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Alternating Direction Implicit Finite Element Method for Multi-Dimensional Black-Scholes Models

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Abstract. A new numerical method is proposed and investigated for solving two-dimensional Black-Scholes option pricing model. This model is represented by Dirichlet initial-boundary value problem in a rectangular domain for a parabolic equation with advection-diffusion operator containing mixed derivatives. It is approximated by using a finite element method in spatial variables and alternating direction implicit (ADI) method in time variable. The ADI scheme is based on the semi-implicit approximation. The stability and convergence of the constructed scheme is proved rigorously. The provided computational results demonstrate the efficiency and high accuracy of the proposed method.

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Key words: Black-Scholes models, finite element method, semi-implicit approximation, alternating direction method.

1 Introduction

A traded option in a financial market is a contract which gives its owner the right to buy (*call option*) or to sell (*put option*) a fixed quantity of assets of a specified stock at a fixed price (exercise or strike price) on (*European option*) or before (*American option*) a given date (*expiry date*). The market prices of the rights to buy and to sell are called, respectively *call prices* and *put prices*. How to valuate an option has long been a hot topic for financial engineers, economists and mathematicians. In the case of a European option, it

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was shown by Black and Scholes [1] that the price satisfies a second-order parabolic partial differential equation with respect to the time horizon and the underlying asset price, known as the Black and Scholes equation. Although an analytical solution to a simplified model is given [2], the equation is, in general, not analytically solvable. Hence, the people in the financial industry have used various numerical methods to solve option pricing problems. Under the Black-Scholes partial differential equation framework, the numerical solution of this equation has been discussed by various authors (cf. for example, [3–21] and the references therein). The fast and accurate numerical computation for the option pricing problem is a crucial matter in the financial industry. Most of the cited publications deal with one dimensional in space problems and some of them are devoted to the numerical solution of high-dimensional Black-Scholes equation (cf. [3, 18–21]).

Let $S_i(t)$ for $i=1,2,\cdots,d$ denote the value of i-th asset at time t, $\mathbf{s}=(S_1,S_2,\cdots,S_d)$ and $u(\mathbf{s},t)$ denote the price of an option. In the Black-Scholes model each underlying asset $S_i(t)$ satisfies the following stochastic differential equation:

$$dS_i(t)=\mu_i S_i(t)dt+\sigma_i S_i(t)dW_i(t), \quad i=1,2,\cdots,d.$$

The above μ_i, σ_i and $W_i(t)$ are the expected instantaneous rate of return, constant volatility and standard Brownian motion on the underlying asset S_i, respectively. The term dW_i contains the randomness which is certainly a feature of asset prices and is assumed to be a Wiener process. The Wiener processes are correlated by $\langle dW_i,dW_j\rangle=\rho_{ij}dt$. Then, the generalized Black-Scholes partial differential equation can be derived by using Ito's lemma and the no-arbitrage principle:

$$\begin{cases} \dfrac{\partial u(\mathbf{s},t)}{\partial t}+\displaystyle\sum_{i=1}^{d} rS_i\dfrac{\partial u(\mathbf{s},t)}{\partial S_i}+\dfrac{1}{2}\sum_{i,j=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j\dfrac{\partial^2 u(\mathbf{s},t)}{\partial S_i\partial S_j}-ru(\mathbf{s},t)=0, \\ 0<t\le T, \quad 0<S_i<+\infty, \quad i=1,2,\cdots,n, \\ u(\mathbf{s},T)=\wedge(\mathbf{s}), \end{cases}$$

where $r>0$ is a constant riskless interest rate and $\wedge(\mathbf{s})$ is the payoff function.

In this paper, we consider the two-dimensional Black-Scholes equation for the sake of simplicity. We change the variables $\tau=T-t$ and $S_1=e^x$, $S_2=e^y$, and derive the following Cauchy problem in $\mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial u}{\partial \tau}-\dfrac{1}{2}\sigma_1^2\dfrac{\partial^2 u}{\partial x^2}-\dfrac{1}{2}\sigma_2^2\dfrac{\partial^2 u}{\partial y^2}-\rho\sigma_1\sigma_2\dfrac{\partial^2 u}{\partial x\partial y}-\Big(r-\dfrac{1}{2}\sigma_1^2\Big)\dfrac{\partial u}{\partial x} \\ \quad -\Big(r-\dfrac{1}{2}\sigma_2^2\Big)\dfrac{\partial u}{\partial y}+ru=0, \quad 0<\tau\le T, \quad -\infty<x,y<+\infty, \end{cases}$$

where $\sigma_1>0$ and $\sigma_2>0$ are the volatilities of the two assets, constant ρ with $|\rho|<1$ is the correlation of the two assets and $r>0$ is the risk-free interest rate.

To solve this problem numerically by finite difference or finite element methods, an artificial boundary is usually introduced to make the computational domain bounded.