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A Hybrided Trapezoidal-Difference Scheme for Nonlinear Time-Fractional Fourth-Order Advection-Dispersion Equation Based on Chebyshev Spectral Collocation Method

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Abstract. In this paper, we firstly present a novel simple method based on a Picard integral type formulation for the nonlinear multi-dimensional variable coefficient fourth-order advection-dispersion equation with the time fractional derivative order $\alpha\in(1,2)$. A new unknown function $v(\mathbf{x},t)=\partial u(\mathbf{x},t)/\partial t$ is introduced and $u(\mathbf{x},t)$ is recovered using the trapezoidal formula. As a result of the variable $v(\mathbf{x},t)$ are introduced in each time step, the constraints of traditional plans considering the non-integer time situation of $u(\mathbf{x},t)$ is no longer considered. The stability and solvability are proved with detailed proofs and the precise describe of error estimates is derived. Further, Chebyshev spectral collocation method supports accurate and efficient variable coefficient model with variable coefficients. Several numerical results are obtained and analyzed in multi-dimensional spatial domains and numerical convergence order are consistent with the theoretical value $3-\alpha$ order for different α under infinite norm.

AMS subject classifications: 65M60, 65N30, 65N15

Key words: Trapezoidal-difference scheme, time-fractional order, variable coefficient fourth-order advection-dispersion equation, Chebyshev spectral collocation method, nonlinearity.

1 Introduction

Fractional calculus is a natural generalization of integer order operator. Utilizing the models based on derivatives of fractional orders in several branches of science and engineering is a major study of many mathematicians and physicians [1–5]. Roughly speaking time fractional derivative is designed to characterize physical processes and dynamic systems with history memory. As a counterpart of traditional integer order differential

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equation, fractional differential equation can be obtained by replacing the integer order derivatives with fractional ones in integer order differential equation. Fractional partial differential equations(FPDEs), particularly space and time-fractional equations, have been widely studied to construct the existence of solution and validity of these problems [6–8]. In addition, the reliable and powerful numerical and analytical methods for solving FPDEs has been focused in the last two decades. According to the mathematical literature, fractional partial differential equations have been progressed in various problems in science and engineering such as the Schrödinger, diffusion and telegraph fractional equations [6, 9–14].

In several applications, the fourth-order model system [15, 16] is an important part of the fractional order system and can be found in physics, engineering, statistics, and other fields, such as wave propagation in beam problems [17], A flat surface system of grooves [5,18], several mathematical models of fourth-order subdiffusion systems [18–21] and so on. Here we will consider the following the nonlinear multi-dimensional variable coefficient time-fractional fourth-order advection-dispersion equation:

$$\begin{aligned} &{}_0^c\mathfrak{D}_t^{\alpha}u(\mathbf{x},t)-(A(\mathbf{x},t)+{}_0^c\mathfrak{D}_t^{\alpha})\Delta u(\mathbf{x},t)\\ &=-B(\mathbf{x},t)\Delta^2u(\mathbf{x},t)+\mathcal{N}(u(\mathbf{x},t))+f(\mathbf{x},t),\quad \mathbf{x}\in\Omega,\quad t\in(0,T], \end{aligned} \tag{1.1}$$

where $A(\mathbf{x},t)$ and $B(\mathbf{x},t)$ are positive variable coefficients with the following initial and boundary conditions:

$$u(\mathbf{x},0)=u_0(\mathbf{x}),\quad u_t(\mathbf{x},0)=v_0(\mathbf{x}),\qquad \mathbf{x}\in\Omega, \tag{1.2a}$$

$$u(\mathbf{x},t)=\Delta u(\mathbf{x},t)=0,\qquad \mathbf{x}\in\partial\Omega,\quad t\in(0,T], \tag{1.2b}$$

where $u(\mathbf{x},t)$ is unknown functions. Here ${}_0^c\mathfrak{D}_t^{\beta(\mathbf{x},t)}$ denotes the higher order Caputo fractional derivative of variable order $\beta(\mathbf{x},t)$ with respect to t in [4, 5, 11, 15–21]

$${}_0^c\mathfrak{D}_t^{\beta(\mathbf{x},t)}u(\mathbf{x},t)=\frac{1}{\Gamma(n-\beta(\mathbf{x},t))}\int_0^t\frac{\partial^n u(\mathbf{x},\eta)}{\partial\eta^n}\frac{d\eta}{(t-\eta)^{\beta(\mathbf{x},t)+1-n}},\quad n-1\le\beta(\mathbf{x},t)\le n, \tag{1.3}$$

where $\Gamma(\cdot)$ is the Gamma function. The nonlinear term $\mathcal{N}(u(\mathbf{x},t))$ is assumed to satisfy the following conditions: a) $|\mathcal{N}(u(\mathbf{x},t))|\le C|u|$, b) The first-order derivative function of $\mathcal{N}(u(\mathbf{x},t))$ with respect to u is bounded, i.e., $|\mathcal{N}'(u(\mathbf{x},t))|\le a$, a is a positive constant.

Most of fractional partial differential equations do not have the analytic solutions, many researchers in the last two decades have focused on the approximation or numerical methods of these fractional order systems in [22, 23]. Lots of the researchers focus their attention on the strong format. This format is directly obtained by the original discrete equation. So, it is also called the collocation method. Strong formulation is reliable, simple in structure, and easy to erect the algebra system. The homotopy analysis method was utilized to approximate some FPDEs in [24, 25]. The finite difference scheme and fractional predictor-corrector method are introduced for simulating the multi-term time-fractional wave-diffusion equations with computationally effective results by Adams-Bashforth method [26]. Also, some fractional differential equations utilized for modeling dynamical systems are investigated by an implicit difference approximation in [27].