

Lower Bounds of Eigenvalues of the Stokes Operator by Nonconforming Finite Elements on Local Quasi-Uniform Grids

Youai Li*

School of Science, Beijing Technology and Business University, Beijing 100048, China

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Abstract. This paper is a generalization of some recent results concerned with the lower bound property of eigenvalues produced by both the enriched rotated Q_1 and Crouzeix–Raviart elements of the Stokes eigenvalue problem. The main ingredient are a novel and sharp L^2 error estimate of discrete eigenfunctions, and a new error analysis of nonconforming finite element methods.

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1 Introduction

It was observed from numerical examples that some nonstandard finite element methods including nonconforming finite element methods and mass lumping finite element methods are able to yield lower bounds of eigenvalues of partial differential eigenvalue problems, see, Zienkiewicz et al. [32], for the Morley element, Rannacher [25], for the Morley and Adini elements, Liu and Yan [22], for the Wilson, enriched rotated Q_1 , and rotated Q_1 elements. However, it was only very recent that these phenomena were rigorously analyzed. Indeed, Armentano and Duran [1] proposed an identity of errors of eigenvalues and proved that the Crouzeix–Raviart element produces lower bounds of eigenvalues for the Laplace operator provided that eigenfunctions $u \in H^{1+r}(\Omega) \cap H_0^1(\Omega)$ with $0 < r < 1$. The idea was generalized to the enriched rotated Q_1 element by the author of this paper in [14], and to the Wilson element in Zhang, Yang and Chen [31]. The extension to the Morley element was carried out in [29]. However, all of those papers are based on the saturation condition of finite element solutions by piecewise polynomials. The saturation condition can be from consistency errors and approximation errors as

*Corresponding author.
Email: lya@lsec.cc.ac.cn (Y. A. Li)

well as regularity of exact solutions, and was proved based on a *direct argument* in Hu, Huang and Lin [9]. If only the saturation condition of approximation errors is concerned, it was independently showed based on a *contradiction argument* in Lin, Xie and Xu [21]; see its application to nonconforming finite elements of the Laplace eigenvalue problem in [23]. In Hu, Huang and Lin [9], a more general result of the lower bound property of eigenvalues by nonconforming finite elements was established that if local approximation properties of nonconforming finite element spaces are better than total errors (sums of global approximation errors and consistency errors) of nonconforming finite element methods, corresponding methods will produce lower bounds for eigenvalues. For expansion methods based on superconvergence or extrapolation we refer interested readers to [17, 18, 30, 31], where the lower bound property of eigenvalues by nonconforming elements was analyzed on uniform rectangular meshes. We also refer interested readers to [10] for mass lumping finite element methods of eigenvalue problems.

The lower bound property of the eigenvalue by the nonconforming methods of the Stokes eigenvalue problem was first analyzed in [20], where a numerical result indicated that conforming finite elements of the Stokes eigenvalue problem are also possible to yield lower bounds of eigenvalues. In a recent paper by Hu and Huang [8], a more general framework is established for both conforming and nonconforming finite element methods for the Stokes operator. In particular, it was proved that the conforming $P_2 - P_0$ element yields lower bounds of eigenvalues for the Stokes operator. However, all of these papers can only provide (asymptotic) lower bounds for eigenvalues on quasi-uniform grids.

In this paper we give a refined analysis for the lower bound property of eigenvalues by both the enriched rotated Q_1 [19] and Crouzeix–Raviart elements [9] of the Stokes eigenvalue problem. The main idea is to combine a series of new techniques: the element-wise Poincaré-like inequality of the canonical interpolation operators of these two elements, a novel L^2 error estimate of discrete eigenfunctions from [8], a new error analysis of nonconforming finite element methods, plus the commuting property of the canonical interpolation operators.

In this paper, we use the standard gradient operator:

$$\nabla r := (\partial r / \partial x_1, \dots, \partial r / \partial x_n)^T.$$

Given any n dimensional vector function $\psi = (\psi_1, \dots, \psi_n)$, its divergence reads

$$\operatorname{div} \psi := \partial \psi_1 / \partial x_1 + \partial \psi_2 / \partial x_2 + \dots + \partial \psi_n / \partial x_n.$$

The spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$ are defined as usual,

$$\begin{aligned} H_0^1(\Omega) &:= \{v \in H^1(\Omega), v = 0 \text{ on } \partial\Omega\}, \\ L_0^2(\Omega) &:= \left\{q \in L^2(\Omega), \int_{\Omega} q dx = 0\right\}. \end{aligned}$$

This paper is organized as follows. In the next section, we present the Stokes eigenvalue problem. In Section 3, we get its nonconforming finite element methods. We present a

new error analysis of nonconforming finite element methods in Section 4. In Section 5, we give a sharp L^2 error estimate of the discrete velocity and prove the final lower bound property of the discrete eigenvalues. We present the numerical experiments in Section 6.

2 The Stokes eigenvalue problem

The Stokes eigenvalue problem is defined as follows: Find $(\lambda, u, p) \in \mathbb{R} \times V \times Q := \mathbb{R} \times H_0^1(\Omega)^n \times L_0^2(\Omega)$ such that

$$a(u, v) + b(v, p) + b(u, q) = \lambda(u, v) \quad \text{and} \quad \|u\| = 1 \quad \text{for any } (v, q) \in V \times Q, \quad (2.1)$$

where the bilinear forms $a(u, v)$ and $b(v, q)$ are defined as, respectively,

$$a(u, v) := (\nabla u, \nabla v) \quad \text{and} \quad b(v, q) := -(\operatorname{div} v, q).$$

The kernel space of the divergence operator consists of all divergence free functions in V , which reads

$$V_0 := \{v \in V, b(v, q) = 0 \text{ for any } q \in Q\}.$$

Let (λ, u, p) be the solution of the problem (2.1). It follows from (2.1) that $u \in V_0$ and that

$$a(u, v) = \lambda(u, v) \quad \text{for any } v \in V_0. \quad (2.2)$$

Then, we have that the eigenvalue problem (2.1) has a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \nearrow +\infty,$$

and corresponding eigenfunctions

$$(u_1, p_1), (u_2, p_2), (u_3, p_3), \dots,$$

which can be chosen to satisfy

$$(u_i, u_j) = \delta_{ij}, \quad i, j = 1, 2, \dots.$$

We define

$$E_\ell := \operatorname{span}\{u_1, u_2, \dots, u_\ell\}.$$

Then, the eigenvalues and eigenfunctions satisfy the following well-known minimum-maximum principle:

$$\lambda_k = \min_{\dim V_k = k, V_k \subset V_0} \max_{v \in V_k} \frac{a(v, v)}{(v, v)} = \max_{u \in E_k} \frac{a(u, u)}{(u, u)}. \quad (2.3)$$

3 Two stable nonconforming finite element methods

In this section we introduce two stable nonconforming finite elements from [19] and [9], respectively.

3.1 Enriched rotated Q_1 elements in any dimension

Let \mathcal{T}_h be regular n -rectangular triangulations of domains $\Omega \subset \mathbb{R}^n$ with $2 \leq n$ in the sense that $\bigcup_{K \in \mathcal{T}_h} K = \bar{\Omega}$, two distinct elements K and K' in \mathcal{T}_h are either disjoint, or share an ℓ -dimensional hyper-plane, $\ell = 0, \dots, n-1$. Let \mathcal{H}_h denote the set of all $n-1$ dimensional hyper-planes in \mathcal{T}_h with the set of interior $n-1$ dimensional hyper-planes $\mathcal{H}_h(\Omega)$ and the set of boundary $n-1$ dimensional hyper-planes $\mathcal{H}_h(\partial\Omega)$. \mathcal{N}_h is the set of nodes of \mathcal{T}_h with the set of internal nodes $\mathcal{N}_h(\Omega)$ and the set of boundary nodes $\mathcal{N}_h(\partial\Omega)$.

For each $K \in \mathcal{T}_h$, introduce the following affine invertible transformation

$$F_K: \hat{K} \rightarrow K, \quad x_i = h_{x_i, K} \xi_i + x_i^0,$$

with the center $(x_1^0, x_2^0, \dots, x_n^0)$ and the lengths $2h_{x_i, K}$ of K in the directions of the x_i -axis, and the reference element $\hat{K} = [-1, 1]^n$. In addition, set $h = \max_{1 \leq i \leq n} h_{x_i}$.

Denote by $\text{EQ}(K)$ the enriched rotated Q_1 element space defined by [19]

$$\text{EQ}(K) := P_1(K) + \text{span}\{x_1^2, x_2^2, \dots, x_n^2\}. \quad (3.1)$$

The enriched rotated Q_1 element space V_h^{EQ} is then defined by

$$\begin{aligned} V_h^{EQ} := \{ & v \in L^2(\Omega)^n : v|_K \in \text{EQ}(K)^n \text{ for each } K \in \mathcal{T}_h, \int_E [v] dE = 0, \\ & \text{for all internal } n-1 \text{ dimensional hyper-planes } E, \\ & \text{and } \int_E v dE = 0 \text{ for all } E \text{ on } \partial\Omega \}. \end{aligned} \quad (3.2)$$

Here and throughout this paper, $[v]$ denotes the jump of v across E .

3.2 Enriched Crouzeix–Raviart elements in any dimension

Suppose that $\bar{\Omega}$ is covered exactly by shape-regular partitions \mathcal{T}_h consisting of n -simplices in n dimensions. Given $E \in \mathcal{H}_h$, let ν be unit normal vector.

To obtain a nonconforming finite element method that is able to produce lower bounds of eigenvalues of second order elliptic operators, it was proposed in [9] to enrich the shape function space $P_1(K)$ by $\text{span}\left\{\sum_{i=1}^n x_i^2\right\}$ on each element. This leads to the following shape function space

$$\text{ECR}(K) := P_1(K) + \text{span}\left\{\sum_{i=1}^n x_i^2\right\} \quad \text{for any } K \in \mathcal{T}.$$

The enriched Crouzeix-Raviart element space V_h^{ECR} is then defined by

$$\begin{aligned} V_h^{ECR} := \{ & v \in L^2(\Omega)^n : v|_K \in \text{ECR}(K)^n \text{ for each } K \in \mathcal{T}, \int_E [v] dE = 0, \\ & \text{for all } E \in \mathcal{H}_h(\Omega), \text{ and } \int_E v dE = 0 \text{ for all } E \in \mathcal{H}_h(\partial\Omega) \}. \end{aligned} \quad (3.3)$$

The discrete pressure space Q_h is defined by

$$Q_h := \{q \in Q, q|_K \in P_0(K) \text{ for any } K \in \mathcal{T}_h\}, \tag{3.4}$$

where $P_\ell(K)$ denote the space of polynomials of degree $\leq \ell$.

In this paper, we take the nonconforming finite element spaces V_h^{nc} as V_h^{EQ} or V_h^{ECR} . Then, the discrete eigenvalue problem is to seek $(\lambda_h, u_h, p_h) \in \mathbb{R} \times V_h^{nc} \times Q_h$ such that $\|u_h\| = 1$ and that

$$a_h(u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = \lambda_h(u_h, v_h) \text{ for all } (v_h, q_h) \in V_h^{nc} \times Q_h. \tag{3.5}$$

We define the semi-norm over $V_h^{nc} + V$ by $\|\cdot\|_h := a_h(\cdot, \cdot)^{1/2}$, it follows from the definition of V_h^{nc} that $\|\cdot\|_h$ is a norm over the discrete velocity space V_h^{nc} under consideration.

We define the kernel space of the discrete divergence operator by

$$V_{0,h} := \{v_h \in V_h^{nc}, b_h(v_h, q_h) = 0 \text{ for any } q_h \in Q_h\}.$$

Let (λ_h, u_h, p_h) be the solution of the problem (3.5), we have $u_h \in V_{0,h}$ and

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h) \text{ for any } v_h \in V_{0,h}.$$

Let $N := \dim V_{0,h}$. The discrete problem (3.5) admits a sequence of discrete eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \dots \leq \lambda_{N,h},$$

and corresponding eigenfunctions

$$(u_{1,h}, p_{1,h}), (u_{2,h}, p_{2,h}), \dots, (u_{N,h}, p_{N,h}).$$

In the case where (V_h, Q_h) is an conforming approximation in the sense $V_{0,h} \subset V_0$, it immediately follows from the minimum-maximum principle (2.3) that

$$\lambda_k \leq \lambda_{k,h}, \quad k = 1, 2, \dots, N,$$

which indicates that $\lambda_{k,h}$ is an approximation above to λ_k .

We define the discrete counterpart of E_ℓ by

$$E_{\ell,h} := \text{span}\{u_{1,h}, u_{2,h}, \dots, u_{\ell,h}\}.$$

Then, we have the following discrete minimum-maximum principle:

$$\lambda_{k,h} = \min_{\dim V_{k,h} = k, V_{k,h} \subset V_{0,h}} \max_{v \in V_{k,h}} \frac{a_h(v, v)}{(v, v)} = \max_{u \in E_{k,h}} \frac{a_h(u, u)}{(u, u)}. \tag{3.6}$$

4 A new error analysis of nonconforming finite element methods

For the analysis, we need a new error analysis of nonconforming finite element methods of the Stokes problem, which can be regarded as the generalization to the current case of those results from [6, 11, 16, 24].

Consider the Stokes problem: Given $f \in L^2(\Omega)$ find $(u, p) \in V \times Q$ such that

$$a(u_f, v) + b(v, p_f) + b(u_f, q) = (f, v) \quad \text{for any } (v, q) \in V \times Q. \quad (4.1)$$

The discrete problem of (4.1) is to seek $(u_{f,h}, p_{f,h}) \in V_h \times Q_h$ such that

$$a_h(u_{f,h}, v_h) + b_h(v_h, p_{f,h}) + b_h(u_{f,h}, q_h) = (f, v_h) \quad \text{for all } (v_h, q_h) \in V_h \times Q_h. \quad (4.2)$$

We need some quasi-interpolation/approximation operator $\pi_h : V_h^{nc} \rightarrow V_h^c$, where $V_h^c \subset H_0^1(\Omega)$ is some conforming finite element space over the triangulation T_h . We take V_h^c as the conforming linear finite element space for the triangulation mesh and the conforming n-linear finite element space for the quadrilateral mesh. Given $v_h \in V_h^{nc}$, the interpolation function $\pi_h v_h \in V_h^c$ is defined as

$$(\pi_h v_h)(\mathcal{P}) := \frac{1}{|\omega_{\mathcal{P}}|} \int_{\omega_{\mathcal{P}}} v_h d\sigma \quad \text{for any interior vertex } \mathcal{P}, \quad (4.3a)$$

$$(\pi_h v_h)(\mathcal{P}) := 0 \quad \text{for any boundary vertex } \mathcal{P}, \quad (4.3b)$$

where $\omega_{\mathcal{P}}$ is the node patch defined as $\omega_{\mathcal{P}} = \{K \in T_h, \mathcal{P} \text{ is a vertex of } K\}$. Such a kind of operators was proposed in [26]. For this quasi-interpolation operator, we have the following properties, whose proof can be found in, for instance, [2].

Lemma 4.1. *There holds for any $v_h \in V_h^{nc}$ and $K \in T_h$ that*

$$\|v_h - \pi_h v_h\|_{0,K} \lesssim h_K \|\nabla_h v_h\|_{0,\omega_K},$$

$$\|v_h - \pi_h v_h\|_{0,E} \lesssim h_E^{\frac{1}{2}} \|\nabla_h v_h\|_{0,\omega_E},$$

$$\|\nabla_h(v_h - \pi_h v_h)\|_{0,\Omega} \lesssim \|\nabla_h v_h\|_{0,\Omega}.$$

Here and throughout the paper, we shall use the notation

$$A_1 \lesssim B_1 \quad \text{and} \quad A_2 \approx B_2,$$

to denote that there exist constants C_1, c_2 and C_2 such that

$$A_1 \leq C_1 B_1 \quad \text{and} \quad c_2 B_2 \leq A_2 \leq C_2 B_2.$$

Before we present a new error analysis of the approximate solution $(u_{f,h}, p_{f,h})$, we need the following result.

Lemma 4.2. Let (u_f, p_f) be the solution of Problem (4.1). Then, it holds, for any $s_h \in V_h^{nc}$ and $q_h \in Q_h$, $K \in T_h$, and $E \in \mathcal{E}$, that

$$h_K \|f + \Delta s_h + \nabla q_h\|_{0,K} \lesssim \|\nabla(u_f - s_h)\|_{0,K} + \|p_f - q_h\|_{0,K} + h_K \|f - w\|_{0,K}, \tag{4.4a}$$

$$h_E^{\frac{1}{2}} \left\| \left[\frac{\partial s_h}{\partial \nu} + q_h \nu \right] \right\|_{0,E} \lesssim \|\nabla_h(u_f - s_h)\|_{0,\omega_E} + \|p_f - q_h\|_{0,\omega_E} + \sum_{K \in \omega_E} h_E \|f - w\|_{0,K}, \tag{4.4b}$$

for any $w \in P_2(K)^2$

Proof. We will use the bubble function technique from [28]. We first prove the efficiency of $\|f + \Delta s_h + \nabla q_h\|_{0,K}$.

$$\begin{aligned} \|f + \Delta s_h + \nabla q_h\|_{0,K}^2 &= \|f - w + w + \Delta s_h + \nabla q_h\|_{0,K}^2 \\ &\lesssim \|f - w\|_{0,K}^2 + \|w + \Delta s_h + \nabla q_h\|_{0,K}^2. \end{aligned}$$

Let $\lambda_i, i=1,2,3$, be the barycenter coordinate of the triangle K . Let $v_K = b_K(w + \Delta_h s_h + \nabla q_h)$ with the element bubble function $b_K = 27\lambda_1\lambda_2\lambda_3$. Let id be an identity matrix. We consider the following term

$$\begin{aligned} &\|w + \Delta s_h + \nabla q_h\|_{0,K}^2 \\ &\approx (b_K(w + \Delta s_h + \nabla q_h), w + \Delta s_h + \nabla q_h)_{0,K} \\ &= (v_K, w - f)_{0,K} + (v_K, f + \Delta s_h + \nabla q_h)_{0,K} \\ &= (v_K, w - f)_{0,K} + (\nabla u_f + p_f id, \nabla v_K)_{0,K} + (v_K, \Delta s_h + \nabla q_h)_{0,K} \\ &= (v_K, w - f)_{0,K} + ((\nabla u_f - \nabla s_h) + (p_f - q_h)id, \nabla v_K)_{0,K} \\ &\leq \|v_K\|_{0,K} \|w - f\|_{0,K} + (\|\nabla(u_f - s_h)\|_{0,K} + \|p_f - q_h\|_{0,K}) \|\nabla v_K\|_{0,K}. \end{aligned} \tag{4.5}$$

This inequality and the inverse estimate

$$\|\nabla v_K\|_{0,K} \lesssim h_K^{-1} \|v_K\|_{0,K},$$

lead to

$$\|w + \Delta s_h + \nabla q_h\|_{0,K} \lesssim \|w - f\|_{0,K} + h_K^{-1} (\|\nabla(u_f - s_h)\|_{0,K} + \|p_f - q_h\|_{0,K}),$$

which proves (4.4a).

We turn to (4.4b). Set

$$v_E = \left[\frac{\partial s_h}{\partial \nu} + q_h \nu \right] b_E$$

with the usual edge bubble function b_E from [28]. Then, the Poincare inequality and the

inverse estimate give

$$\begin{aligned}
 & \left\| \left[\frac{\partial s_h}{\partial v} + q_h v \right] \right\|_{0,E}^2 \\
 & \approx \left(b_E \left[\frac{\partial s_h}{\partial v} + q_h v \right], \left[\frac{\partial s_h}{\partial v} + q_h v \right] \right)_{0,E} \\
 & = (v_E, \left[\frac{\partial s_h}{\partial v} + q_h v \right])_{0,E} \\
 & = (\nabla_h s_h + q_h id, \nabla v_E)_{0,\omega_E} + (v_E, \Delta_h s_h + \nabla_h q_h)_{0,\omega_E} \\
 & = (\nabla_h (s_h - u_f) + (q_h - p_f) id, \nabla v_E)_{0,\omega_E} + (\nabla u_f + p_f id, \nabla v_E)_{0,\omega_E} + (\Delta_h s_h + \nabla_h q_h, v_E)_{0,\omega_E} \\
 & \leq (\|\nabla_h (s_h - u_f)\|_{0,\omega_E} + \|q_h - p_f\|_{0,\omega_E}) \|\nabla_h v_E\|_{0,\omega_E} + \|f + \Delta_h s_h + \nabla_h q_h\|_{0,\omega_E} \|v_E\|_{0,\omega_E} \\
 & \lesssim (\|\nabla_h (s_h - u_f)\|_{0,\omega_E} + \|q_h - p_f\|_{0,\omega_E} + h_E \|f + \Delta_h s_h + \nabla_h q_h\|_{0,\omega_E}) h_E^{-\frac{1}{2}} \left\| \left[\frac{\partial s_h}{\partial v} \right] \right\|_{0,E}'
 \end{aligned}$$

which implies

$$h_E^{\frac{1}{2}} \left\| \left[\frac{\partial s_h}{\partial v} + q_h v \right] \right\|_{0,E} \lesssim \|\nabla_h (s_h - u_f)\|_{0,\omega_E} + \|q_h - p_f\|_{0,\omega_E} + h_E \|f + \Delta_h s_h + \nabla_h q_h\|_{0,\omega_E}. \quad (4.6)$$

This completes the proof. □

Theorem 4.1. *Let (u_f, p_f) and $(u_{f,h}, p_{f,h})$ be the solutions of Problems (4.1) and (4.2), respectively. Then, it holds that*

$$\begin{aligned}
 & \|\nabla_h (u_f - u_{f,h})\|_0 + \|p_f - p_{f,h}\|_0 \\
 & \lesssim \inf_{v_h \in V_h^{nc}} \|\nabla_h (u_f - v_h)\|_0 + \inf_{q_h \in Q_h} \|p_f - q_h\|_0 + \left(\sum_{K \in \mathcal{T}_h} h_K^2 \inf_{w \in P_2(K)^2} \|f - w\|_{0,K}^2 \right)^{1/2}. \quad (4.7)
 \end{aligned}$$

Proof. By the theories of mixed finite elements and nonconforming finite elements, see for instance, [3,4], it follows that

$$\begin{aligned}
 & \|\nabla_h (u_f - u_{f,h})\|_0 + \|p_f - p_{f,h}\|_0 \\
 & \lesssim \inf_{v_h \in V_h^{nc}} \|\nabla_h (u_f - v_h)\|_0 + \inf_{q_h \in Q_h} \|p_f - q_h\|_0 + \sup_{0 \neq v_h \in V_h^{nc}} \frac{a_h(u_f, v_h) + b_h(v, p_f) - (f, v_h)}{\|\nabla_h v_h\|_0}. \quad (4.8)
 \end{aligned}$$

Next we shall bound the consistent error term on the right-hand side of (4.8) by the approximation error term plus a oscillation term (up to some multiplicative constant). In fact, for any $v_h, s_h \in V_h^{nc}$, it follows from (4.1) that

$$\begin{aligned}
 & a_h(u_f, v_h) + b_h(v_h, p_f) - (f, v_h) \\
 & = a_h(u_f, v_h - \pi_h v_h) + b_h(v_h - \pi_h v_h, p_f) - (f, v_h - \pi_h v_h) \\
 & = a_h(u_f - s_h, v_h - \pi_h v_h) + b_h(v_h - \pi_h v_h, p_f - q_h) \\
 & \quad - (f, v_h - \pi_h v_h) + a_h(s_h, v_h - \pi_h v_h) + b_h(v_h - \pi_h v_h, q_h). \quad (4.9)
 \end{aligned}$$

By Lemma 4.1, the first and second term can be estimated as

$$a_h(u_f - s_h, v_h - \pi_h v_h) + b_h(v_h - \pi_h v_h, p_f - q_h) \lesssim (\|\nabla_h(u_f - s_h)\|_0 + \|p_f - q_h\|_0) \|\nabla_h v_h\|_0.$$

After an integration by parts, the last three terms can be bounded by Lemma 4.1 and Lemma 4.2 as

$$\begin{aligned} & -(f, v_h - \pi_h v_h) + a_h(s_h, v_h - \pi_h v_h) + b_h(v_h - \pi_h v_h, q_h) \\ & \lesssim \|\nabla_h(u_f - s_h)\|_0 + \|p_f - q_h\|_0 + \left(\sum_{K \in \mathcal{T}_h} h_K^2 \inf_{w \in P_2(K)^2} \|f - w\|_{0,K}^2 \right)^{1/2}, \end{aligned}$$

which completes the proof. □

5 Lower bounds of eigenvalues

We need to bound the L^2 error of the discrete velocity. To this end, we follow [8] to introduce the quasi-Ritz-Galerkin projection $(\mathcal{P}_h u, \mathcal{P}_h p) \in V_h^{nc} \times Q_h$ as

$$a_h(\mathcal{P}_h u, v_h) + b_h(v_h, \mathcal{P}_h p) + b_h(\mathcal{P}_h u, q_h) = (\lambda u, v_h) \quad \text{for any } (v_h, q_h) \in V_h^{nc} \times Q_h, \quad (5.1)$$

where (λ, u, p) is the eigenpair of Problem (2.1).

Lemma 5.1 (see [8]). *Let (λ, u, p) and (λ_h, u_h, p_h) be the solutions of Problems (2.1) and (3.5), respectively. Then,*

$$\|u - u_h\|_0 \leq 2(1+d) \|u - \mathcal{P}_h u\|_0, \quad (5.2)$$

where d is the separation constant, see [8] for more details.

Let (w_d, p_d) be the solution of the following dual problem: Find $(w_d, p_d) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$ such that

$$a(w_d, v) + b(v, p_d) + b(w_d, q) = (u - \mathcal{P}_h u, v) \quad \text{for any } (v, q) \in H_0^1(\Omega)^n \times L_0^2(\Omega). \quad (5.3)$$

Assume the solution (w_d, p_d) has the following regularity:

$$\|w_d\|_{1+\sigma} + \|p_d\|_\sigma \lesssim \|u - \mathcal{P}_h u\|_0, \quad (5.4)$$

where $0 < \sigma \leq 1$. Then a standard dual argument of nonconforming finite elements, see for instance, [3, 4], shows that

$$\|u - \mathcal{P}_h u\|_0 \lesssim h^\sigma \|\nabla_h(u - \mathcal{P}_h u)\|_0. \quad (5.5)$$

For both the enriched rotated Q_1 and Crouzeix-Raviart elements, we can define the interpolation operator $\Pi_h: H_0^1(\Omega)^n \rightarrow V_h^{nc}$ by

$$\int_E \Pi_h v dE = \int_E v dE \quad \text{for any } v \in H_0^1(\Omega)^n, \quad E \in \mathcal{H}_h, \quad (5.6a)$$

$$\int_K \Pi_h v dx = \int_K v dx \quad \text{for any } K \in \mathcal{T}_h. \quad (5.6b)$$

For this interpolation operator, since $u - \Pi_h u$ has vanishing mean on K , it follows from the Poincare inequality that

$$\|u - \Pi_h u\|_{0,K} \leq Ch_K \|\nabla(u - \Pi_h u)\|_{0,K}. \tag{5.7}$$

Before state the second property of the interpolation operator, we introduce the following space, for any $K \in \mathcal{T}_h$,

$$C_K^{EQ} = \begin{pmatrix} a_{11} + a_{12}x_1 \\ a_{21} + a_{22}x_2 \\ \dots \\ a_{n1} + a_{n2}x_n \end{pmatrix},$$

for $a_{11}, a_{12}, \dots, a_{n1}, a_{n2} \in \mathbb{R}$. Let P_K^{EQ} be the L^2 projection operator from $L^2(K)$ onto C_K^{EQ} . Define

$$C_h^{EQ} := \{\psi \in L^2(\Omega)^n, \psi|_K \in C_K^{EQ} \text{ for any } K \in \mathcal{T}_h\},$$

and

$$P_h^{EQ}|_K = P_K^{EQ} \quad \text{for any } K \in \mathcal{T}_h.$$

For the enriched rotated Q_1 element, it holds the following commuting property:

$$\nabla_h(\Pi_h u)_i = P_h^{EQ} \nabla u_i, \quad i = 1, \dots, n, \tag{5.8}$$

where $(\Pi_h u)_i$ and u_i are the i -th components of $\Pi_h u$ and u , respectively. Such a commuting property which is proved in [14] for two dimension, in [9] for any dimension for the scale case, which can be adopt to the vector case.

For the enriched Crouzeix–Raviart element, we introduce the following space, for any $K \in \mathcal{T}_h$,

$$C_K^{ECR} = \begin{pmatrix} a_{11} + a_0x_1 \\ a_{21} + a_0x_2 \\ \dots \\ a_{n1} + a_0x_n \end{pmatrix},$$

for $a_{11}, a_{21}, \dots, a_{n1}, a_0 \in \mathbb{R}$. Let P_K^{ECR} be the L^2 projection operator from $L^2(K)$ onto C_K^{ECR} . Define

$$C_h^{ECR} := \{\psi \in L^2(\Omega)^n, \psi|_K \in C_K^{ECR} \text{ for any } K \in \mathcal{T}_h\},$$

and

$$P_h^{ECR}|_K = P_K^{ECR} \quad \text{for any } K \in \mathcal{T}_h.$$

For the enriched Crouzeix–Raviart element, a similar argument of (5.8) is able to show the following commuting property:

$$\nabla_h(\Pi_h u)_i = P_h^{ECR} \nabla u_i, \quad i = 1, \dots, n. \tag{5.9}$$

Theorem 5.1. *Let (λ, u, p) and (λ_h, u_h, p_h) be eigenpairs of (2.1) and (3.5), respectively. Then,*

$$\|u - u_h\|_0 \lesssim (1+d)h^\sigma \left((1+\lambda h^2) \|\nabla_h(u - u_h)\|_0 + \inf_{q_h \in Q_h} \|p - q_h\|_0 \right). \tag{5.10}$$

Proof. Let $f = \lambda u$ in (4.1) and (4.2), which implies that

$$(u_f, p_f) = (u, p) \quad \text{and} \quad (u_{f,h}, p_{f,h}) = (\mathcal{P}_h u, \mathcal{P}_h p).$$

Further we let $w|_K = \Pi_h \lambda u|_K$ for any $K \in \mathcal{T}_h$ in Theorem 4.1. Then it follows from Theorem 4.1, (5.7), (5.8), and (5.9) that

$$\|\nabla_h(u - \mathcal{P}_h u)\|_0 + \|p - \mathcal{P}_h p\|_0 \lesssim (1 + \lambda h^2) \|\nabla_h(u - u_h)\|_0 + \inf_{q_h \in Q_h} \|p - q_h\|_0. \quad (5.11)$$

The desired result follows from Lemma 5.1, and (5.5). □

We need the following error identity from [8, 20].

Lemma 5.2. *Let (λ, u, p) and (λ_h, u_h, p_h) be eigenpairs of (2.1) and (3.5), respectively. It holds that*

$$\begin{aligned} & \lambda - \lambda_h \\ &= \|\nabla_h(u - u_h)\|_0^2 - \lambda_h \|u - u_h\|_0^2 - 2\lambda_h(u - \Pi_h u, u_h) \\ & \quad + 2a_h(u - \Pi_h u, u_h) - 2b_h(\Pi_h u, p_h). \end{aligned} \quad (5.12)$$

Theorem 5.2. *Let (λ, u, p) and (λ_h, u_h, p_h) be eigenpairs of (2.1) and (3.5), respectively. Suppose that*

$$1 - \epsilon^2(h) > 0 \quad \text{and} \quad \|\nabla_h(u - u_h)\|_0 \geq \delta(h) / (1 - \epsilon^2(h)).$$

Then

$$\lambda \geq \lambda_h, \quad (5.13)$$

where

$$\epsilon^2(h) := C_1 \lambda_h (1+d)^2 (1 + \lambda h^2)^2 h^{2\sigma},$$

and

$$\delta^2(h) = C_2 \sum_{K \in \mathcal{T}_h} (\lambda_h^2 h_K^4 \|\nabla u\|_{0,K}^2 + \lambda_h (1+d)^2 h^{2\sigma} h_K^{2\sigma} |p|_{\sigma,K}^2),$$

provided that $p \in H^\sigma(\Omega)$.

Proof. We need to analyze the terms on the right-hand side of (5.12). It follows from (5.8) and (5.9) that both the fourth and fifth terms vanish. Let Π_0 be the piecewise constant L^2 projection operator with respect to \mathcal{T}_h . The definition of Π_h , the elementwise Poincare inequality, and (5.7) yield

$$\begin{aligned} & 2\lambda_h(u - \Pi_h u, u_h) = 2\lambda_h(u - \Pi_h u, u_h - \Pi_0 u_h) \\ & \lesssim \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla(u - \Pi_h u)\|_{0,K} (\|\nabla(u_h - u)\|_{0,K} + \|\nabla u\|_{0,K}). \end{aligned} \quad (5.14)$$

The final result follows from (5.10). □

6 Numerical experiments

In this section, we give the first six eigenvalues of Stokes eigenvalue problem on unit square by enriched rotated Q_1 element and enriched Crouzeix–Raviart element.

Table 1: The eigenvalues and the orders of convergence on a unit square domain, by enriched rotated Q_1 element.

level	λ_1	h^n	λ_2	h^n	λ_3	h^n
2	0.4167E+02	0.00	0.4167E+02	0.00	0.4167E+02	0.00
3	0.4714E+02	1.04	0.7811E+02	1.85	0.7811E+02	1.85
4	0.5063E+02	1.60	0.8783E+02	1.71	0.8783E+02	1.71
5	0.5188E+02	1.88	0.9094E+02	1.86	0.9094E+02	1.86
6	0.5223E+02	1.97	0.9182E+02	1.96	0.9182E+02	1.96
7	0.5231E+02	2.00	0.9205E+02	1.99	0.9205E+02	1.99
8	0.5234E+02	1.96	0.9211E+02	1.98	0.9211E+02	1.98
exact	0.5234E+02		0.9212E+02		0.9212E+02	
level	λ_4	h^n	λ_5	h^n	λ_6	h^n
2	0.4167E+02	0.00	0.4167E+02	0.00	0.6857E+02	0.00
3	0.9537E+02	1.40	0.1195E+03	1.70	0.1310E+03	1.45
4	0.1166E+03	1.51	0.1460E+03	2.10	0.1623E+03	2.94
5	0.1249E+03	1.84	0.1519E+03	1.84	0.1655E+03	1.61
6	0.1273E+03	2.10	0.1535E+03	1.95	0.1666E+03	1.89
7	0.1280E+03	2.84	0.1540E+03	1.99	0.1669E+03	1.97
8	0.1282E+03	0.84	0.1541E+03	1.99	0.1670E+03	1.99
exact	0.1281E+03		0.1541E+03		0.1670E+03	

It is clear from Table 1 that all eigenvalues converge from below. The computer error for λ_4 on the eighth-level grid is too big to violate the theory of lower bound.

Next, we use the enriched Crouzeix–Raviart element to compute the Stokes' eigenvalues, on uniform triangular grids where each square is cut by its upper-left to lower-right diagonal line. It is clear from Table 2 that all discrete eigenvalues converge from below.

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Table 2: The eigenvalues and the orders of convergence on a unit square domain, by enriched Crouzeix–Raviart element.

level	λ_1	h^n	λ_2	h^n	λ_3	h^n
2	0.2648E+02	0.00	0.2702E+02	0.00	0.3199E+02	0.00
3	0.4308E+02	1.48	0.5497E+02	0.81	0.6045E+02	0.92
4	0.4956E+02	1.73	0.8093E+02	1.73	0.8203E+02	1.65
5	0.5159E+02	1.88	0.8917E+02	1.92	0.8940E+02	1.89
6	0.5215E+02	1.96	0.9137E+02	1.97	0.9143E+02	1.97
7	0.5230E+02	1.99	0.9194E+02	1.99	0.9195E+02	1.99
8	0.5233E+02	1.98	0.9208E+02	1.99	0.9208E+02	1.99
exact	0.5234E+02		0.9212E+02		0.9212E+02	
level	λ_4	h^n	λ_5	h^n	λ_6	h^n
2	0.3200E+02	0.00	0.3849E+02	0.00	0.6857E+02	0.00
3	0.7461E+02	0.85	0.7666E+02	0.58	0.8252E+02	0.22
4	0.1110E+03	1.64	0.1235E+03	1.34	0.1410E+03	1.70
5	0.1234E+03	1.86	0.1458E+03	1.88	0.1599E+03	1.88
6	0.1270E+03	2.05	0.1520E+03	1.97	0.1652E+03	1.96
7	0.1279E+03	2.49	0.1536E+03	1.99	0.1666E+03	1.99
8	0.1281E+03	2.57	0.1540E+03	2.00	0.1669E+03	2.00
exact	0.1281E+03		0.1541E+03		0.1670E+03	

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