Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 2, pp. 338-359

DOI: 10.4208/aamm.OA-2018-0143
April 2019

Fractional Integro-Differential Equations Involving ψ-Hilfer Fractional Derivative

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Received 13 June 2018; Accepted (in revised version) 27 August 2018

Abstract. Considering a fractional integro-differential equation involving a general form of Hilfer fractional derivative with respect to another function. We show that weighted Cauchy-type problem is equivalent to a Volterra integral equation, we also prove the existence, uniqueness of solutions and Ulam-Hyers stability of this problem by employing a variety of tools of fractional calculus including Banach fixed point theorem. An example is provided to illustrate our main results.

AMS subject classifications: 34K37, 26A33, 34A12, 47H10

Key words: Fractional integro-differential equations, ψ-Hilfer fractional derivative and ψ-fractional integral, existence, uniqueness and Ulam-Hyers stability, Fixed point theorem.

1 Introduction

Fractional calculus has grasped the attention of many researchers through the recent decades as it is a solid and growing work each in the theoretical and applied concept, more info see Kilbas et al. [22] and Samko et al. [30]. The importance of fractional calculus development is due to applied mathematics such as physics, mechanics chemistry, biology, engineering, see the papers [6,16,18,24,25,29,40].

Fractional calculus can be considered a generalization of classical calculus, there are numerous of various definitions of integrals and derivatives of arbitrary order. So in the literature several studies dealing with similar topics for different operators, for instance, the Riemann-Liouville [13, 41], the Caputo [2, 28, 37], the Hilfer [14, 15], the Erdélyi–Kober [7, 38], the Hadamard [23, 39] and the $\psi-$fractional derivatives and integrals [3, 4, 31, 32], etc. And over time, other types of new fractional derivatives and integrals arise

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have a various kernel and this makes the number of definitions wide see [3,12,17,19–22, 26,27,30,32,33,36], and the references therein.

Theory of fractional differential equations has grown as a new field of applied mathematics and many applications, due to the evolution of fractional calculus. Further, the existence, uniqueness, and stability of solutions are among the most important qualitative properties of fractional differential equations. The existence and uniqueness of solutions of differential equations involving the fractional derivatives with respect to other function were tackled by few researchers (see [4,11,14,26,31,34,35]). For example in [4], Almeida investigated the existence and uniqueness of solutions to fractional differential equations with mixed boundary conditions involving the $\psi-$Caputo derivative with respect to another function. Furati et al. in [14], considered nonlinear fractional differential equation involving Hilfer fractional derivative

$$D_{a^+}^{\alpha,\beta}u(t)=f(t,u(t)), \qquad t>a, \quad 0<\alpha<1, \quad 0\leq\beta\leq 1, \tag{1.1a}$$

$$I_{a^+}^{1-\gamma}u(a^+)=u_a, \qquad \gamma=\alpha+\beta-\alpha\beta, \tag{1.1b}$$

where $D_{a^+}^{\alpha,\beta}(\cdot)$, $I_{a^+}^{1-\gamma}(\cdot)$ are Hilfer fractional derivative and Riemann-Liouville fractional integral, respectively, $u_a\in\mathbb{R}$. The author used the Banach fixed point theorem to investigate the existence and uniqueness and stability of global solutions in the weighted space on the problem Eq. (1.1a) and Eq. (1.1b).

Dheigude and Bhairat in [11], disccused the existence, uniqueness and continuous dependence of solution for problem Eq. (1.1a) and Eq. (1.1b) by using successive approximations and generalized Gronwall inequality.

Oliveira and de Oliveira in [26], proposed a new fractional derivative the Hilfer–Katugampola fractional derivatives ${}^\rho D_{a^+}^{\alpha,\beta}(\cdot)$ and generalized fractional integral ${}^\rho I_{a^+}^{1-\gamma}(\cdot)$. The authors used Banach fixed point theorem to obtain the existence and uniqueness of solution for a weighted Cauchy-type problem Eq. (1.1a) and Eq. (1.1b).

In [31], Sousa and de Oliveira proposed a generalized Gronwall inequality through the fractional integral with respect to another function $I_{a^+}^{1-\gamma;\psi}(\cdot)$. They considered Cauchy-type problem Eq. (1.1a) and Eq. (1.1b) involving the ψ-Hilfer fractional derivative $D_{a^+}^{\alpha,\beta;\psi}(\cdot)$ to obtain the existence uniqueness and continuous dependence of solutions.

The integro-differential equations emerge in many engineering and scientific specializations, they often are as an approximation to partial differential equations, which represent much of the continuum phenomena. For details, see [1,5,8–10].

This paper is devoted to study a fractional integro-differential equation with left generalized Hilfer fractional derivatives with respect to another function in a weighted space of continuous functions and by relying Banach fixed point theorem, we prove the existence, uniqueness, and Ulam-Hyers stability of the following nonlinear fractional integro-differential equation:

$$D_{a^+}^{\alpha,\beta;\psi}u(t)=f\left(t,u(t),\int_0^t h(t,s,u(s))ds\right), \qquad 0<\alpha<1, \quad 0\leq\beta\leq 1, \quad t>a, \tag{1.2a}$$