

A Hybrid Trapezoidal-Difference Scheme for Nonlinear Time-Fractional Fourth-Order Advection-Dispersion Equation Based on Chebyshev Spectral Collocation Method

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Abstract. In this paper, we firstly present a novel simple method based on a Picard integral type formulation for the nonlinear multi-dimensional variable coefficient fourth-order advection-dispersion equation with the time fractional derivative order $\alpha \in (1, 2)$. A new unknown function $v(\mathbf{x}, t) = \partial u(\mathbf{x}, t) / \partial t$ is introduced and $u(\mathbf{x}, t)$ is recovered using the trapezoidal formula. As a result of the variable $v(\mathbf{x}, t)$ are introduced in each time step, the constraints of traditional plans considering the non-integer time situation of $u(\mathbf{x}, t)$ is no longer considered. The stability and solvability are proved with detailed proofs and the precise describe of error estimates is derived. Further, Chebyshev spectral collocation method supports accurate and efficient variable coefficient model with variable coefficients. Several numerical results are obtained and analyzed in multi-dimensional spatial domains and numerical convergence order are consistent with the theoretical value $3 - \alpha$ order for different α under infinite norm.

AMS subject classifications: 65M60, 65N30, 65N15

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1 Introduction

Fractional calculus is a natural generalization of integer order operator. Utilizing the models based on derivatives of fractional orders in several branches of science and engineering is a major study of many mathematicians and physicians [1–5]. Roughly speaking time fractional derivative is designed to characterize physical processes and dynamic systems with history memory. As a counterpart of traditional integer order differential

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equation, fractional differential equation can be obtained by replacing the integer order derivatives with fractional ones in integer order differential equation. Fractional partial differential equations (FPDEs), particularly space and time-fractional equations, have been widely studied to construct the existence of solution and validity of these problems [6–8]. In addition, the reliable and powerful numerical and analytical methods for solving FPDEs has been focused in the last two decades. According to the mathematical literature, fractional partial differential equations have been progressed in various problems in science and engineering such as the Schrödinger, diffusion and telegraph fractional equations [6, 9–14].

In several applications, the fourth-order model system [15, 16] is an important part of the fractional order system and can be found in physics, engineering, statistics, and other fields, such as wave propagation in beam problems [17], A flat surface system of grooves [5, 18], several mathematical models of fourth-order subdiffusion systems [18–21] and so on. Here we will consider the following the nonlinear multi-dimensional variable coefficient time-fractional fourth-order advection-dispersion equation:

$$\begin{aligned} & {}_0^c \mathcal{D}_t^\alpha u(\mathbf{x}, t) - (A(\mathbf{x}, t) + {}_0^c \mathcal{D}_t^\alpha) \Delta u(\mathbf{x}, t) \\ & = -B(\mathbf{x}, t) \Delta^2 u(\mathbf{x}, t) + \mathcal{N}(u(\mathbf{x}, t)) + f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad t \in (0, T], \end{aligned} \quad (1.1)$$

where $A(\mathbf{x}, t)$ and $B(\mathbf{x}, t)$ are positive variable coefficients with the following initial and boundary conditions:

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (1.2a)$$

$$u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, T], \quad (1.2b)$$

where $u(\mathbf{x}, t)$ is unknown functions. Here ${}_0^c \mathcal{D}_t^{\beta(\mathbf{x}, t)}$ denotes the higher order Caputo fractional derivative of variable order $\beta(\mathbf{x}, t)$ with respect to t in [4, 5, 11, 15–21]

$${}_0^c \mathcal{D}_t^{\beta(\mathbf{x}, t)} u(\mathbf{x}, t) = \frac{1}{\Gamma(n - \beta(\mathbf{x}, t))} \int_0^t \frac{\partial^n u(\mathbf{x}, \eta)}{\partial \eta^n} \frac{d\eta}{(t - \eta)^{\beta(\mathbf{x}, t) + 1 - n}}, \quad n - 1 \leq \beta(\mathbf{x}, t) \leq n, \quad (1.3)$$

where $\Gamma(\cdot)$ is the Gamma function. The nonlinear term $\mathcal{N}(u(\mathbf{x}, t))$ is assumed to satisfy the following conditions: a) $|\mathcal{N}(u(\mathbf{x}, t))| \leq C|u|$, b) The first-order derivative function of $\mathcal{N}(u(\mathbf{x}, t))$ with respect to u is bounded, i.e., $|\mathcal{N}'(u(\mathbf{x}, t))| \leq a$, a is a positive constant.

Most of fractional partial differential equations do not have the analytic solutions, many researchers in the last two decades have focused on the approximation or numerical methods of these fractional order systems in [22, 23]. Lots of the researchers focus their attention on the strong format. This format is directly obtained by the original discrete equation. So, it is also called the collocation method. Strong formulation is reliable, simple in structure, and easy to erect the algebra system. The homotopy analysis method was utilized to approximate some FPDEs in [24, 25]. The finite difference scheme and fractional predictor-corrector method are introduced for simulating the multi-term time-fractional wave-diffusion equations with computationally effective results by Adams-Bashforth method [26]. Also, some fractional differential equations utilized for modeling dynamical systems are investigated by an implicit difference approximation in [27].

The Jacobi collocation method is used to find the numerical solution of the fractional advection-diffusion equation with a nonlinear source term by Parvizi et al. in [28] and so on.

Underlying model of time appropriate to the application of discrete schemes, almost these schemes need to consider the half time step situation of u because it is easy to obtain the representations of the derivative situation and the integration process. Meanwhile the energy method of the u commonly considers the variable value coupled with v . So, in this work the situation of v is changed as the main consideration. The stability, solvability of the system are proved in detail and the convergence of u is given with $3-\alpha$ order. Furthermore, the derivative operation is an unbounded operator, by constant, the integration is a refinement operator. In this paper, the integral formula coupled with the difference scheme is proved to be a good stable scheme in Section 4. Finally, the high temporal schemes, such as high order Lagrange scheme, high order Runge-Kutta method and so on, have high accuracy in time direction. However, it is regrettable that these schemes are hard to get the regularity in the time process. The conservation of energy is merely proved on these scheme and only on special conditions or semi-analytical method, such as, mass-conservative Fourier spectral methods [26] and so on.

Thus, we use the $v(t)$ to replace the the unknown value $u'(t)$ in this paper. Let $\sigma(\mathbf{x},t) = \Delta u(\mathbf{x},t)$ and this problem is rewritten as

$$\begin{aligned} & {}_0^c \mathcal{D}_t^\alpha \left(u_0(\mathbf{x}) + \int_0^t v'(\mathbf{x},\eta) d\eta \right) - (A(\mathbf{x},t) + {}_0^c \mathcal{D}_t^\alpha) \Delta \left(u_0(\mathbf{x}) + \int_0^t v'(\mathbf{x},\eta) d\eta \right) \\ & = -B(\mathbf{x},t) \Delta \sigma + \mathcal{N} \left(u_0(\mathbf{x}) + \int_0^t v'(\mathbf{x},\eta) d\eta \right) + f(\mathbf{x},t), \end{aligned} \quad (1.4a)$$

$$\sigma(\mathbf{x},t) = \Delta \left(u_0(\mathbf{x}) + \int_0^t v'(\mathbf{x},\eta) d\eta \right), \quad t \in (0,T], \quad (1.4b)$$

with the following initial and boundary conditions:

$$v(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (1.5a)$$

$$\sigma(\mathbf{x},t) = v(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0,T]. \quad (1.5b)$$

On space discrete schemes, the Lagrange approach is a choice for dealing with polynomial interpolations in most cases. Taken as one of the high order lagrange schemes, Chebyshev collocation method is chosen as the priority scheme in this paper. Firstly, the Lemma 4.1 of this paper illustrates the hybridized trapezoidal-difference scheme based on Chebyshev collocation method keeping the regularity and conservation of energy in the time-difference process. Secondly, the Chebyshev collocation method has a good numerical stability and high accuracy. The key is that the Lagrange polynomial must be manipulated through the formulas of barycentric interpolation [29]. When the nodes are of uniform distribution, the weight functions becomes extremely big lead to the Runge phenomenon and ruins the merits of the lagrange interpolation. But, the family of Chebyshev points following the density proportion $(1-x^2)^{-1/2}$ has good numerical stability [30–35].

The collocation method based on Chebyshev polynomial interpolation was recently extended to solve various ordinary and partial differential equations including 1D high order initial and boundary values problems [36] and nonlinear Burgers' equation [37]. In fact, there are few reports about the application of barycentric lagrange interpolation in the literature, especially for the high dimensional problems.

In summary, the hybridized trapezoidal-difference scheme based on Chebyshev spectral collocation method is a stable, high accuracy scheme. The rest of the paper is organized as follows: In Section 2, the formula of a compact finite difference coupled with spectral collocation method is derived. In Section 3, some preliminary information and lemmas are presented. In Section 4, the stability, convergence and solvability are proofed. In Section 5, computational results for some numerical experiments are illustrated. Finally, some conclusions are discussed in Section 6.

2 Formulation of time-fractional equation

Let $\{t_n | n \geq 0\}$ is the uniform time interval, where $t_n = n\tau$, $\tau > 0$. And the trapezoidal-difference scheme suppose

$$u_i^n = u_i^0 + \tau \sum_{k=1}^n (v_i^{k-1} + v_i^k) / 2, \quad \int_0^{t_n} g(\eta) \frac{\partial v_i}{\partial \eta} d\eta = \sum_{k=1}^n \left[\frac{v_i^k - v_i^{k-1}}{\tau} \int_{t_{k-1}}^{t_k} g(\eta) d\eta \right], \quad (2.1)$$

where u_i^n , v_i^n , A_i^n and B_i^n are the values of function, the first derivative function, variable coefficients $A(\mathbf{x}, t)$ and $B(\mathbf{x}, t)$ at the time t_n of the point \mathbf{x}_i , and $g(t)$ is the smooth function in $(0, T]$.

The difference scheme we will consider for (1.4) is as follows:

Case I: $n = 1$

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^n - a_{n-1} v_i^0 \right] \\ & - \frac{1}{\tau\Gamma(2-\alpha)} \Delta \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^n - a_{n-1} v_i^0 \right] \\ & = \tau A_i^n \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) - B_i^n \Delta \sigma_i^n + \mathcal{N}(u_i^0) + F_i^n, \end{aligned} \quad (2.2)$$

Case II: $n \geq 2$

$$\frac{1}{\tau\Gamma(2-\alpha)} \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^n - a_{n-1} v_i^0 \right]$$

$$\begin{aligned}
 & -\frac{1}{\tau\Gamma(2-\alpha)}\Delta\left[a_0v_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})v_i^n-a_{n-1}v_i^0\right] \\
 & =\tau A_i^n\Delta\left(\sum_{k=1}^{n-1}v_i^k+\frac{v_i^0+v_i^n}{2}\right)-B_i^n\Delta\sigma_i^n+2\mathcal{N}(u_i^{n-1})-\mathcal{N}(u_i^{n-2})+F_i^n, \tag{2.3}
 \end{aligned}$$

where

$$a_l=\int_{t_l}^{t_{l+1}}\frac{dt}{t^{\alpha-1}}=\frac{1}{2-\alpha}[(t_{l+1})^{2-\alpha}-(t_l)^{2-\alpha}]=\frac{\tau^{3-\alpha}}{2-\alpha}[(l+1)^{2-\alpha}-l^{2-\alpha}], \quad l\geq 0, \tag{2.4}$$

and

$$F_i^n=\Delta(u_i^0)-\Delta^2(u_i^0)+f_i^n.$$

It is easy to verify that $\{a_l, 0\leq l\leq n\}$ is a monotone decreasing sequence for each n with $a_0=\tau^{2-\alpha}/(2-\alpha)$.

In the process of Chebyshev spectral collocation method, the essential part is the generation of the spectral differentiation matrix. We first give the Chebyshev spectral differentiation matrix on the interval $[-1,1]$. Other intervals can be easily handled by the pan and zoom.

The collocation points are chosen as Chebyshev-Gauss-Lobatto points $x_k=\cos((k-1)\pi/(m-1))$, $k=1,2,\dots,m$. We assume that $u(x)$ is a smooth function on $[-1,1]$. Then $u(x)$ can be interpolated by a sum of smooth shape functions $\phi_j(x)$

$$u_m(x)=\sum_{j=1}^m u(x_j)\phi_j(x), \tag{2.5}$$

where

$$\phi_j(x)=\left(\frac{\omega_j}{x-x_j}/\sum_{k=1}^m\frac{\omega_k}{x-x_k}\right) \quad \text{with} \quad \omega_j^{-1}=\prod_{i=1,i\neq j}^m(x_j-x_i).$$

The derivatives of the interpolant $u_m(x)$ are then estimated at collocation points by differentiating (2.5)

$$u'_m(x)=\sum_{j=1}^m u(x_j)\phi'_j(x). \tag{2.6}$$

Define the vector of function and derivatives as

$$U=[u(x_1),u(x_2),\dots,u(x_m)]^T$$

and

$$U^{(1)}=[u'(x_1),u'(x_2),\dots,u'(x_m)]^T,$$

respectively. We can get the matrix formulation of (2.6) on collocation points

$$U^{(1)}=D_m U, \tag{2.7}$$

where D_m is the first order Chebyshev spectral differentiation matrix. The entry of D_m is

$$(D_m)_{ij} = \begin{cases} \frac{\omega_j}{\omega_i} \frac{1}{x_i - x_j}, & \text{if } i \neq j, \\ \sum_{i=1, i \neq j}^m \left(\frac{\omega_i}{\omega_j} \frac{1}{x_i - x_j} \right), & \text{if } i = j. \end{cases} \tag{2.8}$$

From the above analysis, we can also get the second order Chebyshev spectral differentiation matrix $D_m^{(2)}$ by differentiating the interpolant $u_m(x)$ twice. However, in this work, in order to construct a conservative scheme for the fourth order differential system, we compute the second order spectral differentiation matrix by $D_m^{(2)} = D_m^2$, the square of D_m . The matrix $D_m^{(2)}$ can map the vector U to vector $U^{(2)}$:

$$U^{(2)} = D_m^2 U, \tag{2.9}$$

where $U^{(2)} = [u''(x_1), u''(x_2), \dots, u''(x_m)]^T$ represents the second-derivative on collocation points. These spectral differentiation matrices have some useful properties.

The procedure given above for the calculation of the derivatives in one-dimension can be readily extended to two dimensions. If an unknown matrix U is defined as $U(x_i, y_k) = u_{ik}$, then its partial derivatives evaluated at the collocation points can be expressed in terms of the matrix-matrix products, where the differentiation with respect to x_k corresponds to multiplying the rows of D_x (the collocation derivative matrix in the x -direction) by the columns of U , and the differentiation with respect to y_k corresponds to multiplying the rows of U by the columns of D_y (the collocation derivative matrix transpose in the y -direction).

We also define

$$\|g^n\|_\infty = \max_{1 \leq i \leq m} |g_i^n|, \quad |\nabla g^n| = \sqrt{D \sum_{i=1}^m (g_i^n)^2}, \tag{2.10}$$

where D is the measurement of the space domain Ω . In addition, if $g(\Gamma) = 0$, we have

$$\|g^n\|_\infty \leq \frac{\sqrt{D}}{2} |\nabla g^n|. \tag{2.11}$$

Then, we have the following error estimate from the interpolate process

Lemma 2.1 (see [38]). *If $u \in H^p(\Omega)$ and $u_{\hat{m}} \in H^{\hat{m}}(\Omega)$, where $\Omega \in \mathbb{R}^d$ is a nonempty, open bounded set with a Lipschitz continuous boundary. Then the following error estimates for the lagrange interpolation collocation method holds*

$$\|\partial_x^l (u - u_{\hat{m}})\|_l \leq \hat{n}^{l - \min\{\hat{m}, p\}} \|u\|_l, \quad 0 \leq l \leq \min\{\hat{m}, p\},$$

where \hat{m} and \hat{n} are the minimum polynomial order and the minimum number of nodes in spatial coordinates.

3 Preliminaries and some lemmas

For the fractional derivation of the present scheme, we introduce the following lemmas from [39].

Lemma 3.1. For any $v = \{v(t_0), v(t_1), v(t_2), \dots\}$, we have

$$\begin{aligned} & \sum_{n=1}^m \left[a_0 v(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v(t_k) - a_{n-1} v(t_0) \right] v(t_n) \\ & \geq \frac{t_m^{1-\alpha}}{2} \tau \sum_{n=1}^N v(t_n)^2 - \frac{t_N^{2-\alpha}}{2(2-\alpha)} v(t_0)^2, \end{aligned} \quad (3.1)$$

where a_1 is defined in (2.4).

Lemma 3.2. For $n \geq 1$ and $t_k = k\tau$, $0 \leq k \leq n$, we have

$$\begin{aligned} 0 & \leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left[(t_n - t)^{2-\alpha} - \frac{(t - t_{k-1})(t_n - t_k)^{2-\alpha} + (t_k - t)(t_n - t_{k-1})^{2-\alpha}}{\tau} \right] dt \\ & \leq \left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha}) \right] \tau^{3-\alpha}. \end{aligned} \quad (3.2)$$

Lemma 3.3. Suppose $v(t) \in C^2([0, t_n])$. Then

$$\begin{aligned} & \left| \int_0^{t_n} v'(t) \frac{dt}{(t_n - t)^{\alpha-1}} - \sum_{k=1}^n \frac{v(t_k) - v(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} \frac{dt}{(t_n - t)^{\alpha-1}} \right| \\ & \leq \frac{\tau^{3-\alpha}}{2-\alpha} \left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha}) \right] \|v(t)\|_{L^2, \infty([0, t_n])}. \end{aligned} \quad (3.3)$$

Lemma 3.4. Suppose $v(t) \in C^2[0, t_n]$. Then

$$\begin{aligned} & \left| \int_0^{t_n} v'(t) \frac{dt}{(t_n - t)^{\alpha-1}} - \frac{1}{\tau} \left[a_0 v(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v(t_k) - a_{n-1} v(t_0) \right] \right| \\ & \leq \frac{\tau^{3-\alpha}}{2-\alpha} \left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha}) \right] \|v(t)\|_{L^1, \infty([0, t_n])}, \end{aligned} \quad (3.4)$$

where a_1 is defined in (2.4) and $1 < \alpha < 2$.

Based on Lemmas 2.1 and 3.4, introducing Eqs. (2.2)-(2.3), we have

$$\begin{aligned}
 {}^c\mathcal{D}_0^\alpha u_i^n &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial^2 u(\mathbf{x}_i, \tau)}{\partial \tau^2} \frac{d\tau}{(t_n - \tau)^{\alpha-1}} \\
 &= \frac{1}{\tau \Gamma(2-\alpha)} \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] + c_1 \left(\tau^{3-\alpha} + \hat{n}^{-\min\{\hat{m}, p\}} \right), \tag{3.5a}
 \end{aligned}$$

$$\begin{aligned}
 {}^c\mathcal{D}_0^\alpha \Delta u_i^n &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 \Delta u(\mathbf{x}_i, \tau)}{\partial \tau^2} \frac{d\tau}{(t_n - \tau)^{\alpha-1}} \\
 &= \frac{1}{\tau \Gamma(2-\alpha)} \left[\Delta \left(a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right) \right] + c_1 \left(\tau^{3-\alpha} + \hat{n}^{2-\min\{\hat{m}, p\}} \right), \tag{3.5b}
 \end{aligned}$$

and

$$A_i^n \Delta u_i^n - B_i^n \Delta^2 u_i^n = \tau A_i^n \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) - B_i^n \Delta \sigma_i^n + c_2 \left(\tau^{3-\alpha} + \hat{n}^{4-\min\{\hat{m}, p\}} \right). \tag{3.6}$$

Considering the difference property of the nonlinear term $\mathcal{N}(u(\mathbf{x}, t))$,

$$\mathcal{N}(u_i^n) = \left\{ \begin{array}{ll} \mathcal{N}(u(\mathbf{x}_i, t_0)) + c_3(\tau), & n=1, \\ 2\mathcal{N}(u(\mathbf{x}_i, t_{n-1})) - \mathcal{N}(u(\mathbf{x}_i, t_{n-2})) + c'_3(\tau^2), & n \geq 2. \end{array} \right\}. \tag{3.7}$$

Substituting above results into (1.1) and noticing the initial value condition, we obtain

$$\begin{aligned}
 &\frac{1}{\tau \Gamma(2-\alpha)} \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] \\
 &+ \frac{1}{\tau \Gamma(2-\alpha)} \left[\Delta \left(a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right) \right] \\
 &= \tau A_i^n \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) - B_i^n \Delta \sigma_i^n + \hat{F}_i^n + R_i^n, \tag{3.8}
 \end{aligned}$$

where

$$|R_i^n| \leq c(\tau^{3-\alpha} + \hat{n}^{4-\min\{\hat{m}, p\}}), \quad n > 1, \tag{3.9}$$

and \hat{F}_i^n involves the original F_i^n and the approximate value of $\mathcal{N}(u_i^n)$.

4 Analysis of the hybridized trapezoidal-difference scheme

Before we prove the solvability, stability and convergence, we first give the notation the inner product and the bilinear form by

$$(u, w) = \int_\Omega u w d\Omega, \quad \forall u, w \in L^2(\Omega), \quad \text{especially, } (u, u) = \|u\|^2 = \sum_{i=1}^m \mathbf{w}_i (u_i)^2, \tag{4.1}$$

w_i denotes the gauss weight at the corresponding gauss point x_i . Then, we give some lemmas.

Lemma 4.1. Suppose $\{v^n\}$ is the solution of

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] \\ & + \frac{1}{\tau\Gamma(2-\alpha)} \left[\Delta \left(a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right) \right] \\ & = \tau A_i^n \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) - B_i^n \Delta \sigma_i^n + R_i^n, \quad u_i^n(\partial\Omega) = 0, \quad i = 1, 2, \dots, \quad m, n = 1, 2, \dots \end{aligned} \quad (4.2)$$

We have

$$\sum_{n=1}^N \|\nabla u^n\|^2 \leq \sum_{n=1}^N \|\nabla u^0\|^2 + \frac{t_N^{2-\alpha}}{4\hat{C}\Gamma(3-\alpha)} (\|v^0\|^2 + \|\nabla v^0\|^2) + \frac{\Gamma(2-\alpha)t_N^{\alpha-1}}{4\hat{C}} \tau \sum_{n=1}^N \|R^n\|^2 \quad (4.3)$$

with $A_i^n \geq \hat{A} > 0$, $B_i^n \geq \hat{B} > 0$ and $\hat{C} = \min\{\hat{A}, \hat{B}\}$.

Proof. Multiplying both sides of (4.2) by $w_i v_i^n$ and summing up for i from 1 to m and for n from 1 to N , we obtain

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \sum_{i=1}^m \left\{ \sum_{n=1}^N w_i \left(\left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] \right. \right. \\ & \left. \left. - \Delta \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] \right) v_i^n \right\} \\ & = \sum_{n=1}^N \sum_{i=1}^m \left(A_i^n w_i \left[\sum_{x \in X} \delta_x^2 \left(\sum_{k=1}^{n-1} v^k + \frac{v^0 + v^n}{2} \right) \right] v_i^n \right) \\ & \quad - \sum_{n=1}^N \sum_{i=1}^m \left(B_i^n w_i \left[\sum_{x \in X} \delta_x^2 \left(\sum_{k=1}^{n-1} \Delta v^k + \frac{\Delta v^0 + \Delta v^n}{2} \right) \right] v_i^n \right) + \sum_{n=1}^N \sum_{i=1}^m w_i R_i^n v_i^n. \end{aligned} \quad (4.4)$$

Using Lemma 3.1, we have

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \sum_{n=1}^N \sum_{i=1}^m \left(w_i \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] v_i^n \right) \\ & \geq \frac{1}{2\Gamma(2-\alpha)} t_N^{1-\alpha} \sum_{n=1}^N \|v^n\|^2 - \frac{t_N^{2-\alpha}}{2\tau\Gamma(3-\alpha)} \|v^0\|^2, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \sum_{n=1}^N \sum_{i=1}^m \left(\mathfrak{w}_i \Delta \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] v_i^n \right) \\ & \geq \frac{1}{2\Gamma(2-\alpha)} t_N^{1-\alpha} \sum_{n=1}^N \|\nabla v^n\|^2 - \frac{t_N^{2-\alpha}}{2\tau\Gamma(3-\alpha)} \|\nabla v^0\|^2. \end{aligned} \tag{4.6}$$

Applying the boundary conditions in (4.2), we have $v_i^n(\partial\Omega) = \nabla v_i^n(\partial\Omega) = 0$. Consequently,

$$\tau \sum_{i=1}^m \left(\mathfrak{w}_i \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) v_i^n \right) = -\tau \sum_{i=1}^m \left(\mathfrak{w}_i \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) v_i^n \right) = -\frac{\tau}{2} \left\| \sum_{k=1}^n \nabla v^k \right\|^2, \tag{4.7a}$$

$$\begin{aligned} & -\tau \sum_{i=1}^m \left(\mathfrak{w}_i \Delta^2 \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) v_i^n \right) = -\tau \sum_{i=1}^m \left(\mathfrak{w}_i \Delta^2 \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) v_i^n \right) \\ & = -\frac{\tau}{2} \left\| \sum_{k=1}^n \Delta v^k \right\|^2. \end{aligned} \tag{4.7b}$$

In addition,

$$\sum_{n=1}^N \sum_{i=1}^m \mathfrak{w}_i R_i^n v_i^n \leq \frac{1}{2} \frac{1}{\Gamma(2-\alpha)} t_N^{1-\alpha} \sum_{n=1}^N \|v^n\|^2 + \frac{\Gamma(2-\alpha)}{2} t_N^{\alpha-1} \sum_{n=1}^N \|R^n\|^2. \tag{4.8}$$

Substituting (4.5)-(4.8) into (4.4), we obtain

$$\begin{aligned} & \frac{1}{2\Gamma(2-\alpha)} t_N^{1-\alpha} \sum_{n=1}^N \|v^n\|^2 - \frac{t_N^{2-\alpha}}{2\tau\Gamma(3-\alpha)} \sum_{n=1}^N \|v^0\|^2 + \frac{1}{2\Gamma(2-\alpha)} t_N^{1-\alpha} \sum_{n=1}^N \|\nabla v^n\|^2 \\ & - \frac{t_N^{2-\alpha}}{2\tau\Gamma(3-\alpha)} \sum_{n=1}^N \|\nabla v^0\|^2 \\ & \leq -\frac{\tau\hat{A}}{2} \sum_{n=1}^N \left\| \sum_{k=1}^n \nabla v^k \right\|^2 - \frac{\tau\hat{B}}{2} \sum_{n=1}^N \left\| \sum_{k=1}^n \Delta v^k \right\|^2 + \frac{1}{2\Gamma(2-\alpha)} t_N^{1-\alpha} \sum_{n=1}^N \|v^n\|^2 \\ & + \frac{\Gamma(2-\alpha)}{2} t_N^{\alpha-1} \sum_{n=1}^N \|R^n\|^2. \end{aligned} \tag{4.9}$$

Then,

$$\sum_{n=1}^N \left\| \sum_{k=1}^n \nabla v^k \right\|^2 \leq \frac{t_N^{2-\alpha}}{\hat{C}\tau^2\Gamma(3-\alpha)} (\|v^0\|^2 + \|\nabla v^0\|^2) + \frac{\Gamma(2-\alpha)t_N^{\alpha-1}}{\hat{C}\tau} \sum_{n=1}^N \|R^n\|^2. \tag{4.10}$$

Combing the correction between v and u with (4.10), the following inequality is obtained

$$\begin{aligned} & \sum_{n=1}^N \|\nabla u^n\|^2 \leq \sum_{n=1}^N \|\nabla u^0\|^2 + \left(\frac{\tau}{2}\right)^2 \sum_{n=1}^N \left\| \sum_{k=1}^n \nabla v^k \right\|^2 \\ & \leq \sum_{n=1}^N \|\nabla u^0\|^2 + \frac{t_N^{2-\alpha}}{4\hat{C}\Gamma(3-\alpha)} (\|v^0\|^2 + \|\nabla v^0\|^2) + \frac{\Gamma(2-\alpha)t_N^{\alpha-1}}{4\hat{C}} \tau \sum_{n=1}^N \|R^n\|^2. \end{aligned} \tag{4.11}$$

Thus, we complete the proof. □

Theorem 4.1. *The difference scheme (2.2)-(2.3) is uniquely solvable.*

Proof. Since (2.2)-(2.3) are a system of linear algebraic equations at each iterative process of different time level, it suffices to show that the corresponding homogeneous equations:

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] \\ & - \frac{1}{\tau\Gamma(2-\alpha)} \Delta \left[a_0 v_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) v_i^k - a_{n-1} v_i^0 \right] \\ & = \tau A_i^n \Delta \left(\sum_{k=1}^{n-1} v_i^k + \frac{v_i^0 + v_i^n}{2} \right) - B_i^n \Delta \sigma_i^n, \end{aligned} \tag{4.12a}$$

$$u_i^n(\Gamma) = \nabla u_i^n(\Gamma) = 0, \quad 1 \leq i \leq m, \quad n \geq 1, \tag{4.12b}$$

have only zero solution. Using Lemma 4.1, we have

$$\nabla u^n = 0, \quad n = 1, \dots, N,$$

and combining the above equality with the boundary condition in (2.4), we obtain

$$u_i^n = v_i^n = 0, \quad n \geq 1, \quad 1 \leq i \leq m.$$

This completes the proof. □

Theorem 4.2. *Let $u(\mathbf{x}, t) \in C_{\mathbf{x}, t}^{4,3}(\Omega \times [0, T])$ and $\{v^n | n \geq 0\}$ be the solution of the difference scheme (2.2)-(2.3). Then, for $\tau < n\tau \leq T$, we have*

$$\|u(\mathbf{x}_i, t_n) - u_i^n\|_\infty \leq C^* D \sqrt{\Gamma(2-\alpha) T^\alpha (\tau^{3-\alpha} + \hat{h}^{4-\min\{\hat{m}, p\}})},$$

where C^* is a constant number.

Proof. Denote

$$\hat{v}_i^n = v(\mathbf{x}_i, t_n) - v_i^n, \tag{4.13a}$$

$$\hat{u}_i^n = u(\mathbf{x}_i, t_n) - u_i^n, \quad n \geq 0. \tag{4.13b}$$

Subtracting (2.2-2.3) from (3.8) and (3.9) respectively, we have the error equations

$$\begin{aligned} & \frac{1}{\tau\Gamma(2-\alpha)} \left[a_0 \hat{v}_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \hat{v}_i^k - a_{n-1} \hat{v}_i^0 \right] \\ & - \frac{1}{\tau\Gamma(2-\alpha)} \Delta \left[a_0 \hat{v}_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \hat{v}_i^k - a_{n-1} \hat{v}_i^0 \right] \\ & = \tau A_i^n \Delta \left(\sum_{k=1}^{n-1} \hat{v}_i^k + \frac{\hat{v}_0 + \hat{v}_n}{2} \right) - B_i^n \Delta \hat{\sigma}_i^n, \end{aligned} \tag{4.14a}$$

$$\hat{u}_i^n(\Gamma) = \hat{v}_i^n(\Gamma) = \hat{\sigma}_i^n(\Gamma) = 0, \quad n \geq 1. \tag{4.14b}$$

Using Lemma 3.3, we have

$$\|\nabla u^n\|^2 \leq \frac{\Gamma(2-\alpha)t_n^{\alpha-1}}{4\hat{C}} \tau \sum_{k=1}^n \|R^k\|^2, \quad \tau < n\tau \leq T. \quad (4.15)$$

Inserting (3.8) into the right hand of the above inequality and considering the correction (2.10), we get

$$|\nabla \hat{u}^n| \leq c^* \sqrt{\frac{D\Gamma(2-\alpha)T^\alpha}{4\hat{C}} (\tau^{3-\alpha} + \hat{n}^{4-\min\{\hat{m}, p\}})}, \quad \tau < n\tau \leq T. \quad (4.16)$$

Noticing (2.11), we have the result:

$$\|\hat{u}^n\|_\infty \leq C^* D \sqrt{\Gamma(2-\alpha)T^\alpha (\tau^{3-\alpha} + \hat{n}^{4-\min\{\hat{m}, p\}})}, \quad \tau < n\tau \leq T, \quad (4.17)$$

where C^* are constant. This completes the proof. \square

Remark 4.1. Obviously, when n is equal to 1, the error order of the spatial variable t is equal to $\mathcal{O}(\tau)$. But, by the increasing of n , the final error order will tend to $\mathcal{O}(\tau^{3-\alpha})$ at $t = T$. The detailed conclusions are considered in the following numerical examples.

5 Numerical experiments

In this section, some comparative examples are provided to show the strength of the proposed method in approximating the solution of multi-dimensional time-fractional diffusion equations with a non-local boundary condition

In all numerical experiments, the presented method are local approximate schemes in spatial dimensions Ω . The numerical results are performed in MATLAB 2014a on an Intel core i5(8G RAM) Windows Win10 system. The L_2 error which will be reported in those examples are defined as

$$E_n = \max_{x_i \in \Omega} |u^n(x_i) - u_i^n| \quad \text{and} \quad \text{Order} = \frac{\log_2(E_n)}{\log_2(E_{2n})}, \quad (5.1)$$

where u_i and $u(x_i)$ denote the numerical and exact solution of the problem respectively.

5.1 1D space fourth order fractional system

Consider the following problem:

$$\begin{aligned} & {}_0^c \mathcal{D}_t^\alpha u(x,t) + ({}_0^c \mathcal{D}_t^\alpha - \cos(xt)) \Delta u(x,t) + \exp(x+t) \Delta^2 u(x,t) \\ & = u^3(x,t) + f(x,t), \quad x \in (0,1), \quad t \in (0,1], \quad \alpha \in (1,2), \end{aligned} \quad (5.2a)$$

$$\begin{aligned} & u(0,t) = 0; \quad u(1,t) = 0, \quad \frac{\partial^2 u(0,t)}{\partial x^2} = 0, \quad \frac{\partial^2 u(1,t)}{\partial x^2} = 0, \\ & u(x,0) = \sin(\pi x), \quad \frac{\partial u(x,0)}{\partial t} = \sin(\pi x), \quad x \in (0,1), \quad t \in (0,1]. \end{aligned} \quad (5.2b)$$

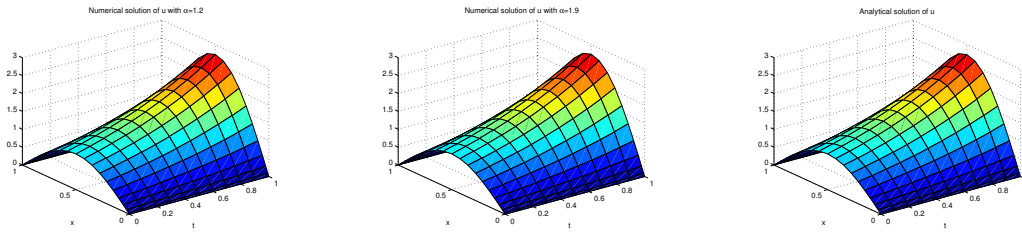


Figure 1: The surfaces for the numerical solutions (left and middle) and exact solutions (right).

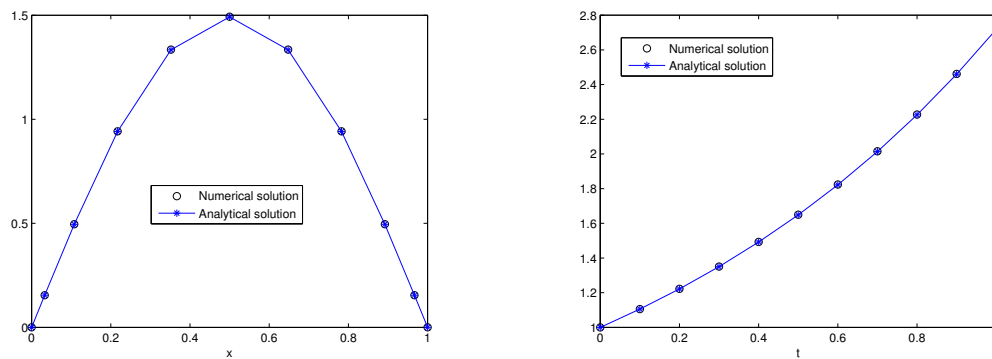


Figure 2: the curves of the numerical solutions and exact solutions along with $t=0.4$ (left) and $x=0.5$ (right).

The exact solution of the system is

$$u(x,t) = \exp(t) \sin(\pi x). \tag{5.3}$$

Take $\tau = 1/10$ and $m = 11$, Fig. 1 shows the numerical solutions (left and middle) of the proposed method and the exact solutions (right) at $t = 1$ for the fourth order fractional diffusion-wave system for $\alpha = 1.2$ (left) and $\alpha = 1.9$ (middle) with $m = 11$.

Fig. 2 plots the curves of the numerical solutions of the compact difference scheme and the analytical solutions of the same α , ($\alpha = 1.4$) at the time $t = 0.4$ and $x = 0.5$. From these figures, it is easy to see that the proposed methods meets the analytical solution effectively.

Take $\alpha = 1.7$, Table 1 represents some numerical and exact solutions on the points $(0.5,0.5)$, $(0.5,1)$ for different mesh sizes with fixed temporal step. From there, we can see the presented scheme converges to the exact solutions quickly.

Table 2 give the temporal convergence order of the presented scheme at $t = 1$ with $\alpha = 1.9$, which are close to our theoretical values.

Table 1: Some numerical results of the different grid schemes ($\tau=1/160$).

(x,t)	$m=5$	7	9	11	13	Exact solution
(0.5,0.5)	1.6530	1.6489	1.6489	1.6489	1.6489	1.6487
(0.5,1)	2.7245	2.7184	2.7184	2.7184	2.7184	2.7183

Table 2: Temporal convergence order of the presented scheme ($m=21$).

τ	1/10	1/20	1/40	1/80	1/160	1/320	Theoretical value
E_n	1.0316e-2	3.6200e-3	1.3246e-3	5.1886e-4	2.1650e-4	9.4577e-5	
Order		1.5109	1.4504	1.3521	1.2610	1.1948	1.1000

5.2 2D space fourth order fractional system

To implement the numerical effectiveness, we consider the following two dimensional nonlinear fourth-order equation

$$\begin{aligned} & {}^c_0\mathcal{D}_t^\alpha u(x,y,t) + ({}^c_0\mathcal{D}_t^\alpha - \cos(xy))\Delta u(x,y,t) + (t^2+1)\Delta^2 u(x,y,t) \\ & = u^2(x,y,t) - u(x,y,t) + f(x,y,t), \end{aligned} \quad (5.4a)$$

$$\alpha \in (1,2), \quad (x,y) \in (0,1)^2, \quad t \in (0,1], \quad (5.4b)$$

with initial conditions and boundary conditions

$$u(\Omega,0)=0; \quad u_t(\Omega,0)=\sin(2\pi x)\sin(2\pi y); \quad u(\Gamma,t)=0; \quad \sigma(\Gamma,t)=0, \quad t \in (0,1]. \quad (5.5)$$

The exact solution of the system is

$$u(x,y,t) = (t^{3+\alpha} + t) \sin(2\pi x) \sin(2\pi y). \quad (5.6)$$

Take $m=21^2$, $\alpha=1.3, 1.5, 1.7$, respectively. Table 3 gives some numerical results of the maximum errors and the corresponding temporal convergence order of the different temporal mesh size at $t=1$, from which we can see the temporal convergence order is nearly and close to $\mathcal{O}(\tau^{3-\alpha})$.

In Fig. 3, the surfaces of the numerical solution and the absolute error of u are shown with $m=21^2$, $\tau=1/10$ and $\alpha=1.4$ at $t=1$. It not hard to see that the numerical solutions is vary close to the analytical solutions. Compared with the two surfaces of u , the surface of the absolute error coincides with the surface of numerical results. The similar conclusion between the numerical solution and absolute error of the σ is obtained in Fig. 3.

5.3 3D space fourth order fractional system

To implement the numerical effectiveness, we consider the following three dimensional nonlinear fourth-order equation

$$\begin{aligned} & {}^c_0\mathcal{D}_t^\alpha u(x,y,z,t) + ({}^c_0\mathcal{D}_t^\alpha - 1)\Delta u(x,y,z,t) + \Delta^2 u(x,y,z,t) \\ & = u^3(x,y,z,t) - u(x,y,z,t) + f(x,y,z,t), \quad \alpha \in (1,2), \quad (x,y,z) \in (0,1)^3, \quad t \in (0,1], \end{aligned} \quad (5.7)$$

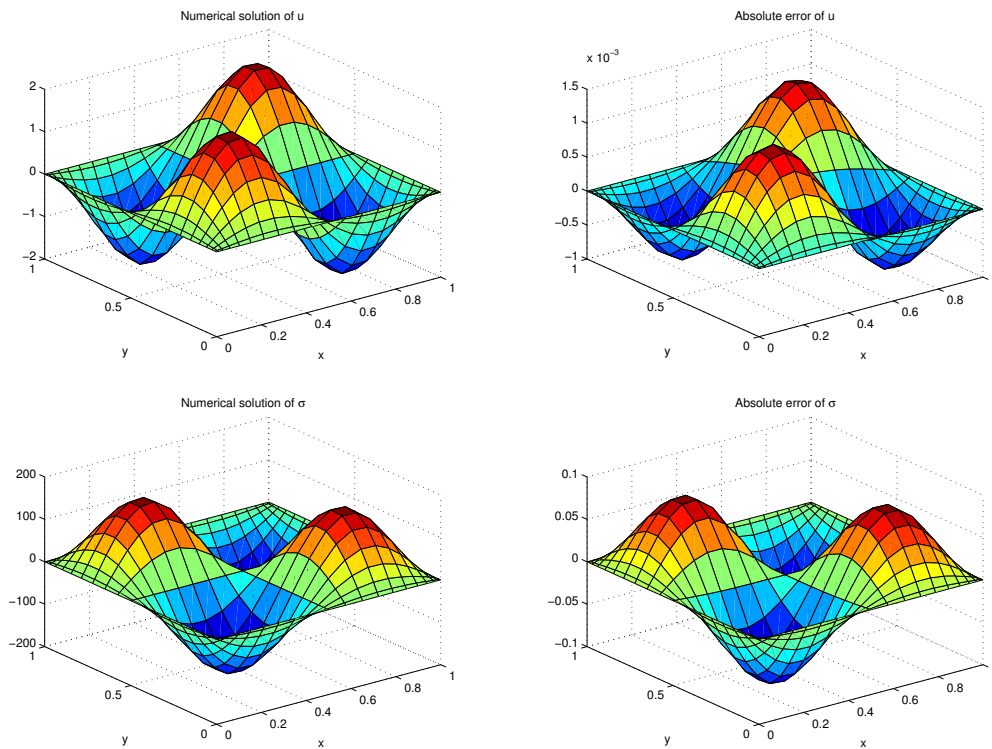


Figure 3: The surfaces of the numerical solution (left) and absolute error (right) of u (up) and σ (down).

Table 3: Some numerical results of the maximum errors and the corresponding temporal order at $t=1$ for the fourth order fractional equation.

τ	$\alpha = 1.7$		$\alpha = 1.5$		$\alpha = 1.3$	
	E_n	Order	E_n	Order	E_n	Order
1/10	3.3835e-2		1.4740e-2		5.7372e-3	
1/20	1.5782e-2	1.1002	5.2540e-3	1.4882	1.7887e-3	1.6814
1/40	6.4484e-3	1.2913	1.8611e-3	1.4973	5.4500e-4	1.7164
1/80	2.6267e-3	1.2957	6.8512e-4	1.4997	1.6497e-4	1.7241
1/160	1.0685e-3	1.2977	2.3267e-4	1.5001	4.9872e-5	1.7259
1/320	4.3429e-4	1.2989	8.2263e-5	1.5000	1.5149e-5	1.7190
1/640	1.7645e-4	1.2994	2.9087e-5	1.4999	4.6189e-6	1.7136
1/1280	7.1680e-5	1.2996	1.0285e-5	1.4998	1.4126e-6	1.7092
1/2560	2.9114e-5	1.2999	3.6367e-6	1.4998	4.3292e-7	1.7062
Theoretical value		1.3000		1.5000		1.7000

with initial conditions and boundary conditions

$$u(\Omega,0)=0; \quad u_t(\Omega,0)=0; \quad u(\Gamma,t)=0; \quad \sigma(\Gamma,t)=0, \quad t \in (0,1]. \tag{5.8}$$

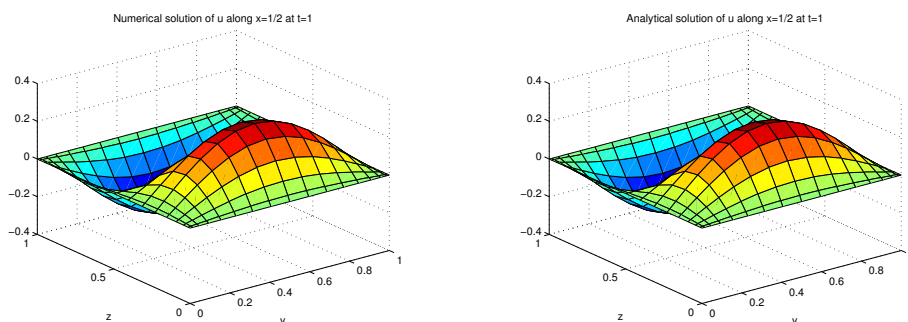


Figure 4: The surfaces for the numerical solutions (left) and exact solutions (right).

Table 4: Some numerical results of the maximum errors and the corresponding temporal order at $t=1$ for the fourth order fractional equation.

τ	$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$	
	E_n	Order	E_n	Order	E_n	Order
1/10	7.7200e-4		1.8211e-3		4.1648e-3	
1/20	2.3963e-4	1.6878	6.4596e-4	1.4953	1.7144e-3	1.2805
1/40	7.4216e-5	1.6910	2.2917e-4	1.4950	6.9422e-4	1.3042
1/80	2.2945e-5	1.6936	8.1211e-5	1.4967	2.8068e-4	1.3065
1/160	7.0933e-6	1.6937	2.8757e-5	1.4978	1.1369e-4	1.3038
1/320	2.2197e-6	1.6961	1.0182e-5	1.4979	4.6115e-5	1.3018

The exact solution of the system is

$$u(x, y, z, t) = \frac{t^3}{3} \sin(\pi x) \sin(\pi y) \sin(2\pi z). \quad (5.9)$$

For the convenience of the numerical calculation, we now take $M=17^3$ and $\tau=1/20$.

In Fig. 4, Three comparisons for the surfaces of the exact solution and the numerical solution are shown along with $x=1/2$ at $t=1$ considering $\alpha=1.6$.

Take $\alpha=1.3, 1.5, 1.7$, respectively. Table 4 gives some numerical results of the maximum errors and the corresponding temporal convergence order of the different temporal mesh size at $t=1$, from which we also take the conclusion that the temporal convergence order is nearly and close to $\mathcal{O}(\tau^{3-\alpha})$.

6 Conclusions

In this paper, based on the Picard integral formulation, we present a compact finite difference scheme coupled with Chebyshev spectral collocation method to solve the multi-dimensional nonlinear variable coefficient time-fractional fourth-order partial differential

system. Different from other many schemes, the proposed method consider the regularity of the derivative function v on the integer time step. The stability and solvability are proved with detailed proofs and the precisely describe of error estimates is derived. To confirm the practicability and accuracy of the proposed method. several numerical results are calculated and analyzed in one-dimensional, two-dimensional and even three dimensional spatial domain. Numerical convergence rate consistent with the theoretical value $3-\alpha$ in L_∞ norm.

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