Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 1, pp. 241-254

DOI: 10.4208/aamm.OA-2018-0061
February 2019

Lower Bounds of Eigenvalues of the Stokes Operator by Nonconforming Finite Elements on Local Quasi-Uniform Grids

Youai Li*

School of Science, Beijing Technology and Business University, Beijing 100048, China

Received 27 February 2018; Accepted (in revised version) 18 October 2018

Abstract. This paper is a generalization of some recent results concerned with the lower bound property of eigenvalues produced by both the enriched rotated Q_1 and Crouzeix–Raviart elements of the Stokes eigenvalue problem. The main ingredient are a novel and sharp L^2 error estimate of discrete eigenfunctions, and a new error analysis of nonconforming finite element methods.

AMS subject classifications: 65N30, 65N15, 35J25

Key words: Lower bound, eigenvalue, nonconforming finite element method, Stokes operator.

1 Introduction

It was observed from numerical examples that some nonstandard finite element methods including nonconforming finite element methods and mass lumping finite element methods are able to yield lower bounds of eigenvalues of partial differential eigenvalue problems, see, Zienkiewicz et al. [32], for the Morley element, Rannacher [25], for the Morley and Adini elements, Liu and Yan [22], for the Wilson, enriched rotated Q_1, and rotated Q_1 elements. However, it was only very recent that these phenomena were rigorously analyzed. Indeed, Armentano and Duran [1] proposed an identity of errors of eigenvalues and proved that the Crouzeix–Raviart element produces lower bounds of eigenvalues for the Laplace operator provided that eigenfunctions $u\in H^{1+r}(\Omega)\cap H_0^1(\Omega)$ with $0<r<1$. The idea was generalized to the enriched rotated Q_1 element by the author of this paper in [14], and to the Wilson element in Zhang, Yang and Chen [31]. The extension to the Morley element was carried out in [29]. However, all of those papers are based on the saturation condition of finite element solutions by piecewise polynomials. The saturation condition can be from consistency errors and approximation errors as

*Corresponding author.
Email: lya@lsec.cc.ac.cn (Y. A. Li)

well as regularity of exact solutions, and was proved based on *a direct argument* in Hu, Huang and Lin [9]. If only the saturation condition of approximation errors is concerned, it was independently showed based on *a contradiction argument* in Lin, Xie and Xu [21]; see its application to nonconforming finite elements of the Laplace eigenvalue problem in [23]. In Hu, Huang and Lin [9], a more general result of the lower bound property of eigenvalues by nonconforming finite elements was established that if local approximation properties of nonconforming finite element spaces are better than total errors (sums of global approximation errors and consistency errors) of nonconforming finite element methods, corresponding methods will produce lower bounds for eigenvalues. For expansion methods based on superconvergence or extrapolation we refer interested readers to [17, 18, 30, 31], where the lower bound property of eigenvalues by nonconforming elements was analyzed on uniform rectangular meshes. We also refer interested readers to [10] for mass lumping finite element methods of eigenvalue problems.

The lower bound property of the eigenvalue by the nonconforming methods of the Stokes eigenvalue problem was first analyzed in [20], where a numerical result indicated that conforming finite elements of the Stokes eigenvalue problem are also possible to yield lower bounds of eigenvalues. In a recent paper by Hu and Huang [8], a more general framework is established for both conforming and nonconforming finite element methods for the Stokes operator. In particular, it was proved that the conforming P_2-P_0 element yields lower bounds of eigenvalues for the Stokes operator. However, all of these papers can only provide (asymptotic) lower bounds for eigenvalues on quasi-uniform grids.

In this paper we give a refined analysis for the lower bound property of eigenvalues by both the enriched rotated Q_1 [19] and Crouzeix–Raviart elements [9] of the Stokes eigenvalue problem. The main idea is to combine a series of new techniques: the elementwise Poincare–like inequality of the canonical interpolation operators of these two elements, a novel L^2 error estimate of discrete eigenfunctions from [8], a new error analysis of nonconforming finite element methods, plus the commuting property of the canonical interpolation operators.

In this paper, we use the standard gradient operator:

$$\nabla r := (\partial r/\partial x_1, \cdots, \partial r/\partial x_n)^T.$$

Given any n dimensional vector function $\psi = (\psi_1, \cdots, \psi_n)$, its divergence reads

$$\operatorname{div}\psi := \partial\psi_1/\partial x_1 + \partial\psi_2/\partial x_2 + \cdots + \partial\psi_n/\partial x_n.$$

The spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$ are defined as usual,

$$\begin{aligned} &H_0^1(\Omega) := \{v \in H^1(\Omega),\ v = 0 \ \text{ on } \ \partial\Omega\}, \\ &L_0^2(\Omega) := \Big\{q \in L^2(\Omega),\ \int_\Omega q\,dx = 0\Big\}. \end{aligned}$$

This paper is organized as follows. In the next section, we present the Stokes eigenvalue problem. In Section 3, we get its nonconforming finite element methods. We present a