag{3.3a}$$

$$\|u-I_N^{-\mu,-\mu}u\|_{L^\infty(I)}\le\begin{cases}CN^{1-\mu-m}|u|_{H^{m,N}_{\omega^c}(I)}, & 0\le\mu<\frac{1}{2},\\ CN^{\frac{1}{2}-m}\log N|u|_{H^{m,N}_{\omega^c}(I)}, & \frac{1}{2}\le\mu<1,\end{cases} \tag{3.3b}$$

where $\omega^c=\omega^{-\frac{1}{2},-\frac{1}{2}}$ *denotes the Chebyshev weight function.*

Lemma 3.3 (see [20]). *Assume that* $\{F_j(x)\}_{j=0}^N$ *are the* $N-th$ *degree Lagrange basis polynomials associated with the Gauss points of the Jacobi polynomials. Then,*

$$\|I_N^{\alpha,\beta}\|_{L^\infty(I)}\le\max_{x\in[-1,1]}\sum_{j=0}^N|F_j(x)|=\begin{cases}\mathcal{O}(\log N), & -1<\alpha,\beta\le-\frac{1}{2},\\ \mathcal{O}(N^{\gamma+\frac{1}{2}}), & \gamma=\max(\alpha,\beta),\ otherwise.\end{cases} \tag{3.4}$$

Lemma 3.4 (Gronwall inequality, see Lemma 7.1.1 in [21]). *Suppose* $L\ge 0$, $0<\mu<1$, *u and v are a non-negative, locally integrable functions defined on* $[-1,1]$ *satisfying*

$$u(x)\le v(x)+L\int_{-1}^x(x-\tau)^{-\mu}u(\tau)d\tau.$$

Then there exists a constant $C=C(\mu)$ *such that*

$$u(x)\le v(x)+CL\int_{-1}^x(x-\tau)^{-\mu}v(\tau)d\tau \quad for \ \ -1\le x<1.$$

If a nonnegative integrable function $E(x)$ *satisfies*

$$E(x)\le L\int_{-1}^x E(s)ds+J(x), \quad -1<x\le 1,$$

where $J(x)$ *is an integrable function, then*

$$\|E\|_{L^\infty(-1,1)}\le C\|J\|_{L^\infty(-1,1)}, \tag{3.5a}$$

$$\|E\|_{L^p_{\omega^{-\alpha,\beta}}(-1,1)}\le C\|J\|_{L^p_{\omega^{-\alpha,\beta}}(-1,1)}, \quad q\ge 1. \tag{3.5b}$$

Lemma 3.5 (see [22, 23]). *For a nonnegative integer r and* $\kappa\in(0,1)$, *there exists a constant* $C_{r,\kappa}>0$ *such that for any function* $v\in C^{r,\kappa}([-1,1])$, *there exists a polynomial function* $\mathcal{T}_N v\in\mathcal{P}_N$ *such that*

$$\|v-\mathcal{T}_N v\|_{L^\infty(I)}\le C_{r,\kappa}N^{-(r+\kappa)}\|v\|_{r,\kappa}, \tag{3.6}$$

where $\|\cdot\|_{r,\kappa}$ *is the standard norm in* $C^{r,\kappa}([-1,1])$, $\mathcal{T}_N$ *is a linear operator from* $C^{r,\kappa}([-1,1])$ *into* $\mathcal{P}_N$, *as stated in [22,23].*

Lemma 3.6 (see [24]). *Let $\kappa\in(0,1)$ and let $\mathcal{M}$ be defined by*

$$(\mathcal{M}v)(x)=\int_{-1}^{x}(x-\tau)^{-\mu}K(x,\tau)v(\tau)d\tau.$$

Then, for any function $v\in C([-1,1])$, there exists a positive constant C such that

$$\frac{|\mathcal{M}v(x')-\mathcal{M}v(x'')|}{|x'-x''|}\le C\max_{x\in[-1,1]}|v(x)|,$$

under the assumption that $0<\kappa<1-\mu$, for any $x',x''\in[-1,1]$ and $x'\neq x''$. This implies that

$$\|\mathcal{M}v\|_{0,\kappa}\le C\max_{x\in[-1,1]}|v(x)|,\quad 0<\kappa<1-\mu.$$

Lemma 3.7 (see [25]). *For every bounded function v, there exists a constant C, independent of v, such that*

$$\sup_{N}\Big\|\sum_{j=0}^{N}v(x_j)F_j(x)\Big\|_{L^2_{\omega^{\alpha,\beta}}(I)}\le C\max_{x\in[-1,1]}|v(x)|,$$

where $F_j(x)$, $j=0,1,\cdots,N$, are the Lagrange interpolation basis functions associated with the Jacobi collocation points $\{x_j\}_{j=0}^{N}$.

Lemma 3.8 (see [26]). *For all measurable functions $f\ge 0$, the following generalized Hardy's inequality*

$$\Big(\int_a^b|(Tf)(x)|^q u(x)dx\Big)^{1/q}\le\Big(\int_a^b|f(x)|^p v(x)dx\Big)^{1/p}$$

holds if and only if

$$\sup_{a<x<b}\Big(\int_x^b u(t)dt\Big)^{1/q}\Big(\int_a^x v^{1-p'}(t)dt\Big)^{1/p'}<\infty,\quad p'=\frac{p}{p-1},$$

for the case $1<p\le q<\infty$. Here, T is an operator of the form

$$(Tf)(x)=\int_a^x k(x,t)f(t)dt$$

with $k(x,t)$ a given kernel, u, v are nonnegative weight functions and $-\infty\le a<b\le\infty$.

4 Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed approximations. Firstly, we will carry our convergence analysis in L^∞ space.

Theorem 4.1. *Let $u(x)$ and $u^\gamma(x)$ be the exact solution of the fractional integro-differential equation* (2.3)*, which is assumed to be sufficiently smooth. Assume that $U(x)$ and $U^\gamma(x)$ are obtained by using the spectral collocation scheme* (2.12) *together with a polynomial interpolation* (2.11)*. If γ associated with the weakly singular kernel satisfies $0<\gamma<1$ and $u\in H^{m+1}_{\omega^{-\mu,-\mu}}(I)$, then*

$$\begin{aligned}&\|U^\gamma(x)-u^\gamma(x)\|_{L^\infty(I)}\\ \le&\begin{cases}CN^{\gamma-\frac{1}{2}-m}\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & \frac{1}{2}<\gamma<1,\\ CN^{-m}\log N\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & 0<\gamma\le\frac{1}{2},\end{cases}\end{aligned} \tag{4.1a}$$

$$\begin{aligned}&\|U(x)-u(x)\|_{L^\infty(I)}\\ \le&\begin{cases}CN^{\gamma-\frac{1}{2}-m}\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & \frac{1}{2}<\gamma<1,\\ CN^{-m}\log N\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & 0<\gamma\le\frac{1}{2},\end{cases}\end{aligned} \tag{4.1b}$$

provided that N is sufficiently large, where C is a constant independent of N,

$$K^*=\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}, \tag{4.2a}$$

$$\mathcal{U}=|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)}. \tag{4.2b}$$

Proof. The numerical scheme (2.12) can be written as

$$\begin{aligned}u_i^\gamma=&f(x_i^{-\mu,-\mu},u_i)+\int_{-1}^1\widetilde{K}\Big(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta),\sum_{j=0}^N u_jF_j(s(x_i^{-\mu,-\mu},\theta))\Big)ds\\ &+g(x_i^{-\mu,-\mu})+I_{i,1},\end{aligned} \tag{4.3a}$$

$$u_i=u_{-1}+\frac{1}{\Gamma(\gamma)}\left(\frac{1+x_i^{-\mu,-\mu}}{2}\right)^\gamma\int_{-1}^1(1-\theta)^{-\mu}U^\gamma(s)ds, \tag{4.3b}$$

where

$$\begin{aligned}I_{i,1}=&\sum_{k=0}^N\widetilde{K}\left(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta_k),\sum_{j=0}^N u_jF_j(s(x_i^{-\mu,-\mu},\theta_k))\right)\omega_k^{0,0}\\ &-\int_{-1}^1\widetilde{K}(x_i^{-\mu,-\mu},s(x_i^{-\mu,-\mu},\theta),U(s(x_i^{-\mu,-\mu},\theta)))ds.\end{aligned} \tag{4.4}$$

Using the integration error estimates from Jacobi-Gauss polynomials quadrature in Lemma 3.1, we have

$$\begin{aligned}|I_{i,1}(x)|\le&CN^{-m}|\widetilde{K}(x,s(x,\theta),U(s(x,\theta)))|_{H^{m,N}_{\omega^{0,0}}(I)}\\ \le&CN^{-m}\left(|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+|K(x,s,U(s))-K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}\right).\end{aligned} \tag{4.5}$$

Using the definition of $|\cdot|_{H^{m,N}(I)}$ in (3.2) and the Lipschitz conditions (1.3), we have

$$\begin{aligned}
&|K(x,s,U(s))-K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}\\
=&\left(\sum_{j=\min(m,N+1)}^{m}\left\|\frac{\partial^j K(x,s,U)}{\partial U^j}-\frac{\partial^j K(x,s,u)}{\partial u^j}\right\|^2_{L^2_{\omega^{0,0}}(I)}\right)^{\frac{1}{2}}\\
\leq&\sum_{j=\min(m,N+1)}^{m} L_i\|U-u\|_{L^2_{\omega^{0,0}}(I)}\\
\leq& C\|U-u\|_{L^2_{\omega^{0,0}}(I)}.
\end{aligned}\tag{4.6}$$

Then (4.5) can be rewritten as

$$\begin{aligned}
|I_{i,1}(x)|\leq& CN^{-m}|K(x,s(x,\theta),U(s(x,\theta)))|_{H^{m,N}_{\omega^{0,0}}(I)}\\
\leq& CN^{-m}\left(|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+\|U-u\|_{L^2_{\omega^{0,0}}(I)}\right)\\
\leq& CN^{-m}\left(|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+\|U-u\|_{L^\infty(I)}\right).
\end{aligned}\tag{4.7}$$

It follows from (2.5) and (2.7) that

$$u_i^\gamma=f(x_i^{-\mu,-\mu},u_i)+\int_{-1}^{x_i^{-\mu,-\mu}}K(x_i^{-\mu,-\mu},s,U(s))ds+g(x_i^{-\mu,-\mu})+I_{i,1},\tag{4.8a}$$

$$u_i=u_{-1}+\frac{1}{\Gamma(\gamma)}\int_{-1}^{x_i^{-\mu,-\mu}}(x_i^{-\mu,-\mu}-s)^{-\mu}U^\gamma(s)ds.\tag{4.8b}$$

Multiplying $F_i(x)$ on both sides of (4.8) and summing up from 0 to N yield

$$U^\gamma(x)=I_N^{-\mu,-\mu}f(x,U(x))+I_N^{-\mu,-\mu}\int_{-1}^{x}K(x,s,U(s))ds+I_N^{-\mu,-\mu}g(x)+J_1(x),\tag{4.9a}$$

$$U(x)=I_N^{-\mu,-\mu}(u_{-1})+I_N^{-\mu,-\mu}\left(\frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{-\mu}U^\gamma(s)ds\right),\tag{4.9b}$$

where

$$J_1(x)=\sum_{i=0}^{N}I_{i,1}F_i(x).$$

It follows from (2.4) that

$$\begin{aligned}
U^\gamma(x)=&I_N^{-\mu,-\mu}f(x,U(x))+I_N^{-\mu,-\mu}\left(\int_{-1}^{x}K(x,s,U(s))ds\right)\\
&+I_N^{-\mu,-\mu}\left(D^\gamma u-f(x,u)-\int_{-1}^{x}K(x,s,u(s))ds\right)+J_1(x)
\end{aligned}$$

$$
\begin{aligned}
&= I_N^{-\mu,-\mu} D^\gamma u + I_N^{-\mu,-\mu}\big(f(x,U(x)) - f(x,u(x))\big) \\
&\quad + I_N^{-\mu,-\mu}\int_{-1}^{x}\big(K(x,s,U(s)) - K(x,s,u(s))\big)ds + J_1(x), \qquad (4.10a)
\end{aligned}
$$

$$
U(x) = I_N^{-\mu,-\mu}(u_{-1}) + I_N^{-\mu,-\mu}\left(\frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{-\mu}U^\gamma(s)ds\right). \qquad (4.10b)
$$

Let e and e^γ denotes the error function,

$$
e(x) = U(x) - u(x), \quad e^\gamma(x) = U^\gamma(x) - u^\gamma(x),
$$

using the Lipschitz conditions, which gives

$$
\begin{aligned}
|e^\gamma(x)| &\le M I_N^{-\mu,-\mu}|U(x)-u(x)| + L_0 I_N^{-\mu,-\mu}\int_{-1}^{x}|U(s)-u(s)|ds + |J_1(x)| + |J_2(x)| \\
&\le M|e(x)| + L_0\int_{-1}^{x}|e(s)|ds + |J_1(x)| + |J_2(x)| + |J_3(x)| + |J_4(x)|, \qquad (4.11a)
\end{aligned}
$$

$$
e(x) = \frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{-\mu}e^\gamma(s)ds + J_5(x) + J_6(x), \qquad (4.11b)
$$

and

$$
\begin{aligned}
J_2(x) &= I_N^{-\mu,-\mu}u^\gamma(x) - u^\gamma(x), \\
J_3(x) &= M I_N^{-\mu,-\mu}e(x) - Me(x), \\
J_4(x) &= L_0 I_N^{-\mu,-\mu}\int_{-1}^{x}e(s)ds - L_0\int_{-1}^{x}e(s)ds, \\
J_5(x) &= I_N^{-\mu,-\mu}u(x) - u(x), \\
J_6(x) &= I_N^{-\mu,-\mu}\left(\frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{-\mu}e^\gamma(s)ds\right) - \frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{-\mu}e^\gamma(s)ds,
\end{aligned}
$$

using the Dirichlet's formula which states

$$
\int_{-1}^{x}\int_{-1}^{\tau}\Phi(\tau,s)dsd\tau = \int_{-1}^{x}\int_{s}^{x}\Phi(\tau,s)d\tau ds
$$

provided the integral exists, we obtain

$$
\begin{aligned}
e^\gamma(x) \le\, & M\frac{1}{\Gamma(\gamma)}\int_{-1}^{x}(x-s)^{-\mu}|e^\gamma(s)|ds \\
&+ L_0\int_{-1}^{x}\left(\int_{s}^{\tau}\frac{1}{\Gamma(\gamma)}(\tau-s)^{-\mu}ds\right)|e^\gamma(\tau)|d\tau \\
&+ M|J_5(x)| + M|J_6(x)| + L_0\int_{-1}^{x}(J_5(s)+J_6(s))ds \\
&+ |J_1(x)| + |J_2(x)| + |J_3(x)| + |J_4(x)|. \qquad (4.12)
\end{aligned}
$$

It follows from the Gronwall inequality in Lemma 3.4 that

$$\|e^{\gamma}(x)\|_{L^{\infty}(I)} \leq C\sum_{i=1}^{6}\|J_i\|_{L^{\infty}(I)}, \tag{4.13a}$$

$$\|e(x)\|_{L^{\infty}(I)} \leq C\sum_{i=1}^{6}\|J_i\|_{L^{\infty}(I)}. \tag{4.13b}$$

Using Lemma 3.3 and the estimates (4.7), we have

$$\begin{aligned}\|J_1\|_{L^{\infty}(I)} &\leq \begin{cases} CN^{\frac{1}{2}-\mu}\max_{0\leq i\leq N}|I_{i,1}|, & 0<\mu<\dfrac{1}{2}, \\ C\log N\max_{0\leq i\leq N}|I_{i,1}|, & \dfrac{1}{2}\leq\mu<1, \end{cases} \\ &\leq \begin{cases} CN^{\frac{1}{2}-\mu-m}(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+\|e^{\gamma}(x)\|_{L^{\infty}(I)}), & 0<\mu<\dfrac{1}{2}, \\ CN^{-m}\log N(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+\|e^{\gamma}(x)\|_{L^{\infty}(I)}), & \dfrac{1}{2}\leq\mu<1. \end{cases}\end{aligned} \tag{4.14}$$

Due to Lemma 3.2,

$$\|J_2\|_{L^{\infty}(I)} \leq \begin{cases} CN^{1-\mu-m}|u^{\gamma}|_{H^{m,N}_{\omega^c}(I)}, & 0<\mu<\dfrac{1}{2}, \\ CN^{\frac{1}{2}-m}\log N|u^{\gamma}|_{H^{m,N}_{\omega^c}(I)}, & \dfrac{1}{2}\leq\mu<1, \end{cases} \tag{4.15a}$$

$$\|J_5\|_{L^{\infty}(I)} \leq \begin{cases} CN^{1-\mu-m}|u|_{H^{m,N}_{\omega^c}(I)}, & 0<\mu<\dfrac{1}{2}, \\ CN^{\frac{1}{2}-m}\log N|u|_{H^{m,N}_{\omega^c}(I)}, & \dfrac{1}{2}\leq\mu<1. \end{cases} \tag{4.15b}$$

By virtue of Lemma 3.2 (3.3b) with $m=1$,

$$\begin{aligned}\|J_4\|_{L^{\infty}(I)} &\leq \begin{cases} CN^{-\mu}|\int_{-1}^{x}e(s)ds|_{H^1_{\omega^c}(I)} \leq CN^{-\mu}\|e\|_{L^{\infty}(I)}, & 0<\mu<\dfrac{1}{2}, \\ CN^{-\frac{1}{2}}|\int_{-1}^{x}e(s)ds|_{H^1_{\omega^c}(I)} \leq CN^{-\mu}\|e\|_{L^{\infty}(I)}, & \dfrac{1}{2}\leq\mu<1, \end{cases} \\ &\leq \begin{cases} CN^{-\mu}\Big(\|e^{\gamma}\|_{L^{\infty}(I)}+\|J_3\|_{L^{\infty}(I)}+\|J_5\|_{L^{\infty}(I)}\Big), & 0<\mu<\dfrac{1}{2}, \\ CN^{-\frac{1}{2}}\Big(\|e^{\gamma}\|_{L^{\infty}(I)}+\|J_3\|_{L^{\infty}(I)}+\|J_5\|_{L^{\infty}(I)}\Big), & \dfrac{1}{2}\leq\mu<1. \end{cases}\end{aligned} \tag{4.16}$$

We now estimate the term $J_6(x)$. It follows from Lemma 3.5 and Lemma 3.6, that

$$\begin{aligned}\|J_6\|_{L^{\infty}(I)} &= \|(I_N^{-\mu,-\mu}-I)\mathcal{M}e^{\gamma}\|_{L^{\infty}(I)} \\ &= \|(I_N^{-\mu,-\mu}-I)(\mathcal{M}e^{\gamma}-\mathcal{T}_N\mathcal{M}e^{\gamma})\|_{L^{\infty}(I)} \\ &\leq \Big(1+\|I_N^{-\mu,-\mu}\|_{L^{\infty}(I)}\Big)CN^{-k}\|\mathcal{M}e^{\gamma}\|_{0,\kappa} \\ &\leq \begin{cases} CN^{\frac{1}{2}-\mu-\kappa}\|e^{\gamma}\|_{L^{\infty}(I)}, & 0<\mu<\dfrac{1}{2}, \\ CN^{-\kappa}\log N\|e^{\gamma}\|_{L^{\infty}(I)}, & \dfrac{1}{2}\leq\mu<1, \end{cases}\end{aligned} \tag{4.17}$$

where in the last step we used Lemma 3.6 under the following assumption

$$\begin{cases} \frac{1}{2}-\mu<\kappa<1-\mu, & \text{when } 0<\mu<\frac{1}{2}, \\ 0<\kappa<1-\mu, & \text{when } \frac{1}{2}\le\mu<1. \end{cases}$$

It's easy to obtain that

$$J_3(x)=MJ_6(x)+MI_N^{-\mu,-\mu}(J_5(x)+J_6(x))-(J_5(x)+J_6(x)), \tag{4.18}$$

we have

$$\|J_3\|_{L^\infty(I)}\le\begin{cases} CN^{\frac{1}{2}-\mu-\kappa}\|e^\gamma\|_{L^\infty(I)}, & 0<\mu<\frac{1}{2}, \\ CN^{-\kappa}\log N\|e^\gamma\|_{L^\infty(I)}, & \frac{1}{2}\le\mu<1. \end{cases} \tag{4.19}$$

Provided that N is sufficiently large. Combining (4.14), (4.15), (4.16) and (4.17) gives

$$\begin{aligned} &\|U^\gamma(x)-u^\gamma(x)\|_{L^\infty(I)} \\ \le&\begin{cases} CN^{\frac{1}{2}-\mu-m}\Big(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+N^{\frac{1}{2}}(|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)})\Big), & 0<\mu<\frac{1}{2}, \\ CN^{-m}\log N\Big(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+N^{\frac{1}{2}}(|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)})\Big), & \frac{1}{2}\le\mu<1, \end{cases} \\ &\|U(x)-u(x)\|_{L^\infty(I)} \\ \le&\begin{cases} CN^{\frac{1}{2}-\mu-m}\Big(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+N^{\frac{1}{2}}(|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)})\Big), & 0<\mu<\frac{1}{2}, \\ CN^{-m}\log N\Big(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+N^{\frac{1}{2}}(|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)})\Big), & \frac{1}{2}\le\mu<1. \end{cases} \end{aligned}$$

Using $\gamma=1-\mu$, we have the desired estimates (4.1) and (4.1b). □

Next, we will give the error estimates in $L^2_{\omega^{-\mu,-\mu}}$ space.

Theorem 4.2. *If the hypotheses given in Theorem 4.1 hold, then*

$$\begin{aligned} &\|U^\gamma(x)-u^\gamma(x)\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \\ \le&\begin{cases} CN^{-m}\Big(N^{\frac{1}{2}-\mu-\kappa}K^*+N^{1-\mu-\kappa}\mathcal{U}+\mathcal{V}\Big), & \frac{1}{2}<\gamma<1, \\ CN^{-m}\Big(N^{\frac{1}{2}-\mu-\kappa}K^*+N^{1-\mu-\kappa}\mathcal{U}+\mathcal{V}\Big), & 0<\gamma\le\frac{1}{2}, \end{cases} \end{aligned} \tag{4.20a}$$

$$\begin{aligned} &\|U(x)-u(x)\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \\ \le&\begin{cases} CN^{-m}\Big(N^{\frac{1}{2}-\mu-\kappa}K^*+N^{1-\mu-\kappa}\mathcal{U}+\mathcal{V}\Big), & \frac{1}{2}<\gamma<1, \\ CN^{-m}\Big(N^{\frac{1}{2}-\mu-\kappa}K^*+N^{1-\mu-\kappa}\mathcal{U}+\mathcal{V}\Big), & 0<\gamma\le\frac{1}{2}, \end{cases} \end{aligned} \tag{4.20b}$$

for any $\kappa\in(0,\gamma)$ provided that N is sufficiently large and C is a constant independent of N, where

$$\mathcal{V}=\mathcal{U}+K^*.$$

Proof. By using the generalization of Gronwalls Lemma 3.4 and the Hardy inequality Lemma 3.8, it follows from (4.12) that

$$\|e^\gamma\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \le C\sum_{i=1}^{6}\|J_i\|_{L^2_{\omega^{-\mu,-\mu}}(I)}. \tag{4.21}$$

Now, using Lemma 3.7, we have

$$\begin{aligned}\|J_1\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \le & C\max_{x\in[-1,1]}|I(x)|\\ \le & CN^{-m}\left(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}_{\omega^{0,0}}(I)}+\|e^\gamma(x)\|_{L^\infty(I)}\right).\end{aligned} \tag{4.22}$$

By the convergence result in Theorem 4.1 ($m=1$), we have

$$\|e^\gamma\|_{L^\infty(I)} \le C\left(|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)}\right).$$

So that

$$\|J_1\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \le CN^{-m}\left(K^*+\left(|u^\gamma|_{H^{m,N}_{\omega^c}(I)}+|u|_{H^{m,N}_{\omega^c}(I)}\right)\right). \tag{4.23}$$

Due to Lemma 3.2 (3.3a),

$$\|J_2\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \le CN^{-m}|u^\gamma|_{H^{m;N}_{\omega^c}(I)}, \tag{4.24a}$$

$$\|J_5\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \le CN^{-m}|u|_{H^{m;N}_{\omega^c}(I)}. \tag{4.24b}$$

By virtue of Lemma 3.2 (3.3a) with $m=1$,

$$\|J_4\|_{L^2_{\omega^{-\mu,-\mu}}(I)} \le CN^{-1}\left|\int_{-1}^{x}e(s)ds\right|_{H^1_{\omega^{-\mu,-\mu}}(I)} \le CN^{-1}\|e\|_{L^2_{\omega^{-\mu,-\mu}}(I)}. \tag{4.25}$$

Finally, it follows from Lemma 3.5 and Lemma 3.7 that

$$\begin{aligned}&\|J_6\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\\ =&\|(I_N^{-\mu,-\mu}-I)\mathcal{M}e^\gamma\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\\ =&\|(I_N^{-\mu,-\mu}-I)(\mathcal{M}e^\gamma-\mathcal{T}_Ne^\gamma\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\\ \le&\|I_N^{-\mu,-\mu}(\mathcal{M}e^\gamma-\mathcal{T}_Ne^\gamma)\|_{L^2_{\omega^{-\mu,-\mu}}(I)}+\|\mathcal{M}e^\gamma-\mathcal{T}_Ne^\gamma\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\\ \le&C\|\mathcal{M}e^\gamma-\mathcal{T}_Ne^\gamma\|_{L^\infty(I)}\\ \le&CN^{-\kappa}\|\mathcal{M}e^\gamma\|_{0,\kappa}\\ \le&CN^{-\kappa}\|e^\gamma\|_{L^\infty(I)},\end{aligned} \tag{4.26}$$

where, in the last step we used Lemma 3.6 for any $\kappa\in(0,1-\mu)$. By the convergence result in Theorem 4.1, we obtain that

$$\|J_6\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\le\begin{cases} CN^{\frac{1}{2}-\mu-m-\kappa}\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & 0<\mu<\dfrac{1}{2},\\ CN^{-m-\kappa}\log N\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & \dfrac{1}{2}\le\mu<1.\end{cases} \tag{4.27}$$

Using (4.18), we have

$$\|J_3\|_{L^2_{\omega^{-\mu,-\mu}}(I)}\le\begin{cases} CN^{\frac{1}{2}-\mu-m-\kappa}\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & 0<\mu<\dfrac{1}{2},\\ CN^{-m-\kappa}\log N\left(K^*+N^{\frac{1}{2}}\mathcal{U}\right), & \dfrac{1}{2}\le\mu<1,\end{cases} \tag{4.28}$$

for N sufficiently large and for any $\kappa\in(0,1-\mu)$. The desired estimates (4.20a) and (4.20b) follows from the above estimates and (4.21) and $\gamma=1-\mu$. □

5 Numerical experiments

Example 5.1. Consider the following nonlinear fractional integro-differential equation

$$D^{\alpha}y(t)=1+2t-e^{-t^2}y^2(t)+t(1+2t)\int_0^t e^{\tau(t-\tau)}y(\tau)d\tau, \tag{5.1a}$$

$$y(0)=1, \tag{5.1b}$$

when $\alpha=1$, the exact solution of (5.1) is $y(t)=e^{t^2}$.

In the only case of $\alpha=1$, we know the exact solution. We have reported the obtained numerical results for $N=20$ and $\alpha=0.25,0.5,0.75,1$ in Fig. 1. We can see that, as α approaches 1, the numerical solutions converges to the analytical solution $y(t)=e^{t^2}$, i.e., in the limit, the solution of fractional integro differential equations approaches to that of the integer order integro differential equations. In Fig. 2, we plot the resulting errors versus the number N of the steps. This figure shows the exponential rate of convergence predicted by the proposed method.

Example 5.2. Consider the following fractional integro-differential equation,

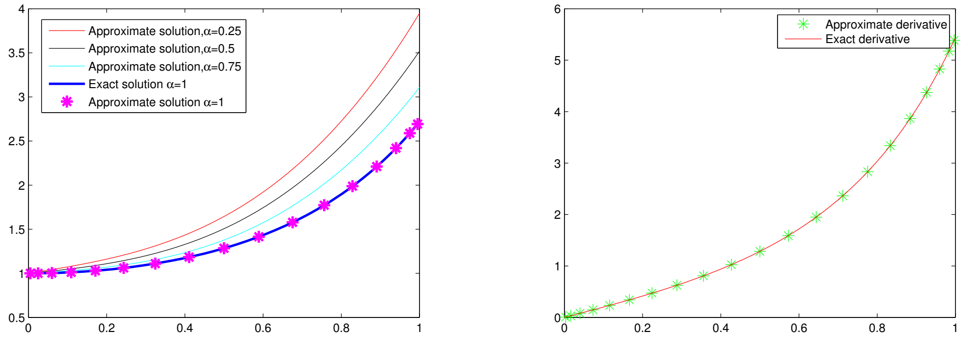
$$D^{0.5}y(t)=f(t)y(t)+g(t)+\sqrt{t}\int_0^t y^2(\tau)d\tau, \tag{5.2a}$$

$$y(0)=0, \tag{5.2b}$$

with

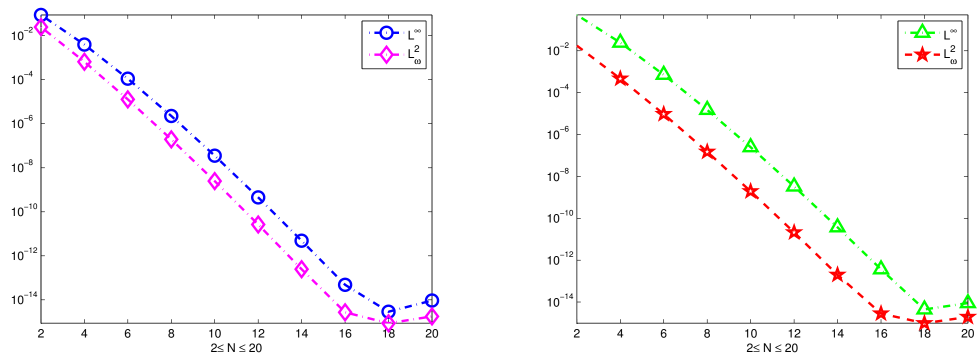
$$f(t)=2\sqrt{t}+2t^{\frac{3}{2}}-\left(\sqrt{t}+t^{\frac{3}{2}}\right)\ln(1+t),\quad g(t)=\frac{2\text{arcsinh}(\sqrt{t})}{\sqrt{\pi(1+t)}}-2t^{\frac{3}{2}}.$$

The exact solution is $y(t)=\ln(1+t)$.



(a) (b)

Figure 1: Example 5.1: Approximation solutions with different α and exact solution of $y(t)$ with $\alpha=1$ (a). Comparison between approximate solution and exact solution of $y'(t)$.



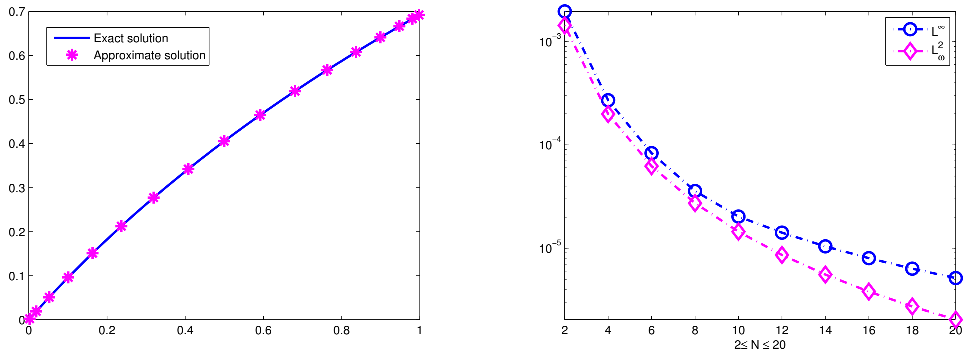
(a) (b)

Figure 2: Example 5.1: The errors of numerical and exact solution $y(t)$ (a) and the errors of numerical and exact solution $y'(t)$ (b) versus the number of collocation points in L^∞ and L^2_ω norms.

The numerical results can be seen from Fig. 3. These results indicate that the spectral accuracy is obtained for this problem, although the given functions $f(t)$ and $g(t)$ are not very smooth.

Acknowledgements

The author would like to thank the referees for the helpful suggestions. The work was supported by NSFC Project (Nos. 11671342, 91430213, 11671157 and 11771369), Project of Scientific Research Fund of Hunan Provincial Science and Technology Department (No. 2018JJ2374) and Key Project of Hunan Provincial Department of Education (No. 17A210).



(a) (b)

Figure 3: Example 5.2: Comparison between approximate solution and exact solution of $y(t)$ (a). The errors of numerical and exact solution $y(t)$ versus the number of collocation points in L^∞ and L^2_ω norms (b).

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