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Fundamentals of Lax-Wendroff Type Approach to Hyperbolic Problems with Discontinuities

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Abstract. This paper presents the understanding of the fundamentals when designing a numerical schemes for hyperbolic problems with discontinuities as parts of their solutions. The fundamentals include the consistency with hyperbolic balance laws in integral form rather than PDE form, spatial-temporal coupling, thermodynamic consistency for computing compressible fluid flows, convergence arguments and multidimensionality etc. Some numerical results are shown to display the performance.

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Key words: Hyperbolic problems, ccmpressible fluid flows, shocks, material interfaces, Lax-Wendroff type methods, generalized Riemann problem (GRP) method.

1 Introduction

This paper presents the recent progress we made when hyperbolic problems, if their solutions contain discontinuities such as shocks and material interfaces, are computed. Due to the presence of discontinuities, the governing equations of the hyperbolic problems have to be understood in integral form (weak sense, distributional sense etc.), rather than in purely differential form. Prototype examples are problems around compressible fluid flows, in which shocks are ubiquitous. Most of traditional numerical methods for such a family of problems are based on the differential form with various techniques near discontinuities. We can refer to [1,28] and references therein for going over the development.

One of most fundamental methods representing the solution of hyperbolic problems can trace back to Cauchy-Kowalevski in 1700's [4], and the approximate solutions are represented in terms of power series using the prescribed data on non-characteristic surfaces.

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The numerical realization of this approach is made by Lax and Wendroff in 1960's [15], mainly for one-dimensional hyperbolic conservation laws. The resulting scheme is the celebrated Lax-Wendroff scheme and has irreplaceable values at least in the following sense.

(i) It is a unique derivation with second order accuracy both in space and time, leading to a three-point second order (either upwind or central) scheme. Any high order scheme should be consistent with the Lax-Wendroff method when it reduces to the second order version. Therefore the Lax-Wendroff method is the reference of all high order accurate methods.

(ii) It uses the least stencils (just three points for each solution value at each time step) and is the most compact. Useless information is adopted as least as possible.

(iii) The Lax-Wendroff approach is a temporal-spatial coupled method and the fully useful information of the governing equations are incorporated into the scheme. Thus, there is no need to exert extra effort if any other physical or geometrical effects are included.

Nevertheless, the Lax-Wendroff approach just works for smooth flows, and it should be modified to adapt for capturing discontinuities. The currently-used generalized Riemann problem (GRP) method is regarded as the discontinuous and nonlinear version of L-W method, and it uses both the Cauchy-Kowalevski methodology and tracks the singularity [2,3,5,7]. Moreover, the resulting scheme is consistent directly with the corresponding balance law, i.e., the weak form of the underlying governing equations, rather the partial differential equations. Hence the GRP approach avoids the large disparity from the "true" solution if strong discontinuities are present. Hence we will interpret detailed fundamentals behind the GRP approach.

This paper is organized as follows. Besides the introduction section, we revisit the Lax-Wendroff method and the generalized Riemann problem method in Section 2. The spatial-temporal consistency with hyperbolic balance law, thermodynamical consistency, and transversal effect are discussed in Sections 3-5, respectively. The multi-stage method based on the Lax-Wendroff type flow solvers is reviewed in Section 6. Finally we show some applications in Section 7.

2 Lax-Wendroff method and the generalized Riemann problem method

Consider hyperbolic conservation laws,

$$u_t+f(u)_x=0, \tag{2.1}$$

where $f(u)$ is the flux function. We denote by Δx the spatial increment, by Δt the time increment, $I_j=(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}})$ the computational cell interval with $x_{j+\frac{1}{2}}=(j+\frac{1}{2})\Delta x$, $x_j=j\Delta x$,