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A Splitting Method for the Degasperis–Procesi Equation Using an Optimized WENO Scheme and the Fourier Pseudospectral Method

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Abstract. The Degasperis–Procesi (DP) equation is split into a system of a hyperbolic equation and an elliptic equation. For the hyperbolic equation, we use an optimized finite difference weighted essentially non-oscillatory (OWENO) scheme. New smoothness measurement is presented to approximate the typical shockpeakon structure in the solution to the DP equation, which evidently reduces the dissipation arising from discontinuities simultaneously removing nonphysical oscillations. For the elliptic equation, the Fourier pseudospectral method (FPM) is employed to discretize the high order derivative. Due to the combination of the WENO reconstruction and FPM, the splitting method shows an excellent performance in capturing the formation and propagation of shockpeakon solutions. The numerical simulations for different solutions of the DP equation are conducted to illustrate the high accuracy and capability of the method.

AMS subject classifications: 65M10, 78A48

Key words: Degasperis–Procesi equation, discontinuous solution, weighted essentially non-oscillatory method, pseudospectral method.

1 Introduction

In this paper, we consider the Degasperis–Procesi equation

$$u_t+3\kappa^3u_x-u_{xxt}+4f(u)_x=f(u)_{xxx}, \tag{1.1}$$

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where $u(x,t)$ is a real function and $f(u)=u^2/2$. This equation is an approximate model of shallow water wave propagation in small amplitude and long wavelength regime. It was first found by Degasperis and Procesi when they were studying the asymptotic integrability to the third-order dispersive equation [1]

$$u_t-\alpha^2u_{xxt}+\gamma u_{xxx}+c_0u_x=(c_1u^2+c_2u_x^2+c_3uu_{xx})_x, \tag{1.2}$$

with six real constants c_0, c_1, c_2, c_3, γ, $\alpha\in\mathbb{R}$, for which only three of them satisfy the integrability condition, namely, the Korteweg–de Vries (KdV) equation $(\alpha=c_2=c_3=0)$, the Camassa–Holm (CH) equation ($c_1=-\frac{3c_3}{2\alpha^2}$, $c_2=\frac{c_3}{2}$), and the DP equation (1.1). The KdV equation, as the simplest model, has been studied in detail [2,3]. While the DP equation is more complicated, because of the existence of the mixed derivative term u_{xxt} and the nonlinear dispersion terms uu_{xxx} and u_xu_{xx}.

Degasperis proved the integrability of the DP equation by constructing a Lax pair and a bi-Hamiltonian structure [4]. It was related to the AKNS shallow water wave equation by a hodograph transformation [5]. Based on above results, Matsuno obtained the multisoliton solutions of the DP equation for the case $\kappa\neq 0$ [6]. Furthermore, Lundmark and Szmigielski found the explicit form of multipeakon solutions for $\kappa=0$ by solving an inverse scattering problem of a discrete cubic string [7–9]. Additionally, the peakon solutions for these two equations are orbitally stable [10].

One of the important features of the DP equation (κ=0) is that it has not only a peaked solution $u(x,t)=ce^{-|x-ct|}$ [4], but also a shock wave solution of the form [11,12]

$$u(x,t)=ce^{-|x-ct|}+\frac{s}{ts+1}\mathrm{sign}(x-ct)e^{-|x-ct|}, \tag{1.3}$$

where c, s $(s>0)$ are constants. Moreover, the DP equation possesses a periodic shock wave solution [13] given by

$$u(x,t)=\begin{cases}\left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}t+c\right)^{-1}\frac{\sinh\left(x-|x|-\frac{1}{2}\right)}{\sinh\frac{1}{2}}, & x\in\mathbb{R}\backslash\mathbb{Z},\ c>0,\\ 0, & x\in\mathbb{Z}.\end{cases} \tag{1.4}$$

Lundmark further extended the multipeakon solution of the DP equation to the multi-shockpeakon solution [11]

$$u(x,t)=\sum_{i=1}^{n}m_i(t)e^{-|x-x_i(t)|}+\sum_{i=1}^{n}s_i(t)\mathrm{sign}(x-x_i)e^{-|x-x_i(t)|}, \tag{1.5}$$

where $m_i(t)$, $x_i(t)$ and $s_i(t)$ represent the momentum, position and strength of the i-th shockpeakon, respectively. It was proved that (1.5) is a weak solution of the DP equation if and only if $m_i(t)$, $x_i(t)$ and $s_i(t)$ satisfy an ODE system. Inspired by the existence of these discontinuous solutions, Coclite et al. developd a well-posedness theory which depends on some functional spaces containing discontinuous functions [11]. They proved