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A Viscosity-Splitting Method for the Navier-Stokes/ Darcy Problem

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Abstract. In this report, we give a viscosity splitting method for the Navier-Stokes/Darcy problem. In this method, the Navier-Stokes/Darcy equation is solved in three steps. In the first step, an explicit/ implicit formulation is used to solve the nonlinear problem. We introduce an artificial diffusion term $\theta\Delta\mathbf{u}$ in our scheme whose purpose is to enlarge the time stepping and enhance numerical stability, especially for small viscosity parameter ν, by choosing suitable parameter θ. In the second step, we solve the Stokes equation for velocity and pressure. In the third step, we solve the Darcy equation for the piezometric head in the porous media domain. We use the numerical solutions at last time level to give the interface condition to decouple the Navier-Stokes equation and the Darcy's equation. The stability analysis, under some condition $\Delta t \leq k_0$, $k_0 > 0$, is given. The error estimates prove our method has an optimal convergence rates. Finally, some numerical results are presented to show the performance of our algorithm.

AMS subject classifications: 76D05, 35Q30, 65M60, 65N30

Key words: Navier-Stokes/Darcy equations, fractional step method, viscosity-splitting method, stability analysis, optimal error analysis.

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1 Introduction

The free flow coupling with the porous media flow, where the behavior of the fluid can be described by different partial differential equations in different domains, is very important in the computational fluids. In this report, we focus on the model of the Navier-Stokes equation in the surface region coupling the Darcy's law in the subsurface region [16], which is described by a mixed Navier-Stokes/Darcy's model. The first important thing is the interface conditions between the Navier-Stokes flow and the Darcy's flow. This model was firstly studied by Beavers and Joseph in [4], they gave the interface condition (named Beavers-Joseph (BJ) condition) on the interface between the surface flow and the subsurface flow. Then, this condition was simplified by Saffman [37] getting the Beavers-Joseph-Saffman (BJS) condition. This model is used in many areas, e.g., the simulation of flooding in dry areas, petroleum engineering and environmental engineering. In point view of the numerical method, how to deal with the BJS or BJ condition on the interface is a enormous challenge.

For its widely used and enormous challenge, many authors have given some great works, e.g., unified stabilized finite element formulations for the Stokes/Darcy's flow [2]; a strongly conservative finite element method for the Stokes/Darcy's flow [30]; superconvergence analysis of finite element method for the Stokes/Darcy system [1]; a posteriori error estimate for the Stokes/Darcy [15] and so on. In 2012, Layton et al. [31] split the Stokes/Darcy problems into the Stokes and Darcy problems and give four non-iterative, sub-physics, uncoupling methods. The Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition were shown by Cao et al. [9], Chen et al. [12] and Discacciati et al. [17]. In [23], fully-mixed finite element methods was given by Gatica et al. In [36], a decoupled finite element method for the Stokes/Darcy flow was given. In 2012, Shan, Zheng and Layton [38] presented a decoupled method using different time steps in different domains. In [41], a local discontinuous Galerkin (LDG) method for the Stokes/Darcy flow was given by Vassilev and Yotov. For the Navier-Stokes/Darcy coupling problem, several iterative methods were presented by Badea, Discacciatiaw and Quarteroni [3, 16]. The two-level method for the Navier-Stokes/Darcy problem was given by Cai et al. [7] in 2009. Girault and Riviére [25] presented a discontinuous Galerkin approximation of coupled Navier-Stokes/Darcy Equations by BJS Interface Condition. In [43], two decoupling algorithms for the steady Stokes-Darcy model based on two-grid discretizations were shown by Zhang and Yuan. In [39], we gave the decoupled modified characteristics finite element method for the time dependent Navier-Stokes/Darcy problem. The fully-mixed finite element method for Stokes-Darcy problems was given by Gatica et al. [8,24]. A decoupled preconditioning technique for a mixed Stokes-Darcy model was presented by Márquez et al. [32]. A strong coupling of finite element methods for the Stokes-Darcy problem was shown by Márquez et al. [33]. Two-grid finite element for mixed Stokes-Darcy equations was given by Hou et al. [29,44].

The fractional step methods, which split effects due to different operators appeared in the problem, are a widely used classical numerical method for time-dependent differential equations in computational fluids. The origin of this method is given by Chorin [11] and Temam [40]. They developed the important and well known projection method. The main drawbacks of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure should satisfy a so-called 'artificial' boundary conditions. The viscosity-splitting methods is a kind of the fraction step method, where viscosity is not fully decoupled from incompressibility. The fully discrete viscosity-splitting method was called θ-scheme given by Glowinski et al. [27] and Ferná-Cara et al. [20]. In [5,6], Blasco et al. gave a viscosity-splitting fractional step method for the Navier-Stokes equation and a new error estimate for which was give by F. Guillén-González and Redondo-Neble [26]. The fractional step method and operator-splitting scheme combined with the well-known predictor-multi-corrector algorithm for solving the Navier-Stokes problem were also considered in [13, 14, 18, 19]. The advancement of this method is decomposed into a sequence of two steps. At the first step, a linear elliptic problem was solved, while at the second step a Stokes problem was considered. In [42], a large time stepping viscosity-splitting fractional-step method was considered for the viscoelastic flow problem.

In this paper, we given a viscosity splitting method for the Navier-Stokes/Darcy problem. In the first step, the explicit/implicit formulation is used to solve the nonlinear problem. We use the implicit scheme for the linear terms and an explicit scheme for the nonlinear term. The advantage of this method is that a linear system with a constant coefficient matrix can be obtained in order to save computational cost. Furthermore, we introduce an artificial diffusion term $\theta\Delta\mathbf{u}$ in our scheme for solving the small viscosity ν. The purpose of the artificial diffusion term $\theta\Delta\mathbf{u}$ is to enlarge the time stepping and enhance numerical stability especially for the small viscosity parameter by choosing suitable parameter θ. In the second step, a Stokes equation is solved. At last, we solve Darcy problems in the porous media domain. Then, we give the numerical analysis including stability analysis and error analysis. The numerical result show that our method has an optimal convergence order. On the other hand, we can see that our method can solve the Navier-Stokes/Darcy problem with small viscosity parameter ν. The numerical analysis proves that our method is stable and has an optimal convergence rate. The numerical results conform our theoretical analysis.

2 The viscosity splitting method for the time dependent Navier-Stokes/Darcy problems and functional settings

Let $\Omega\subset\mathbb{R}^d(d=2,3)$ be a bounded domain, decomposed into two non intersecting subdomains Ω_f and Ω_p separated by an interface Γ, namely $\Omega=\Omega_f\cup\Omega_p$, $\Omega_f\cap\Omega_p=\emptyset$ and $\bar{\Omega}_f\cap\bar{\Omega}_p=\Gamma$. We suppose the boundaries $\partial\Omega_f$ and $\partial\Omega_p$ have the Lipschitz conditions. From the physical point of view, Γ is a surface separating the domain Ω_f filled by a fluid from a domain Ω_p form by a porous medium.

Let $T>0$ be a finite constant, the fluid flow is governed by the Navier-Stokes equation in Ω_f

$$\begin{cases} \mathbf{u}_t-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=f(x,t), & x\in\Omega_f\times(0,T],\\ \nabla\cdot\mathbf{u}=0, & x\in\Omega_f\times(0,T],\\ \mathbf{u}(x,0)=\mathbf{u}_0, & x\in\Omega_f,\\ \mathbf{u}=0, & x\in\partial\Omega_f\backslash\Gamma\times(0,T], \end{cases} \tag{2.1}$$

where $\mathbf{u}(x,t)$ represents the velocity of the fluid flow in Ω_f, $p(x,t)$ is the pressure, $f(x,t)$ is the external body force and ν is the kinematic viscosity.

The porous media flow is governed by the following equation in Ω_p

$$\begin{cases} S_0\phi_t+\nabla\cdot\mathbf{u}_p=g_p(x,t), & x\in\Omega_p\times(0,T],\\ \mathbf{u}_p=-\mathbf{K}\nabla\phi, & x\in\Omega_p\times(0,T],\\ \phi(x,0)=\phi_0, & x\in\Omega_p,\\ \phi=0, & x\in\partial\Omega_p\backslash\Gamma\times(0,T], \end{cases} \tag{2.2}$$

where ϕ is the piezometric head, $\mathbf{u}_p$ is the fluid velocity in the porous media Ω_p. $\mathbf{K}$ represents the hydraulic conductivity tensor, for simplicity, we assume that $\mathbf{K}=\text{diag}$, $(K_1,\cdots,K_d)$ with $K_i\in L^\infty(\Omega_p)$, $i=1,\cdots,d$ and $K_i>0$, $i=1,\cdots,d$, which means the porous media is homogeneous. We assume that $\mathbf{K}$ is uniformly bounded and positive defined in Ω_p, there exist $K_{\min},K_{\max}>0$ such that

$$K_{\min}|x|^2\le\mathbf{K}x\cdot x\le K_{\max}|x|^2 \qquad \text{a.e. } x\in\Omega_p.$$

Using Darcy's law $\mathbf{u}_p=-\mathbf{K}\nabla\phi$, (2.2) can be rewritten in the parabolic form

$$S_0\phi_t-\nabla\cdot(\mathbf{K}\nabla\phi)=g_p(x,t), \qquad x\in\Omega_p\times(0,T]. \tag{2.3}$$

For the Navier-Stokes/Darcy model, the interface conditions of the conservation of mass, balance of forces and the Beavers-Joseph-Saffman condition are imposed herein

$$\mathbf{u}\cdot\mathbf{n}_f+\mathbf{u}_p\cdot\mathbf{n}_p=0 \qquad \text{on } \Gamma\times(0,T], \tag{2.4a}$$

$$p-\nu\mathbf{n}_f\frac{\partial\mathbf{u}}{\partial\mathbf{n}_f}=g\phi \qquad \text{on } \Gamma\times(0,T], \tag{2.4b}$$

$$-\nu\tau_i\frac{\partial\mathbf{u}}{\partial\mathbf{n}_f}=\frac{\alpha\sqrt{\nu g}}{\sqrt{\text{trace}(\mathbf{K})}}\mathbf{u}\cdot\tau_i, \qquad i=1,2,\cdots,d-1, \qquad \text{on } \Gamma\times(0,T]. \tag{2.4c}$$

Here, g is the gravitational acceleration, α is a positive parameter depending on the properties of the porous medium and must be determined experimentally. The condition (2.4) can be rewritten as

$$\mathbf{u}\cdot\mathbf{n}_f=\mathbf{K}\frac{\partial\phi}{\partial\mathbf{n}_p} \qquad \text{on } \Gamma\times(0,T]. \tag{2.5}$$

2.1 Variational formulation of the continuous problem

Define $W = H_f \times H_p$ and $Q = L^2(\Omega_f)$, where

$$H_f = \left\{ \mathbf{v} \in (H^1(\Omega_f))^d : \mathbf{v} = 0 \text{ on } \partial\Omega_f \backslash \Gamma \right\},$$
$$H_p = \left\{ \phi \in H^1(\Omega_p) : \phi = 0 \text{ on } \partial\Omega_p \backslash \Gamma \right\}.$$

The spaces H_f and H_p are equipped with the following norms

$$\|\mathbf{u}\|_{H_f} = \|\nabla \mathbf{u}\|_{\Omega_f}, \qquad \forall \mathbf{u} \in H_f,$$
$$\|\phi\|_{H_p} = \|\nabla \phi\|_{\Omega_p}, \qquad \forall \phi \in H_p,$$

where $\|\cdot\|_D := \|\cdot\|_{L^2(D)}$ means the L^2-norm on the domain D. We equip the space W with the following norms

$$\|w\|_0 = \sqrt{(\mathbf{u},\mathbf{u})_{\Omega_f} + gS_0(\phi,\phi)_{\Omega_p}}, \qquad \forall w = (\mathbf{u},\phi) \in W,$$
$$\|w\|_W = \sqrt{\nu(\nabla \mathbf{u},\nabla \mathbf{u})_{\Omega_f} + g(\mathbf{K}\nabla\phi,\nabla\phi)_{\Omega_p}}, \qquad \forall w = (\mathbf{u},\phi) \in W,$$

where $(\cdot,\cdot)_D$ refers the inner product in the corresponding domain D for $D = \Omega_f$ or Ω_p. The weak formulation of the time-dependent Navier-Stokes/Darcy model read as follows: find $w = (\mathbf{u},\phi) \in W$ and $p \in Q$ such that

$$[w_t,z] + a(w,z) + B(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (F,z), \qquad \forall z = (\mathbf{v},\psi) \in W, \tag{2.6a}$$
$$b(\mathbf{u},q) = 0, \qquad \forall q \in Q, \tag{2.6b}$$
$$w(0) = w_0, \tag{2.6c}$$

where

$$[w_t,z] = (\mathbf{u}_t,\mathbf{v})_{\Omega_f} + gS_0(\phi_t,\psi)_{\Omega_p}, \qquad a(w,z) = a_f(\mathbf{u},\mathbf{v}) + a_p(\phi,\psi) + a_\Gamma(w,z),$$
$$a_f(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u},\nabla\mathbf{v})_{\Omega_f} + \sum_{i=1}^{d-1}\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{\text{trace}(\mathbf{K})}}(\mathbf{u}\cdot\tau_i)(\mathbf{v}\cdot\tau_i)ds,$$
$$a_p(\phi,\psi) = g(\mathbf{K}\nabla\phi,\nabla\psi)_{\Omega_p}, \qquad a_\Gamma(w,z) = a_\Gamma(\mathbf{u},\phi;\mathbf{v},\psi) = c_\Gamma(\phi,\mathbf{v}) - c_\Gamma(\psi,\mathbf{u}),$$
$$B(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}), \qquad c_\Gamma(\psi,\mathbf{u}) = g\int_\Gamma \psi\mathbf{u}\cdot\mathbf{n}_f,$$
$$b(\mathbf{v},p) = -(p,\nabla\cdot\mathbf{v})_{\Omega_p}, \qquad (F,z) = (f,\mathbf{v})_{\Omega_f} + g(g_p,\psi)_{\Omega_p}.$$

Here, we define the discrete Stokes operator $\mathcal{A}$ (see [42] and the references there) defined by

$$(\mathcal{A}\mathbf{u},\mathbf{v}) = (\mathcal{A}^{1/2}\mathbf{u},\mathcal{A}^{1/2}\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v} \in H_f.$$

Lemma 2.1 ([34]). *The bilinear form $a_f(\cdot,\cdot)$, $a_p(\cdot,\cdot)$ satisfy*

$$a_f(\mathbf{u},\mathbf{v})\le \max\left\{\nu+1,\frac{C\alpha}{2\sqrt{K_{\min}}}\right\}\|\mathbf{u}\|_{H_f}\|\mathbf{v}\|_{H_f}, \tag{2.7a}$$

$$a_f(\mathbf{u},\mathbf{u})\ge \nu\|\mathbf{u}\|_{H_f}^2+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=1}^{d}\int_\Gamma(\mathbf{u}\cdot\tau_i)^2 ds=:\nu\|\mathbf{u}\|_{H_f}^2+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}\cdot\tau\|_\Gamma^2, \tag{2.7b}$$

$$a_p(\phi,\psi)\le K_{\max}\|\phi\|_{H_p}\|\psi\|_{H_p}, \tag{2.7c}$$

$$a_p(\phi,\phi)\ge K_{\min}\|\phi\|_{H_p}^2. \tag{2.7d}$$

The trilinear form $B(\cdot,\cdot,\cdot)$ satisfies the following property (see [21])

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})|\le N\|\mathbf{u}\|_{H_f}\|\mathbf{v}\|_{H_f}\|\mathbf{w}\|_{H_f}, \qquad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in H_f. \tag{2.8}$$

We can get the following lemma

Lemma 2.2. *The trilinear form $B(\cdot,\cdot,\cdot)$ satisfies the following property*

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})|\le C\|\mathbf{u}\|_{H^2(\Omega_f)}\|\nabla\mathbf{v}\|_{\Omega_f}\|\mathbf{w}\|_{\Omega_f}, \qquad \forall \mathbf{u}\in H^2(\Omega_f), \quad \mathbf{v},\mathbf{w}\in H^1(\Omega_f). \tag{2.9}$$

Proof. By Cauchy-Schwarz inequality, we have

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})|\le \|\mathbf{u}\|_{L^\infty(\Omega_f)}\|\nabla\mathbf{v}\|_{\Omega_f}\|\mathbf{w}\|_{\Omega_f}.$$

Using the embedding theorem, there holds $\|\mathbf{u}\|_{L^\infty(\Omega_f)}\le C\|\mathbf{u}\|_{H^2(\Omega_f)}$, then it follows that

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})|\le C\|\mathbf{u}\|_{H^2(\Omega_f)}\|\nabla\mathbf{v}\|_{\Omega_f}\|\mathbf{w}\|_{\Omega_f}.$$

Thus, we complete the proof. □

Remark 2.1. Here, for $\mathbf{u}\notin H_0^1(\Omega_f)\cap H^2(\Omega_f)$, so we can not get

$$\|\mathbf{u}\|_{H^2(\Omega_f)}\le C\|\mathcal{A}\mathbf{u}\|_{\Omega_f}.$$

2.2 The Viscosity-Splitting for the Navier-Stokes/Darcy problems

For each positive integer N, let $\{\mathcal{J}_n:\ 1\le n\le N\}$ be a partition of $[0,T]$ into subintervals $\mathcal{J}_n=(t_{n-1},t_n]$, with $t_n=n\Delta t$, $\Delta t=T/N$. With the previous notations, we get the viscosity splitting method for the Navier-Stokes/Darcy problem (2.1).

Algorithm 1 Viscosity-Splitting method.

Step 1 Find $\mathbf{u}^{n+1/2}\in H_f$, such that

$$\frac{\mathbf{u}^{n+1/2}-\mathbf{u}^n}{\Delta t}-\nu\Delta\mathbf{u}^{n+1/2}+\theta\mathcal{A}\mathbf{u}^{n+1/2}-\theta\mathcal{A}\mathbf{u}^n+(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n=f(t_{n+1}). \tag{2.10}$$

Step 2 Find $(\mathbf{u}^{n+1},p^{n+1})\in(H_f,Q)$, such that

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}}{\Delta t}-\nu(\Delta\mathbf{u}^{n+1}-\Delta\mathbf{u}^{n+1/2})+\theta(\mathcal{A}\mathbf{u}^{n+1}-\mathcal{A}\mathbf{u}^{n+1/2})+\nabla p^{n+1}=0, \tag{2.11a}$$

$$\nabla\cdot\mathbf{u}^{n+1}=0. \tag{2.11b}$$

Step 3 Find $\phi^{n+1}\in H_p$, via

$$S_0\frac{\phi^{n+1}-\phi^n}{\Delta t}+\nabla\cdot(\mathbf{K}\nabla\phi^n)=g_p(x,t_{n+1}). \tag{2.12}$$

Then, we can get the weak form of the method

Algorithm 2 Weak form of the Viscosity-Splitting method.

Step 1 Find $\mathbf{u}^{n+1/2}\in H_f$, such that

$$\begin{aligned}&\left(\frac{\mathbf{u}^{n+1/2}-\mathbf{u}^n}{\Delta t},\mathbf{v}\right)_{\Omega_f}+a_f(\mathbf{u}^{n+1/2},\mathbf{v})+\theta(\nabla\mathbf{u}^{n+1/2},\nabla\mathbf{v})_{\Omega_f}-\theta(\nabla\mathbf{u}^n,\nabla\mathbf{v})_{\Omega_f}\\&\quad+B(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v})+c_\Gamma(\phi^n,\mathbf{v})=(f(t_{n+1}),\mathbf{v})_{\Omega_f}.\end{aligned} \tag{2.13}$$

Step 2 Find $(\mathbf{u}^{n+1},p^{n+1})\in(H_f,Q)$, such that

$$\begin{aligned}&\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}}{\Delta t},\nabla\mathbf{v}\right)_{\Omega_f}+a_f(\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2},\mathbf{v})\\&\quad+\theta(\nabla\mathbf{u}^{n+1}-\nabla\mathbf{u}^{n+1/2},\nabla\mathbf{v})_{\Omega_f}+b(\mathbf{v},p^{n+1})=0,\end{aligned} \tag{2.14a}$$

$$b(\mathbf{u}^{n+1},q)=0. \tag{2.14b}$$

Step 3 Find $\phi^{n+1}\in H_p$, via

$$\begin{aligned}&gS_0\left(\frac{\phi^{n+1}-\phi^n}{\Delta t},\psi\right)_{\Omega_p}+g(\mathbf{K}\nabla\phi^{n+1},\nabla\psi)_{\Omega_p}\\&\quad-c_\Gamma(\psi,\mathbf{u}^n)=g(g_p(x,t_{n+1}),\psi)_{\Omega_p}.\end{aligned} \tag{2.15}$$

3 Stability analysis

Lemma 3.1 ([31,36]). *There exits a constant $C>0$, such that*

$$|c_\Gamma(\phi,\mathbf{u})|\le Cg\|\phi\|_{H_p}\|\mathbf{u}\|_{H_f}. \tag{3.1}$$

Theorem 3.1 (Stability)**.** *The Viscosity-Splitting method is stable, in the sense that, under the condition $k_0>0$, $\Delta t<k_0$, there hold*

$$\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2_{\Omega_f}+\sum_{i=0}^n\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{\Omega_f}+\frac{\Delta t\nu}{2}\sum_{i=0}^n\|\mathbf{u}^{i+1/2}\|^2_{H_f}\\
&\quad+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^n(\|\mathbf{u}^{i+1}\cdot\tau\|^2_\Gamma+\|\mathbf{u}^{i+1}\cdot\tau-\mathbf{u}^{i+1/2}\cdot\tau\|^2_\Gamma)+\theta\Delta t(\|\mathbf{u}^{n+1}\|_{H_f}\\
&\quad+\sum_{i=0}^n\|\mathbf{u}^{i+1/2}-\mathbf{u}^i\|^2_{H_f}+\sum_{i=0}^n\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{H_f})+\Delta t\nu\sum_{i=0}^n(\|\mathbf{u}^{i+1}\|^2_{H_f}+\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{H_f})\\
&\quad+gS_0(\|\phi^{n+1}\|^2_{\Omega_p}+\sum_{i=0}^n\|\phi^{i+1}-\phi^i\|^2_{\Omega_p})+2g\Delta t\sum_{i=0}^n\|\mathbf{K}^{1/2}\nabla\phi^{i+1}\|^2_{\Omega_p}\\
&\quad\le\exp\left(\frac{4TC_2^2\nu^{-1}g^2}{K_{\min}}+\frac{4C_2^2gT}{K_{\min}}\right)\left(4\Delta t\nu^{-1}\sum_{i=0}^n\|f(t_{n+1})\|^2_{\Omega_f}+\frac{4\Delta tg}{K_{\min}}\sum_{i=0}^n\|g_p(t_{i+1})\|^2_{\Omega_f}\right),
\end{aligned}\tag{3.2a}$$

$$\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2_{H_f}+2S_0\|\phi^{n+1}\|^2_{H_p}+\sum_{i=1}^n(\|\mathbf{u}^{i+1}-\mathbf{u}^i\|^2_{H_f}+2S_0\|\phi^{i+1}-\phi^i\|^2_{H_p})\\
&\quad+\frac{3}{2}\Delta t\sum_{i=1}^n(\nu\|\mathcal{A}\mathbf{u}^{i+1}\|^2_{\Omega_f}+K_{\min}\|\mathcal{A}\phi^{i+1}\|^2_{\Omega_p})+2\theta\Delta t\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}+2\theta\Delta t\sum_{i=1}^n\|\mathcal{A}(\mathbf{u}^{i+1}-\mathbf{u}^i)\|^2_{\Omega_f}\\
&\quad\le\exp\left(4g\Delta t\nu^{-1}\sum_{i=1}^n\|\mathbf{u}^i\|^2_{H^2(\Omega_f)}+4\nu^{-1}T+\frac{4T}{K_{\min}}\right)\\
&\qquad\times\left(4\Delta t\nu^{-1}\sum_{i=1}^n\|f(t_{i+1})\|^2_{\Omega_f}+\frac{4\Delta t}{K_{\min}}\sum_{i=0}^n\|g_p(t_{i+1})\|^2_{\Omega_p}\right).
\end{aligned}\tag{3.2b}$$

Furthermore, we have

$$(\nu+\theta)\Delta t\sum_{i=0}^n\|\mathcal{A}\mathbf{u}^{i+1}\|_{\Omega_f}\le C.\tag{3.3}$$

Proof. Here, we use mathematical induction. It is obviously that

$$(\nu+\theta)\|\mathcal{A}\mathbf{u}^0\|_{\Omega_f}\le C.$$

We assume that

$$(\nu+\theta)\|\mathcal{A}\mathbf{u}^i\|_{\Omega_f}\le C,\qquad\forall i\le n.$$

Step 1. we give the proof of (3.2a) under the induction assumption.

Then, letting $\mathbf{v}=2\Delta t\mathbf{u}^{n+1/2}$ in (2.13), it follows by

$$\begin{aligned}
&\|\mathbf{u}^{n+1/2}\|^2_{\Omega_f}-\|\mathbf{u}^n\|^2_{\Omega_f}+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{\Omega_f}+2\Delta t\nu\|\mathbf{u}^{n+1/2}\|^2_{H_f}+\frac{\alpha\Delta t\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}^{n+1/2}\cdot\tau\|^2_\Gamma\\
&\quad+\theta\Delta t(\|\mathbf{u}^{n+1/2}\|_{H_f}-\|\mathbf{u}^n\|^2_{H_f}+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{H_f})\\
&\quad+2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,\mathbf{u}^{n+1/2})+2\Delta tc_\Gamma(\phi^n,\mathbf{u}^{n+1/2})=2\Delta t(f(t_{n+1}),\mathbf{u}^{n+1/2})_{\Omega_f}.
\end{aligned}\tag{3.4}$$

Combining (2.8) and (2.9) gives

$$\begin{aligned}&2\Delta t B(\mathbf{u}^n,\mathbf{u}^n,\mathbf{u}^{n+1/2})\leq C\Delta t\|\mathbf{u}^n\|_{H^2(\Omega_f)}\|\mathbf{u}^{n+1/2}\|_{H_f}\|\mathbf{u}^n\|_{\Omega_f}\\ \leq&\frac{\nu\Delta t}{4}\|\mathbf{u}^{n+1/2}\|^2_{H_f}+C\Delta t\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}\|\mathbf{u}^n\|^2_{\Omega_f}.\end{aligned}\tag{3.5}$$

By (3.1), we arrive at

$$\begin{aligned}&2\Delta t c_\Gamma(\phi^n,\mathbf{u}^{n+1/2})\leq 2\Delta t C_2 g\|\phi^n\|_{H_p}\|\mathbf{u}^{n+1/2}\|_{H_f}\\ \leq&4\Delta t C_2^2\nu^{-1}g^2\|\phi^n\|^2_{H_p}+\frac{\nu\Delta t}{4}\|\mathbf{u}^{n+1/2}\|^2_{H_f}\\ \leq&\frac{4\Delta t C_2^2\nu^{-1}g^2}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\phi^n\|^2_{\Omega_p}+\frac{\nu\Delta t}{4}\|\mathbf{u}^{n+1/2}\|^2_{H_f}.\end{aligned}\tag{3.6}$$

Using the Cauchy-Schwarz inequality, we have

$$2\Delta t(f(t_{n+1}),\mathbf{u}^{n+1/2})_{\Omega_f}\leq 4C\Delta t\nu^{-1}\|f(t_{n+1})\|^2_{\Omega_f}+\frac{\nu\Delta t}{4}\|\mathbf{u}^{n+1/2}\|^2_{H_f}.\tag{3.7}$$

Combining (3.4)-(3.7), we arrive at

$$\begin{aligned}&\|\mathbf{u}^{n+1/2}\|^2_{\Omega_f}-\|\mathbf{u}^n\|^2_{\Omega_f}+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{\Omega_f}+\Delta t\nu\|\mathbf{u}^{n+1/2}\|^2_{H_f}+\frac{\alpha\Delta t\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}^{n+1/2}\cdot\tau\|^2_\Gamma\\ &\quad+\theta\Delta t(\|\mathbf{u}^{n+1/2}\|_{H_f}-\|\mathbf{u}^n\|^2_{H_f}+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{H_f})\\ \leq&4\Delta t\nu^{-1}C_1^2\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}\|\mathbf{u}^n\|^2_{\Omega_f}+\frac{4\Delta t C_2^2\nu^{-1}g^2}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\phi^n\|^2_{\Omega_p}+4C\Delta t\nu^{-1}\|f(t_{n+1})\|^2_{\Omega_f}.\end{aligned}\tag{3.8}$$

Taking $\mathbf{v}=2\Delta t\mathbf{u}^{n+1}$ in (2.14) and noting $b(\mathbf{u}^{n+1},q)=0$, we deduce

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2_{\Omega_f}-\|\mathbf{u}^{n+1/2}\|^2_{\Omega_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}\|_{\Omega_f}\\ &\quad+\Delta t(\theta+\nu)(\|\mathbf{u}^{n+1}\|^2_{H_f}-\|\mathbf{u}^{n+1/2}\|^2_{H_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}\|^2_{H_f})\\ &\quad+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{u}^{n+1}\cdot\tau\|^2_\Gamma-\|\mathbf{u}^{n+1/2}\cdot\tau\|^2_\Gamma+\|\mathbf{u}^{n+1}\cdot\tau-\mathbf{u}^{n+1/2}\cdot\tau\|^2_\Gamma)\leq 0.\end{aligned}$$

Combining it with (3.8), it follows by

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2_{\Omega_f}-\|\mathbf{u}^n\|^2_{\Omega_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}\|^2_{\Omega_f}+\Delta t\nu\|\mathbf{u}^{n+1/2}\|^2_{H_f}+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{\Omega_f}\\ &\quad+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{u}^{n+1}\cdot\tau\|^2_\Gamma+\|\mathbf{u}^{n+1}\cdot\tau-\mathbf{u}^{n+1/2}\cdot\tau\|^2_\Gamma)+\theta\Delta t(\|\mathbf{u}^{n+1}\|_{H_f}-\|\mathbf{u}^n\|^2_{H_f}\\ &\quad+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{H_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{1/2}\|^2_{H_f})+\Delta t\nu(\|\mathbf{u}^{n+1}\|^2_{H_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}\|^2_{H_f})\\ \leq&4\Delta t\nu^{-1}C_1^2\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}\|\mathbf{u}^n\|^2_{\Omega_f}+\frac{4\Delta t C_2^2\nu^{-1}g^2}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\phi^n\|^2_{\Omega_p}+4\Delta t\nu^{-1}\|f(t_{n+1})\|^2_{\Omega_f}.\end{aligned}\tag{3.9}$$

Letting $\psi=2\Delta t\phi^{n+1}$ in (2.15), there holds

$$\begin{aligned}&2gS_0(\phi^{n+1}-\phi^n,\phi^{n+1})_{\Omega_p}+2g\Delta t(\mathbf{K}\nabla\phi^{n+1},\nabla\phi^{n+1})_{\Omega_p}-2\Delta tc_\Gamma(\phi^{n+1},\mathbf{u}^n)\\=&2g\Delta t(g_p(t_{n+1}),\phi^{n+1})_{\Omega_p}.\end{aligned}$$

Using (3.1), we deduce

$$\begin{aligned}2\Delta tc_\Gamma(\phi^{n+1},\mathbf{u}^n)\le&\frac{2\Delta tC_2g}{\sqrt{K_{\min}}}\|\mathbf{K}^{1/2}\nabla\phi^{n+1}\|_{\Omega_p}\|\mathbf{u}^n\|_{H_f}\\\le&\frac{g\Delta t}{4}\|\mathbf{K}^{1/2}\nabla\phi^{n+1}\|^2_{\Omega_p}+\frac{4C_2^2g\Delta t}{K_{\min}}\|\mathbf{u}^n\|^2_{H_f}.\end{aligned}\tag{3.10}$$

Using the Cauchy-Schwarz inequality, it shows that

$$2g\Delta t(g_p(t_{n+1}),\phi^{n+1})_{\Omega_p}\le\frac{4\Delta tg}{K_{\min}}\|g_p(t_{n+1})\|^2_{\Omega_p}+\frac{g\Delta t}{4}\|\mathbf{K}^{1/2}\nabla\phi^{n+1}\|^2_{\Omega_p}.\tag{3.11}$$

Combining (3.9), (3.10) and (3.11) gives

$$\begin{aligned}&gS_0(\|\phi^{n+1}\|^2_{\Omega_p}-\|\phi^n\|^2_{\Omega_p}+\|\phi^{n+1}-\phi^n\|^2_{\Omega_p})+g\Delta t\|\mathbf{K}^{1/2}\nabla\phi^{n+1}\|^2_{\Omega_p}\\\le&\frac{4\Delta tg}{K_{\min}}\|g_p(t_{n+1})\|^2_{\Omega_p}+\frac{4C_2^2g\Delta t}{K_{\min}}\|\mathbf{u}^n\|^2_{H_f}.\end{aligned}\tag{3.12}$$

Combining (3.9) and (3.12), we arrive at

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2_{\Omega_f}-\|\mathbf{u}^n\|^2_{\Omega_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}\|^2_{\Omega_f}+\frac{\Delta t\nu}{2}\|\mathbf{u}^{n+1/2}\|^2_{H_f}\\&+\theta\Delta t(\|\mathbf{u}^{n+1}\|_{H_f}-\|\mathbf{u}^n\|^2_{H_f}+\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|^2_{H_f})+\Delta t\nu(\|\mathbf{u}^{n+1}\|^2_{H_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2}\|^2_{H_f})\\&+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{u}^{n+1}\cdot\tau\|^2_\Gamma+\|\mathbf{u}^{n+1}\cdot\tau-\mathbf{u}^{n+1/2}\cdot\tau\|^2_\Gamma)\\&+gS_0(\|\phi^{n+1}\|^2_{\Omega_p}-\|\phi^n\|^2_{\Omega_p}+\|\phi^{n+1}-\phi^n\|^2_{\Omega_p})+2g\Delta t\|\mathbf{K}^{1/2}\nabla\phi^{n+1}\|^2_{\Omega_p}\\\le&C_1^2\Delta t\nu^{-1}\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}\|\mathbf{u}^n\|^2_{\Omega_f}+\frac{4\Delta tC_2^2\nu^{-1}g^2}{K_{\min}}\|\mathbf{K}^{1/2}\phi^n\|^2_{H_p}+4\Delta t\nu^{-1}\|f(t_{n+1})\|^2_{\Omega_f}\\&+\frac{4\Delta tg}{K_{\min}}\|g_p(t_{n+1})\|^2_{\Omega_p}+\frac{4C_2^2g\Delta t}{K_{\min}}\|\mathbf{u}^n\|^2_{H_f}.\end{aligned}\tag{3.13}$$

Summing (3.13) over all n, there holds

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2_{\Omega_f}+\sum_{i=0}^n\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{\Omega_f}+\frac{\Delta t\nu}{2}\sum_{i=0}^n\|\mathbf{u}^{i+1/2}\|^2_{H_f}\\&+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^n(\|\mathbf{u}^{i+1}\cdot\tau\|^2_\Gamma+\|\mathbf{u}^{i+1}\cdot\tau-\mathbf{u}^{i+1/2}\cdot\tau\|^2_\Gamma)+\theta\Delta t(\|\mathbf{u}^{n+1}\|_{H_f}\\&+\sum_{i=0}^n\|\mathbf{u}^{i+1/2}-\mathbf{u}^i\|^2_{H_f}+\sum_{i=0}^n\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{H_f})+\Delta t\nu\sum_{i=0}^n(\|\mathbf{u}^{i+1}\|^2_{H_f}+\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{H_f})\end{aligned}$$

$$
\begin{aligned}
&+gS_0(\|\phi^{n+1}\|^2_{\Omega_p}+\sum_{i=0}^{n}\|\phi^{i+1}-\phi^i\|^2_{\Omega_p})+2g\Delta t\sum_{i=0}^{n}\|\mathbf{K}^{1/2}\nabla\phi^{i+1}\|^2_{\Omega_p}\\
\leq& C_1^2\Delta t\nu^{-1}\sum_{i=0}^{n}\|\mathcal{A}\mathbf{u}^i\|^2_{\Omega_f}\|\mathbf{u}^i\|^2_{\Omega_f}+\frac{4\Delta tC_2^2\nu^{-1}g^2}{K_{\min}}\sum_{i=0}^{n}\|\mathbf{K}^{1/2}\nabla\phi^i\|^2_{\Omega_p}+4\Delta t\nu^{-1}\sum_{i=0}^{n}\|f(t_{i+1})\|^2_{\Omega_f}\\
&+\frac{4\Delta tg}{K_{\min}}\|g_p(t_{i+1})\|^2_{\Omega_p}+\frac{4C_2^2g\Delta t}{K_{\min}}\sum_{i=0}^{n}\|\mathbf{u}^i\|^2_{H_f}.
\end{aligned}
$$

Via Gronwall's lemma three times, it yields

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2_{\Omega_f}+\sum_{i=0}^{n}\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{\Omega_f}+\frac{\Delta t\nu}{2}\sum_{i=0}^{n}\|\mathbf{u}^{i+1/2}\|^2_{H_f}\\
&+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^{n}(\|\mathbf{u}^{i+1}\cdot\tau\|^2_\Gamma+\|\mathbf{u}^{i+1}\cdot\tau-\mathbf{u}^{i+1/2}\cdot\tau\|^2_\Gamma)+\theta\Delta t(\|\mathbf{u}^{n+1}\|_{H_f}\\
&+\sum_{i=0}^{n}\|\mathbf{u}^{i+1/2}-\mathbf{u}^i\|^2_{H_f}+\sum_{i=0}^{n}\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{H_f})+\Delta t\nu\sum_{i=0}^{n}(\|\mathbf{u}^{i+1}\|^2_{H_f}+\|\mathbf{u}^{i+1}-\mathbf{u}^{i+1/2}\|^2_{H_f})\\
&+gS_0(\|\phi^{n+1}\|^2_{\Omega_p}+\sum_{i=0}^{n}\|\phi^{i+1}-\phi^i\|^2_{\Omega_p})+2g\Delta t\sum_{i=0}^{n}\|\mathbf{K}^{1/2}\nabla\phi^{i+1}\|^2_{\Omega_p}\\
\leq&\exp\left(\frac{C_1^2\Delta t}{\nu}\sum_{i=0}^{n}\|\mathcal{A}\mathbf{u}^i\|^2_{\Omega_f}+\frac{4TC_2^2\nu^{-1}g^2}{K_{\min}}+\frac{4C_2^2gT}{K_{\min}}\right)\\
&\times\left(4\Delta t\nu^{-1}\sum_{i=0}^{n}\|f(t_{n+1})\|^2_{\Omega_f}+\frac{4\Delta tg}{K_{\min}}\sum_{i=0}^{n}\|g_p(t_{i+1})\|^2_{\Omega_p}\right).
\end{aligned}
$$

Step 2: we give the proof of (3.2b) under the induction assumption.

Adding (2.13) and (2.14), it follows by

$$
\begin{aligned}
&\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t},\mathbf{v}\right)_{\Omega_f}+a_f(\mathbf{u}^{n+1},\mathbf{v})_{\Omega_f}+\theta(\nabla(\mathbf{u}^{n+1}-\mathbf{u}^n),\nabla\mathbf{v})_{\Omega_f}+B(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v})\\
&\quad+b(\mathbf{v},p^{n+1})+c_\Gamma(\phi^n,\mathbf{v})=(f(t_{n+1}),\mathbf{v})_{\Omega_f}.
\end{aligned}\tag{3.14}
$$

Letting $\mathbf{v}=2\Delta t\mathcal{A}\mathbf{u}^{n+1}$, we deduce

$$
\begin{aligned}
&2\left(\mathbf{u}^{n+1}-\mathbf{u}^n,\mathcal{A}\mathbf{u}^{n+1}\right)_{\Omega_f}+2\Delta t\nu\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}\\
&\quad+2\theta\Delta t(\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}-\|\mathcal{A}\mathbf{u}^n\|^2_{\Omega_f}+\|\mathcal{A}(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2_{\Omega_f})\\
&\quad+2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,\mathcal{A}\mathbf{u}^{n+1})+2\Delta tc_\Gamma(\phi^n,\mathcal{A}\mathbf{u}^{n+1})\\
\leq&2\Delta t(f(t_{n+1}),\mathcal{A}\mathbf{u}^{n+1})_{\Omega_f}.
\end{aligned}
$$

Using the property of $B(\cdot,\cdot,\cdot)$, we have

$$
\begin{aligned}
&2\Delta t B(\mathbf{u}^n,\mathbf{u}^n,\mathcal{A}\mathbf{u}^{n+1})\le 2C\Delta t\|\mathbf{u}^n\|_{H^2(\Omega_f)}\|\mathbf{u}^n\|_{H_f}\|\mathcal{A}\mathbf{u}^{n+1}\|_{\Omega_f}\\
\le&\frac{\Delta t\nu}{4}\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}+4C\nu^{-1}\Delta t\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}\|\mathbf{u}^n\|^2_{H_f}.
\end{aligned}
$$

On the other hand, there holds that

$$
\begin{aligned}
&2\Delta t c_\Gamma(\phi^n,\mathcal{A}\mathbf{u}^{n+1})=g\int_\Gamma \phi^n\mathcal{A}\mathbf{u}^{n+1}\cdot n ds=-2\Delta t g\int_\Gamma \nabla\phi^n\cdot\nabla\mathbf{u}^{n+1}\cdot n ds\\
\le& 2\Delta t g\|\nabla\phi^n\|_{L^2(\Gamma)}\|\nabla\mathbf{u}^{n+1}\|_{L^2(\Gamma)}\le 2\Delta t g\|\mathcal{A}\phi^n\|_{\Omega_f}\|\mathcal{A}\mathbf{u}^{n+1}\|_{\Omega_f}\\
\le&\frac{\Delta t\nu}{4}\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}+4g\Delta t\nu^{-1}\|\mathcal{A}\phi^n\|^2_{\Omega_p}.
\end{aligned}
$$

Via Cauchy-Schwarz inequality, we get

$$
2\Delta t(f(t_{n+1}),\mathcal{A}\mathbf{u}^{n+1})_{\Omega_f}\le\frac{\Delta t\nu}{4}\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}+4\Delta t\nu^{-1}\|f(t_{n+1})\|^2_{\Omega_f}.
$$

Combining these inequalities, it shows that

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2_{H_f}-\|\mathbf{u}^n\|^2_{H_f}+\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2_{H_f}+2\Delta t\nu\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}\\
&\quad+2\theta\Delta t(\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}-\|\mathcal{A}\mathbf{u}^n\|^2_{\Omega_f}+\|\mathcal{A}(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2_{\Omega_f})\\
\le&\Delta t\nu\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}+4\Delta t\nu^{-1}\|f(t_{n+1})\|^2_{\Omega_f}+4g\Delta t\nu^{-1}\|\mathcal{A}\mathbf{K}^{1/2}\phi^n\|^2_{\Omega_p}\\
&\quad+4\nu^{-1}\Delta t\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}\|\mathbf{u}^n\|^2_{H_f}.
\end{aligned}
$$

Summing the above inequality all over n, we arrive at

$$
\begin{aligned}
&\|\mathbf{u}^{n+1}\|^2_{H_f}+\sum_{i=1}^n\|\mathbf{u}^{i+1}-\mathbf{u}^i\|^2_{H_f}+\frac{3}{2}\Delta t\nu\sum_{i=1}^n\|\mathcal{A}\mathbf{u}^{i+1}\|^2_{\Omega_f}\\
&\quad+2\theta\Delta t\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}+2\theta\Delta t\sum_{i=1}^n\|\mathcal{A}(\mathbf{u}^{i+1}-\mathbf{u}^i)\|^2_{\Omega_f}\\
\le&4\Delta t\nu^{-1}\sum_{i=1}^n\|f(t_{i+1})\|^2_{\Omega_f}+4g\Delta t\nu^{-1}\sum_{i=1}^n\|\mathcal{A}\phi^i\|^2_{\Omega_p}+4\nu^{-1}\Delta t\sum_{i=1}^n\|\mathbf{u}^i\|^2_{H^2(\Omega_f)}\|\mathbf{u}^i\|^2_{H_f}.
\end{aligned}
\tag{3.15}
$$

Letting $\psi=2\Delta t\mathcal{A}\phi^{n+1}$ in (2.15), we have

$$
\begin{aligned}
&2S_0\left(\phi^{n+1}-\phi^n,\mathcal{A}\phi^{n+1}\right)_{\Omega_p}+2\Delta t(\nabla\mathbf{K}\nabla\phi^{n+1},\mathcal{A}\phi^{n+1})\\
=&2\Delta t c_\Gamma(\mathcal{A}\phi^{n+1},\mathbf{u}^n)+2\Delta t(g_p(x,t_{n+1}),\mathcal{A}\phi^{n+1})_{\Omega_p}.
\end{aligned}
\tag{3.16}
$$

Using the definition of $c_\Gamma(\cdot,\cdot)$, there holds that

$$
2\Delta t c_\Gamma(\mathcal{A}\phi^{n+1},\mathbf{u}^n)\le\frac{K_{\min}\Delta t}{4}\|\mathcal{A}\phi^{n+1}\|^2_{\Omega_p}+\frac{4\Delta t}{K_{\min}}\|\mathcal{A}\mathbf{u}^n\|^2_{\Omega_f}.
$$

Using Cauchy-Schwarz inequality, we get

$$2\Delta t(g_p(x,t_{n+1}),\mathcal{A}\phi^{n+1})_{\Omega_p} \leq \frac{K_{\min}\Delta t}{4}\|\mathcal{A}\phi^{n+1}\|^2_{\Omega_p}+\frac{4\Delta t}{K_{\min}}\|g_p(t_{n+1})\|^2_{\Omega_p}.$$

Combining these three inequalities gives

$$\begin{aligned}&2S_0(\|\phi^{n+1}\|^2_{H_p}-\|\phi^n\|^2_{H_p}+\|\phi^{n+1}-\phi^n\|^2_{H_p})+2K_{\min}\Delta t\|\mathcal{A}\phi^{n+1}\|^2_{\Omega_p}\\ \leq&\frac{K_{\min}\Delta t}{2}\|\mathcal{A}\phi^{n+1}\|^2_{\Omega_p}+\frac{4\Delta t}{K_{\min}}\|\mathcal{A}\mathbf{u}^n\|^2_{\Omega_f}+\frac{4\Delta t}{K_{\min}}\|g_p(t_{n+1})\|^2_{\Omega_p}.\end{aligned}$$

Summing over the above inequality, we have

$$\begin{aligned}&2S_0\|\phi^{n+1}\|^2_{H_p}+2S_0\sum_{i=0}^{n}\|\phi^{i+1}-\phi^i\|^2_{H_p}+\frac{3}{2}\Delta t\sum_{i=0}^{n}\|\mathbf{K}^{1/2}\mathcal{A}\phi^{i+1}\|^2_{\Omega_p}\\ \leq&\frac{4\Delta t}{K_{\min}}\sum_{i=0}^{n}\|\mathcal{A}\mathbf{u}^i\|^2_{\Omega_f}+\frac{4\Delta t}{K_{\min}}\sum_{i=0}^{n}\|g_p(t_{i+1})\|^2_{\Omega_p}.\end{aligned} \tag{3.17}$$

Combining (3.15) and (3.17), we arrive at

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2_{H_f}+2S_0\|\phi^{n+1}\|^2_{H_p}+\sum_{i=1}^{n}(\|\mathbf{u}^{i+1}-\mathbf{u}^i\|^2_{H_f}+2S_0\|\phi^{i+1}-\phi^i\|^2_{H_p})\\ &+\frac{3}{2}\Delta t\sum_{i=1}^{n}(\nu\|\mathcal{A}\mathbf{u}^{i+1}\|^2_{\Omega_f}+K_{\min}\|\mathcal{A}\phi^{i+1}\|^2_{\Omega_p})+2\theta\Delta t\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}\\ &+2\theta\Delta t\sum_{i=1}^{n}\|\mathcal{A}(\mathbf{u}^{i+1}-\mathbf{u}^i)\|^2_{\Omega_f}\\ \leq&4\Delta t\nu^{-1}\sum_{i=1}^{n}\|f(t_{i+1})\|^2_{\Omega_f}+4g\Delta t\nu^{-1}\sum_{i=1}^{n}\|\mathcal{A}\phi^i\|^2_{\Omega_f}+4\nu^{-1}\Delta t\sum_{i=1}^{n}\|\mathbf{u}^i\|^2_{H^2(\Omega_f)}\|\mathbf{u}^i\|^2_{H_f}\\ &+\frac{4\Delta t}{K_{\min}}\sum_{i=0}^{n}\|\mathcal{A}\mathbf{u}^i\|^2_{\Omega_f}+\frac{4\Delta t}{K_{\min}}\sum_{i=0}^{n}\|g_p(t_{i+1})\|^2_{\Omega_p}.\end{aligned}$$

Using Gronwall's lemma, it yields

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|^2_{H_f}+2S_0\|\phi^{n+1}\|^2_{H_p}+\sum_{i=1}^{n}(\|\mathbf{u}^{i+1}-\mathbf{u}^i\|^2_{H_f}+2S_0\|\phi^{i+1}-\phi^i\|^2_{H_p})\\ &+\frac{3}{2}\Delta t\sum_{i=1}^{n}(\nu\|\mathcal{A}\mathbf{u}^{i+1}\|^2_{\Omega_f}+K_{\min}\|\mathcal{A}\phi^{i+1}\|^2_{\Omega_p})+2\theta\Delta t\|\mathcal{A}\mathbf{u}^{n+1}\|^2_{\Omega_f}\\ &+2\theta\Delta t\sum_{i=1}^{n}\|\mathcal{A}(\mathbf{u}^{i+1}-\mathbf{u}^i)\|^2_{\Omega_f}\end{aligned}$$

$$
\begin{aligned}
&\le \exp\left(4g\Delta t\nu^{-1}\sum_{i=1}^{n}\|\mathbf{u}^i\|_{H^2(\Omega_f)}^2+4\nu^{-1}T+\frac{4T}{K_{\min}}\right)\\
&\quad\times\left(4\Delta t\nu^{-1}\sum_{i=1}^{n}\|f(t_{i+1})\|_{\Omega_f}^2+\frac{4\Delta t}{K_{\min}}\sum_{i=0}^{n}\|g_p(t_{i+1})\|_{\Omega_p}^2\right).
\end{aligned}\tag{3.18}
$$

Step 3: we give the proof of (3.3) under the induction assumption.

Define $d_t\mathbf{u}^{n+1}=\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}$. Letting $\mathbf{v}=2\Delta t d_t\mathbf{u}^{n+1}$ in (3.14) and using $(\nabla\cdot\mathbf{u}^n,q)=0$, $\forall q\in M$, it follows that

$$
\begin{aligned}
&2\Delta t\|d_t\mathbf{u}^{n+1}\|_{\Omega_f}^2+\nu\|\mathbf{u}^{n+1}\|_{H_f}^2-\nu\|\mathbf{u}^n\|_{H_f}^2+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\left(\|\mathbf{u}^{n+1}\cdot\tau\|_\Gamma^2-\|\mathbf{u}^n\cdot\tau\|_\Gamma^2\right)\\
&+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}^{n+1}\cdot\tau-\mathbf{u}^n\cdot\tau\|_\Gamma^2+\nu\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{H_f}^2+\theta\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{H_f}^2\\
&+2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,d_t\mathbf{u}^{n+1})+2\Delta tc_\Gamma(\phi^n,d_t\mathbf{u}^{n+1})=2\Delta t(f(t_{n+1}),d_t\mathbf{u}^{n+1})_{\Omega_f}.
\end{aligned}
$$

Using (2.9), we have

$$
\begin{aligned}
2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,d_t\mathbf{u}^{n+1})&\le C\Delta t\|\mathbf{u}^n\|_{H^2(\Omega_f)}\|\mathbf{u}^n\|_{H_f}\|d_t\mathbf{u}^{n+1}\|_{\Omega_f}\\
&\le 4\Delta t\|\mathbf{u}^n\|_{H^2(\Omega_f)}^2\|\mathbf{u}^n\|_{H_f}^2+\frac{\Delta t}{4}\|d_t\mathbf{u}^{n+1}\|_{\Omega_f}^2.
\end{aligned}
$$

We also get

$$
\begin{aligned}
&2\Delta tc_\Gamma(\phi^n,d_t\mathbf{u}^{n+1})=2c_\Gamma(\phi^n,\mathbf{u}^{n+1}-\mathbf{u}^n)\\
\le& C\|\nabla\mathbf{K}^{1/2}\phi^n\|_{\Omega_p}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{H_f}\\
\le& \|\nabla\mathbf{K}^{1/2}\phi^n\|_{\Omega_p}^2+\nu\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{H_f}^2.
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we deduce

$$
\begin{aligned}
&2\Delta t(f(t_{n+1}),d_t\mathbf{u}^{n+1})\le 2\Delta t\|f(t_{n+1})\|_{\Omega_f}\|d_t\mathbf{u}^{n+1}\|_{\Omega_f}\\
\le& 4\Delta t\|f(t_{n+1})\|_{\Omega_f}^2+\frac{\Delta t}{4}\|d_t\mathbf{u}^{n+1}\|_{\Omega_f}^2.
\end{aligned}
$$

Then, it follows by

$$
\begin{aligned}
&\Delta t\|d_t\mathbf{u}^{n+1}\|_{\Omega_f}^2+\nu\|\mathbf{u}^{n+1}\|_{H_f}^2-\nu\|\mathbf{u}^n\|_{H_f}^2+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\left(\|\mathbf{u}^{n+1}\cdot\tau\|_\Gamma^2-\|\mathbf{u}^n\cdot\tau\|_\Gamma^2\right)\\
&+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}^{n+1}\cdot\tau-\mathbf{u}^n\cdot\tau\|_\Gamma^2+\theta\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{H_f}^2\\
\le& 4\Delta t\|\mathcal{A}\mathbf{u}^n\|_{\Omega_f}^2\|\mathbf{u}^n\|_{H_f}^2+\|\nabla\phi^n\|_{\Omega_p}^2+4\Delta t\|f(t_{n+1})\|_{\Omega_f}^2.
\end{aligned}
$$

Summing it over all n, there holds

$$\begin{aligned}&\Delta t\sum_{i=0}^{n}\|d_t\mathbf{u}^{i+1}\|^2_{\Omega_f}+\nu\|\mathbf{u}^{n+1}\|^2_{H_f}+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}^{n+1}\cdot\tau\|^2_{\Gamma}\\&\quad+\sum_{i=0}^{n}\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{u}^{i+1}\cdot\tau-\mathbf{u}^{i}\cdot\tau\|^2_{\Gamma}+\theta\sum_{i=0}^{n}\|\mathbf{u}^{i+1}-\mathbf{u}^{i}\|^2_{H_f}\\&\le\exp\left(4\Delta t\sum_{i=0}^{n}\|\mathcal{A}\mathbf{u}^{i}\|^2_{\Omega_f}\right)\left(\sum_{i=0}^{n}\|\nabla\phi^{i}\|^2_{\Omega_p}+4\Delta t\sum_{i=0}^{n}\|f(t_{i+1})\|^2_{\Omega_f}\right).\end{aligned}$$

Via (3.14), we have

$$\begin{aligned}&2(\nu+\theta)\|\mathcal{A}\mathbf{u}^{n+1}\|_{\Omega_f}\\&\le\|d_t\mathbf{u}^{n+1}\|_0+2\theta\|\mathcal{A}\mathbf{u}^{n}\|^2_{\Omega_f}+4\|f(t_{n+1})\|^2_{\Omega_f}\\&\quad+4g\|\mathcal{A}\phi^{n}\|^2_{\Omega_p}+4\|\mathbf{u}^{n}\|_{H^2(\Omega_f)}\|\mathbf{u}^{n}\|_{H_f}.\end{aligned}$$

It means that

$$\begin{aligned}&2(\nu+\theta)\Delta t\sum_{i=0}^{n}\|\mathcal{A}\mathbf{u}^{i+1}\|_{\Omega_f}\\&\le\tau\sum_{i=0^n}\tau\sum_{i=0^n}\|d_t\mathbf{u}^{n+1}\|_0+2\theta\|\mathcal{A}\mathbf{u}^{i}\|^2_{\Omega_f}+4\tau\sum_{i=0^n}\|f(t_{i+1})\|^2_{\Omega_f}\\&\quad+4g\tau\sum_{i=0^n}\|\mathcal{A}\phi^{i}\|^2_{\Omega_p}+4\tau\sum_{i=0^n}\|\mathbf{u}^{i}\|_{H^2(\Omega_f)}\|\mathbf{u}^{i}\|_{H_f}\le C.\end{aligned}$$

So we finish the proof. □

4 Error analysis

We define

$$\begin{aligned}&\mathbf{e}^{n+1}=\mathbf{u}(t_{n+1})-\mathbf{u}^{n+1},\\&\mathbf{e}^{n+1/2}=\mathbf{u}(t_{n+1})-\mathbf{u}^{n+1/2},\\&\eta^{n+1}=\phi(t_{n+1})-\phi^{n+1}.\end{aligned}$$

Theorem 4.1 (The non-optimal error estimates)**.** *If u, $u_t\in H^2(\Omega_f)$, ϕ, $\phi_t\in H^2(\Omega_p)$ and $p\in H^1(\Omega_f)$, there exists $k_0>0$ such that when $\Delta t\le k_0$,*

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}+\sum_{i=0}^{n}\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|^2_{\Omega_f}+\sum_{i=0}^{n}\|\mathbf{e}^{i+1/2}-\mathbf{e}^{i}\|^2_{\Omega_f}+\Delta t\nu\sum_{i=0}^{n}\|\mathbf{e}^{i+1/2}\|^2_{H_f}\\&\quad+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^{n}(\|\mathbf{e}^{i+1}\cdot\tau\|^2_{\Gamma}+\|\mathbf{e}^{i+1}\cdot\tau-\mathbf{e}^{i+1/2}\cdot\tau\|^2_{\Gamma})\end{aligned}$$

$$+\theta\Delta t(\|\mathbf{e}^{n+1}\|^2_{H_f}+\sum_{i=0}^{n}\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|^2_{H_f})+\Delta t\nu\sum_{i=0}^{n}(\|\mathbf{e}^{i+1}\|^2_{H_f}+\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|^2_{H_f})$$
$$+S_0\left(\|\eta^{n+1}\|^2_{\Omega_p}+\sum_{i=0}^{n}\|\eta^{i+1}-\eta^i\|^2_{\Omega_p}\right)+\Delta t\sum_{i=0}^{n}\|\mathbf{K}^{1/2}\nabla\eta^{i+1}\|^2_{\Omega_p}\leq C\Delta t,$$

where C is a positive constant independent of Δt.

Proof. Subtracting (2.13) from (2.6) with $\psi=0$, we have

$$\begin{aligned}&\left(\frac{\mathbf{e}^{n+1/2}-\mathbf{e}^n}{\Delta t},\mathbf{v}\right)_{\Omega_f}+a_f(\mathbf{e}^{n+1/2},\mathbf{v})+\theta(\nabla\mathbf{e}^{n+1/2},\nabla\mathbf{v})_{\Omega_f}\\&\quad-\theta(\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}^n),\nabla\mathbf{v})_{\Omega_f}+B(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v})-B(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v})\\&=b(\mathbf{v},p(t_{n+1}))+(R^n,\mathbf{v})_{\Omega_f}+c_\Gamma(\phi(t_{n+1})-\phi^n,\mathbf{v}),\end{aligned}\tag{4.1}$$

where $R^n=\mathbf{u}_t(t_{n+1})-\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\Delta t}$.

Letting $\mathbf{v}=2\Delta t\mathbf{e}^{n+1/2}$, we can deduce

$$\begin{aligned}&\|\mathbf{e}^{n+1/2}\|^2_{\Omega_f}-\|\mathbf{e}^n\|^2_{\Omega_f}+\|\mathbf{e}^{n+1/2}-\mathbf{e}^n\|^2_{\Omega_f}+2\Delta t\nu\|\mathbf{e}^{n+1/2}\|^2_{H_f}+\frac{2\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\|\mathbf{e}^{n+1/2}\cdot\tau\|^2_\Gamma\\&\quad+\theta\Delta t(\|\mathbf{e}^{n+1/2}\|^2_{H_f}-\|\mathbf{e}^n\|^2_{H_f}+\|\mathbf{e}^{n+1/2}-\mathbf{e}^n\|^2_{H_f})\\&\leq-2\Delta tB(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{e}^{n+1/2})+2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,\mathbf{e}^{n+1/2})+2\Delta tb(\mathbf{e}^{n+1/2},p(t_{n+1}))\\&\quad+2\theta\Delta t(\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)),\nabla\mathbf{e}^{n+1/2})_{\Omega_f}+2\Delta t(R^n,\mathbf{e}^{n+1/2})_{\Omega_f}\\&\quad+2\Delta tc_\Gamma(\phi(t_{n+1})-\phi^n,\mathbf{e}^{n+1/2}).\end{aligned}\tag{4.2}$$

Form (2.14), we have

$$\begin{aligned}&\left(\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}}{\Delta t},\mathbf{v}\right)_{\Omega_f}+a_f(\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2},\mathbf{v})\\&\quad+\theta(\nabla\mathbf{e}^{n+1}-\nabla\mathbf{e}^{n+1/2},\nabla\mathbf{v})_{\Omega_f}+b(\mathbf{v},p^{n+1})=0.\end{aligned}\tag{4.3}$$

Taking $\mathbf{v}=2\Delta t\mathbf{e}^{n+1}$ and noticing the fact that $\nabla\cdot\mathbf{e}^{n+1}=0$, we arrive at

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}^{n+1/2}\|^2_{\Omega_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|^2_{\Omega_f}\\&\quad+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{e}^{n+1}\cdot\tau\|^2_\Gamma-\|\mathbf{e}^{n+1/2}\cdot\tau\|^2_\Gamma+\|\mathbf{e}^{n+1}\cdot\tau-\mathbf{e}^{n+1/2}\cdot\tau\|^2_\Gamma)\\&\quad+(\theta+\nu)\Delta t(\|\mathbf{e}^{n+1}\|^2_{H_f}-\|\mathbf{e}^{n+1/2}\|^2_{H_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|^2_{H_f})\leq 0.\end{aligned}\tag{4.4}$$

Summing the inequalities (4.2) and (4.4), we get

$$\begin{aligned}
&\|\mathbf{e}^{n+1}\|_{\Omega_f}^2-\|\mathbf{e}^n\|_{\Omega_f}^2+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|_{\Omega_f}^2+\|\mathbf{e}^{n+1/2}-\mathbf{e}^n\|_{\Omega_f}^2+\Delta t\nu\|\mathbf{e}^{n+1/2}\|_{H_f}^2\\
&\quad+\frac{2\Delta t\alpha}{\sqrt{K_{\max}}}\|\mathbf{e}^{n+1/2}\cdot\tau\|_\Gamma^2+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{e}^{n+1}\cdot\tau\|_\Gamma^2-\|\mathbf{e}^{n+1/2}\cdot\tau\|_\Gamma^2+\|\mathbf{e}^{n+1}\cdot\tau-\mathbf{e}^{n+1/2}\cdot\tau\|_\Gamma^2)\\
&\quad+\theta\Delta t(\|\mathbf{e}^{n+1}\|_{H_f}^2-\|\mathbf{e}^n\|_{H_f}^2+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|_{H_f}^2)+\Delta t\nu(\|\mathbf{e}^{n+1}\|_{H_f}^2+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|_{H_f}^2)\\
\le&-2\Delta tB(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{e}^{n+1/2})+2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,\mathbf{e}^{n+1/2})+2\Delta tb(\mathbf{e}^{n+1/2},p(t_{n+1}))\\
&\quad+2\theta\Delta t(\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)),\nabla\mathbf{e}^{n+1/2})_{\Omega_f}+2\Delta t(R^n,\mathbf{e}^{n+1/2})_{\Omega_f}\\
&\quad+2\Delta tc_\Gamma(\phi(t_{n+1})-\phi^n,\mathbf{e}^{n+1/2}).
\end{aligned}\tag{4.5}$$

Using (2.9), we deduce

$$\begin{aligned}
&-2\Delta tB(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{e}^{n+1/2})+2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,\mathbf{e}^{n+1/2})\\
=&2\Delta t\Big[-B(\mathbf{e}^n,\mathbf{u}^n,\mathbf{e}^{n+1/2})-B(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{u}(t_n),\mathbf{e}^{n+1/2})-B(\mathbf{u}(t_n),\mathbf{e}^n,\mathbf{e}^{n+1/2})\\
&\quad-B(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\mathbf{e}^{n+1/2})\Big]\\
\le&2c_1\Delta t\|\mathbf{e}^n\|_{\Omega_f}\|\mathbf{u}^n\|_{H^2(\Omega_f)}\|\nabla\mathbf{e}^{n+1/2}\|_{\Omega_f}\\
&\quad+2\Delta tc_1\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{\Omega_f}\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}\|\nabla\mathbf{e}^{n+1/2}\|_{\Omega_f}\\
&\quad+2\Delta tc_1\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}\|\mathbf{e}^n\|_{\Omega_f}\|\nabla\mathbf{e}^{n+1/2}\|_{\Omega_f}\\
&\quad+2\Delta t\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{\Omega_f}\|\mathbf{e}^{n+1/2}\|_{H_f}\\
\le&8\nu^{-1}\Delta tc_1^2\|\mathbf{u}^n\|_{H^2(\Omega_f)}^2\|\mathbf{e}^n\|_{\Omega_f}^2+8\Delta tc_1^2\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}^2\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{\Omega_f}^2\\
&\quad+8\Delta tc_1^2\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}^2\|\mathbf{e}^n\|_{\Omega_f}^2+8c_1^2\Delta t\|\mathcal{A}\mathbf{u}(t_{n+1})\|_{\Omega_f}^2\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{\Omega_f}^2\\
&\quad+\frac{\nu\Delta t}{4}\|\mathbf{e}^{n+1/2}\|_{H_f}^2\\
\le&8\nu^{-1}\Delta tc_1^2(\|\mathbf{u}^n\|_{H^2(\Omega_f)}^2+\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}^2)\|\mathbf{e}^n\|_{\Omega_f}^2\\
&\quad+C\Delta t^3\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}^2\|\mathbf{u}_t(t_{n+1})\|_{\Omega_f}^2+\frac{\nu\Delta t}{8}\|\mathbf{e}^{n+1/2}\|_{H_f}^2.
\end{aligned}\tag{4.6}$$

Using Cauchy-Schwarz inequality and Taylor's formula, we deduce

$$\begin{aligned}
&2\theta\Delta t(\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)),\nabla\mathbf{e}^{n+1/2})_{\Omega_f}\\
\le&2\theta\Delta t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{H_f}\|\mathbf{e}^{n+1/2}\|_{H_f}\\
\le&C\theta^2\Delta t^3\|\mathbf{u}_t(t_{n+1})\|_{H_f}^2+\frac{\nu\Delta t}{8}\|\mathbf{e}^{n+1/2}\|_{H_f}^2.
\end{aligned}\tag{4.7}$$

Using Taylor's Formula, we have

$$2\Delta t(R^n,\mathbf{e}^{n+1/2})_{\Omega_f}\le C\Delta t^3+\frac{\nu\Delta t}{8}\|\mathbf{e}^{n+1/2}\|_{H_f}^2.\tag{4.8}$$

Via (3.1), we can get

$$\begin{aligned}&2\Delta t c_\Gamma(\phi(t_{n+1})-\phi^n,\mathbf{e}^{n+1/2})\le 2C_2 g\Delta t\|\phi(t_{n+1})-\phi^n\|_{H_p}\|\mathbf{e}^{n+1/2}\|_{H_f}\\ \le& 2C_2 g\Delta t(\|\phi(t_{n+1})-\phi(t_n)\|_{H_p}+\|\phi(t_n)-\phi^n\|_{H_p})\|\mathbf{e}^{n+1/2}\|_{H_f}\\ \le& Cg\Delta t^3+\frac{C\Delta t}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\eta^n\|^2_{\Omega_p}+\frac{\nu\Delta t}{8}\|\mathbf{e}^{n+1/2}\|^2_{H_f}. \qquad (4.9)\end{aligned}$$

Noting $\nabla\cdot\mathbf{e}^n=0$ and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}&2\Delta t b(\mathbf{e}^{n+1/2},p(t_n))=2\Delta t b(\mathbf{e}^{n+1/2}-\mathbf{e}^n,p(t_n))\\ \le& 2\Delta t\|\mathbf{e}^{n+1/2}-\mathbf{e}^n\|_{\Omega_f}\|p(t_n)\|_{H_f}\le\frac{1}{2}\|\mathbf{e}^{n+1/2}-\mathbf{e}^n\|_{\Omega_f}+2\Delta t^2\|p(t_n)\|^2_{H_f}. \qquad (4.10)\end{aligned}$$

Combining these inequalities with (4.5), we arrive at

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}^n\|^2_{\Omega_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|^2_{\Omega_f}+\Delta t\nu\|\mathbf{e}^{n+1/2}\|^2_{H_f}+\frac{2\Delta t\alpha}{\sqrt{K_{\max}}}\|\mathbf{e}^{n+1/2}\cdot\tau\|^2_\Gamma\\ &+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{e}^{n+1}\cdot\tau\|^2_\Gamma-\|\mathbf{e}^{n+1/2}\cdot\tau\|^2_\Gamma+\|\mathbf{e}^{n+1}\cdot\tau-\mathbf{e}^{n+1/2}\cdot\tau\|^2_\Gamma)\\ &+\theta\Delta t(\|\mathbf{e}^{n+1}\|^2_{H_f}-\|\mathbf{e}^n\|^2_{H_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|^2_{H_f})+\Delta t\nu(\|\mathbf{e}^{n+1}\|^2_{H_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|^2_{H_f})\\ \le& 8\nu^{-1}\Delta t c_1^2(\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}+\|\mathbf{u}(t_n)\|^2_{H^2(\Omega_f)})\|\mathbf{e}^n\|^2_{\Omega_f}+C\Delta t^3\|\mathbf{u}(t_n)\|^2_{H^2(\Omega_f)}\|\mathbf{u}_t(t_{n+1})\|^2_{\Omega_f}\\ &+C\theta^2\Delta t^3+2\Delta t^2\|p(t_n)\|^2_{H_f}+\frac{C\Delta t}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\eta^n\|^2_{\Omega_p}+\frac{\nu\Delta t}{2}\|\mathbf{e}^{n+1/2}\|^2_{H_f}. \qquad (4.11)\end{aligned}$$

Subtracting (2.15) for the weak form of (2.3), we have

$$gS_0\left(\frac{\eta^{n+1}-\eta^n}{\Delta t},\psi\right)_{\Omega_p}+2g\Delta t(\mathbf{K}\nabla\eta^{n+1},\nabla\psi)_{\Omega_p}+2\Delta t c_\Gamma(\psi,\mathbf{u}(t_{n+1})-\mathbf{u}^n)=0.$$

It follows from the above equality, with $\psi=2\Delta t\eta^{n+1}$, that

$$\begin{aligned}&gS_0\left(\|\eta^{n+1}\|^2_{\Omega_p}-\|\eta^n\|_{\Omega_p}+\|\eta^{n+1}-\eta^n\|^2_{\Omega_p}\right)+2g\Delta t\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|_{\Omega_p}\\ =&-2\Delta t c_\Gamma(\eta^{n+1},\mathbf{u}(t_{n+1})-\mathbf{u}^n). \qquad (4.12)\end{aligned}$$

Using (3.1), we can deduce

$$2\Delta t c_\Gamma(\eta^{n+1},\mathbf{u}(t_{n+1})-\mathbf{u}^n)\le Cg\Delta t^3+Cg\Delta t\|\mathbf{e}^n\|^2_{H_f}+\frac{g\Delta t}{4}\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|^2_{\Omega_p}.$$

Then, we have

$$\begin{aligned}&gS_0\left(\|\eta^{n+1}\|^2_{\Omega_p}-\|\eta^n\|^2_{\Omega_p}+\|\eta^{n+1}-\eta^n\|^2_{\Omega_p}\right)+2g\Delta t\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|_{\Omega_p}\\ \le& Cg\Delta t^3+Cg\Delta t\|\mathbf{e}^n\|^2_{H_f}+\frac{g\Delta t}{4}\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|^2_{\Omega_p}. \qquad (4.13)\end{aligned}$$

Summing (4.11) with (4.13), we get

$$
\begin{aligned}
&\|\mathbf{e}^{n+1}\|_{\Omega_f}^2-\|\mathbf{e}^n\|_{\Omega_f}^2+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|_{\Omega_f}^2+\Delta t\nu\|\mathbf{e}^{n+1/2}\|_{H_f}^2\\
&\quad+\frac{\Delta t\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}(\|\mathbf{e}^{n+1}\cdot\tau\|_\Gamma^2+\|\mathbf{e}^{n+1}\cdot\tau-\mathbf{e}^{n+1/2}\cdot\tau\|_\Gamma^2)\\
&\quad+\theta\Delta t(\|\mathbf{e}^{n+1}\|_{H_f}^2-\|\mathbf{e}^n\|_{H_f}^2+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|_{H_f}^2)\\
&\quad+\Delta t\nu(\|\mathbf{e}^{n+1}\|_{H_f}^2+\|\mathbf{e}^{n+1}-\mathbf{e}^{n+1/2}\|_{H_f}^2)\\
&\quad+gS_0\left(\|\eta^{n+1}\|_{\Omega_p}^2-\|\eta^n\|_{\Omega_p}^2+\|\eta^{n+1}-\eta^n\|_{\Omega_p}^2\right)+g\Delta t\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|_{\Omega_p}\\
\le&8\nu^{-1}\Delta tc_1^2(\|\mathbf{u}^n\|_{H^2(\Omega_f)}^2+\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}^2)\|\mathbf{e}^n\|_{\Omega_f}^2+C\Delta t^3\|\mathbf{u}(t_n)\|_{H^2(\Omega_f)}^2\|\mathbf{u}_t(t_{n+1})\|_{\Omega_f}^2\\
&\quad+C\theta^2\Delta t^3\|\mathbf{u}_t(t_{n+1})\|_{H_f}^2+\frac{\nu\Delta t}{2}\|\mathbf{e}^{n+1/2}\|_{H_f}^2+C\Delta t^2+C\Delta t\|\mathbf{e}^n\|_{H_f}^2\\
&\quad+Cg\Delta t\|\mathbf{K}^{1/2}\nabla\eta^n\|_{\Omega_p}^2. \qquad (4.14)
\end{aligned}
$$

Summing (4.14) form 0 to n and using Gronwall's lemma twice, we can deduce

$$
\begin{aligned}
&\|\mathbf{e}^{n+1}\|_{\Omega_f}^2+\sum_{i=0}^n\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{\Omega_f}^2+\sum_{i=0}^n\|\mathbf{e}^{i+1/2}-\mathbf{e}^i\|_{\Omega_f}^2+\Delta t\nu\sum_{i=0}^n\|\mathbf{e}^{i+1/2}\|_{H_f}^2\\
&\quad+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^n(\|\mathbf{e}^{i+1}\cdot\tau\|_\Gamma^2+\|\mathbf{e}^{i+1}\cdot\tau-\mathbf{e}^{i+1/2}\cdot\tau\|_\Gamma^2)\\
&\quad+\theta\Delta t(\|\mathbf{e}^{n+1}\|_{H_f}^2+\sum_{i=0}^n\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{H_f}^2)+\Delta t\nu\sum_{i=0}^n(\|\mathbf{e}^{i+1}\|_{H_f}^2+\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{H_f}^2)\\
&\quad+S_0\left(\|\eta^{n+1}\|_{\Omega_p}^2+\sum_{i=0}^n\|\eta^{i+1}-\eta^i\|_{\Omega_p}^2\right)+\Delta t\sum_{i=0}^n\|\mathbf{K}^{1/2}\nabla\eta^{i+1}\|_{\Omega_p}\le C\Delta t.
\end{aligned}
$$

So we finish the proof. □

Theorem 4.2 (The optimal error estimates)**.** *If u, $u_t\in H^2(\Omega_f)$, ϕ, $\phi_t\in H^2(\Omega_p)$ and $p\in H^1(\Omega_f)$, there exist $k_0>0$ such that when $\Delta t\le k_0$,*

$$
\begin{aligned}
&\|\mathbf{e}^{n+1}\|_{\Omega_f}^2+\sum_{i=0}^n\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{\Omega_f}^2+\sum_{i=0}^n\|\mathbf{e}^{i+1/2}-\mathbf{e}^i\|_{\Omega_f}^2+\Delta t(\nu+\theta)\sum_{i=0}^n\|\mathbf{e}^{i+1}\|_{H_f}^2\\
&\quad+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^n\|\mathbf{e}^{i+1}\cdot\tau\|_\Gamma^2+\theta\Delta t(\|\mathbf{e}^{n+1}\|_{H_f}^2+\sum_{i=0}^n\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{H_f}^2)\\
&\quad+\Delta t\nu\sum_{i=0}^n(\|\mathbf{e}^{i+1}\|_{H_f}^2+\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{H_f}^2)+gS_0\left(\|\eta^{n+1}\|_{\Omega_p}^2+\sum_{i=0}^n\|\eta^{i+1}-\eta^i\|_{\Omega_p}^2\right)\\
&\quad+\Delta t\sum_{i=0}^n\|\mathbf{K}^{1/2}\nabla\eta^{i+1}\|_{\Omega_p}\le C\Delta t^2.
\end{aligned}
$$

Proof. Summing (4.1) and (4.3), we arrive at

$$\begin{aligned}&\left(\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n}}{\Delta t},\mathbf{v}\right)_{\Omega_f}+a_f(\mathbf{e}^{n+1},\mathbf{v})_{\Omega_f}+\theta(\nabla\mathbf{e}^{n+1},\nabla\mathbf{v})_{\Omega_f}\\&\quad-\theta(\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}^n),\nabla\mathbf{v})_{\Omega_f}+B(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{v})-B(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v})\\&=b(\mathbf{v},p(t_{n+1})-p^{n+1})+(R^n,\mathbf{v})_{\Omega_f}+c_\Gamma(\phi(t_{n+1})-\phi^n,\mathbf{v}).\end{aligned}\tag{4.15}$$

Letting $\mathbf{v}=2\Delta t\mathbf{e}^{n+1}$ in (4.15) and noting that $\nabla\cdot\mathbf{e}^{n+1}=0$, we can deduce

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}^{n}\|^2_{\Omega_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n}\|^2_{\Omega_f}+2\Delta t(\nu+\theta)\|\mathbf{e}^{n+1}\|^2_{H_f}\\&\quad+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^{n}\|\mathbf{e}^{i+1}\cdot\tau\|^2_{\Gamma}-2\Delta t\theta(\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}^n),\nabla\mathbf{e}^{n+1})_{\Omega_f}\\&\quad+2\Delta tB(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\mathbf{e}^{n+1})-2\Delta tB(\mathbf{u}^n,\mathbf{u}^n,\mathbf{e}^{n+1})\\&\le2\Delta t(R^n,\mathbf{e}^{n+1})_{\Omega_f}+2\Delta tc_\Gamma(\phi(t_{n+1})-\phi^n,\mathbf{e}^{n+1}).\end{aligned}$$

Using (4.6)-(4.9), we can deduce

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}^{n}\|^2_{\Omega_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n}\|^2_{\Omega_f}+2\Delta t(\nu+\theta)\|\mathbf{e}^{n+1}\|^2_{H_f}+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^{n}\|\mathbf{e}^{i+1}\cdot\tau\|^2_{\Gamma}\\&\le8\nu^{-1}\Delta tc_1^2(\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}+\|\mathbf{u}(t_n)\|^2_{H^2(\Omega_f)})\|\mathbf{e}^n\|^2_{\Omega_f}+C\Delta t^3\|\mathbf{u}(t_n)\|^2_{H^2(\Omega_f)}\|\mathbf{u}_t(t_{n+1})\|^2_{\Omega_f}\\&\quad+C\theta^2\Delta t^3+\frac{C\Delta t}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\eta^n\|^2_{\Omega_p}+\frac{\nu\Delta t}{2}\|\mathbf{e}^{n+1/2}\|^2_{H_f}.\end{aligned}$$

Summing it with (4.13), we can deduce

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}-\|\mathbf{e}^{n}\|^2_{\Omega_f}+\|\mathbf{e}^{n+1}-\mathbf{e}^{n}\|^2_{\Omega_f}+2\Delta t(\nu+\theta)\|\mathbf{e}^{n+1}\|^2_{H_f}+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^{n}\|\mathbf{e}^{i+1}\cdot\tau\|^2_{\Gamma}\\&\quad+gS_0\left(\|\eta^{n+1}\|^2_{\Omega_p}-\|\eta^n\|_{\Omega_p}+\|\eta^{n+1}-\eta^n\|^2_{\Omega_p}\right)+2g\Delta t\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|_{\Omega_p}\\&\le8\nu^{-1}\Delta tc_1^2(\|\mathbf{u}^n\|^2_{H^2(\Omega_f)}+\|\mathbf{u}(t_n)\|^2_{H^2(\Omega_f)})\|\mathbf{e}^n\|^2_{\Omega_f}+C\Delta t^3\|\mathbf{u}(t_n)\|^2_{H^2(\Omega_f)}\|\mathbf{u}_t(t_{n+1})\|^2_{\Omega_f}\\&\quad+C\theta^2\Delta t^3+\frac{C\Delta t}{K_{\min}}\|\mathbf{K}^{1/2}\nabla\eta^n\|^2_{\Omega_p}+\frac{\nu\Delta t}{2}\|\mathbf{e}^{n+1/2}\|^2_{H_f}+Cg\Delta t\|\mathbf{e}^n\|^2_{H_f}\\&\quad+\frac{g\Delta t}{4}\|\mathbf{K}^{1/2}\nabla\eta^{n+1}\|^2_{H_p}.\end{aligned}$$

Summing it form 0 to n and using Gronwall's lemma twice, we can deduce

$$\begin{aligned}&\|\mathbf{e}^{n+1}\|^2_{\Omega_f}+\sum_{i=0}^{n}\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|^2_{\Omega_f}+\sum_{i=0}^{n}\|\mathbf{e}^{i+1/2}-\mathbf{e}^{i}\|^2_{\Omega_f}+\Delta t(\nu+\theta)\sum_{i=0}^{n}\|\mathbf{e}^{i+1}\|^2_{H_f}\\&\quad+\frac{\alpha\sqrt{\nu g}}{\sqrt{K_{\max}}}\sum_{i=0}^{n}\|\mathbf{e}^{i+1}\cdot\tau\|^2_{\Gamma}+\theta\Delta t\left(\|\mathbf{e}^{n+1}\|^2_{H_f}+\sum_{i=0}^{n}\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|^2_{H_f}\right)\end{aligned}$$

$$+\Delta t\nu\sum_{i=0}^{n}(\|\mathbf{e}^{i+1}\|_{H_f}^2+\|\mathbf{e}^{i+1}-\mathbf{e}^{i+1/2}\|_{H_f}^2)+gS_0\left(\|\eta^{n+1}\|_{\Omega_p}^2+\sum_{i=0}^{n}\|\eta^{i+1}-\eta^i\|_{\Omega_p}^2\right)$$
$$+\Delta t\sum_{i=0}^{n}\|\mathbf{K}^{1/2}\nabla\eta^{i+1}\|_{\Omega_p}\le C\Delta t^2.$$

We finish the proof. □

5 Numerical experiments

In order to show the performance of our method, we give some numerical results in this section. Let $\Omega_f=[0,1]\times[1,2]$ and $\Omega_p=[0,1]\times[0,1]$ with interface $\Gamma=(0,1)\times\{1\}$. The exact solution is

$$\begin{aligned}
u_1(x,y,t)&=[x^2(y-1)^2+y]\cos(t),\\
u_2(x,y,t)&=\left[-\frac{2}{3}x(y-1)^3\right]\cos(t)+[2-\pi\sin(\pi x)]\cos(t),\\
p(x,y,t)&=[2-\pi\sin(\pi x)]\sin(0.5\pi y)\cos(t),\\
\phi(x,y,t)&=[2-\pi\sin(\pi x)][1-y-\cos(\pi y)]\cos(t).
\end{aligned}$$

The initial conditions, boundary conditions and the forcing terms are given by the exact solution. The finite element spaces choosing the MINI elements ($P1b-P1$) for the Navier-Stokes equation in Ω_f and the linear Lagrangian elements ($P1$) for the Darcy flow in Ω_p. Here, we use the software package FreeFEM++ [28] for our program.

We introduce a more accurate approach [36,38] to examine the orders of convergence with respect to the time step Δt or the mesh size h due to the approximation errors $\mathcal{O}(\Delta t^{r_1})+\mathcal{O}(h^{r_2})$. Then, we can assume that

$$v_h^{\Delta t}(x,t_m)\approx v(x,t_m)+C_1(x,t_m)\Delta t^{r_1}+C_2(x,t_m)h^{r_2}.$$

Thus, we can get

$$\begin{aligned}
\rho_{v,h,i}&=\frac{\|v_h^{\Delta t}(x,t_m)-v_{h/2}^{\Delta t}(x.t_m)\|_i}{\|v_{h/2}^{\Delta t}(x,t_m)-v_{h/4}^{\Delta t}(x.t_m)\|_i}\approx\frac{4^{r_2}-2^{r_2}}{2^{r_2}-1},\\
\rho_{v,\Delta t,i}&=\frac{\|v_h^{\Delta t}(x,t_m)-v_h^{\Delta t/2}(x.t_m)\|_i}{\|v_h^{\Delta t/2}(x,t_m)-v_h^{\Delta t/4}(x.t_m)\|_i}\approx\frac{4^{r_1}-2^{r_1}}{2^{r_1}-1}.
\end{aligned}$$

Here, v can be u,p or ϕ and i can be 0 or 1. We can see that $\rho_{v,h,i}$, $\rho_{v,\Delta t,i}$ approach 4.0 or 2.0, the convergence order will be 2.0 or 1.0, respectively.

Firstly, we focus on the convergence orders with respect to the spacing step h, we study the errors with a fixed time step $\Delta t=0.01$ and varying spacing steps $h=1/2,1/4,1/8,1/16$, and $1/32$, respectively. Tables 1, 3 and 5 present the errors between

Table 1: The numerical results at $T=1$ with $\Delta t=0.01$, $\nu=1.0$ and $\theta=0.05$ for different h.

$1/h$	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
2	0.232865	0.401518	0.598926	0.628749
4	0.0746141	0.210431	0.188994	0.35896
8	0.0467415	0.117158	0.0471332	0.187352
16	0.044809	0.077671	0.0114311	0.0950967
32	0.0447816	0.064098	0.011852	0.0483665

Table 2: Convergence order of $\mathcal{O}(h^r)$ $T=1$ with $\Delta t=0.01$, $\nu=1.0$ and $\theta=0.05$ for different h.

$1/h$	$\|u_h^N-u_{h/2}^N\|_0$	$\|\nabla(u_h^N-u_{h/2}^N)\|_0$	$\|\phi_h^N-\phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N-\phi_{h/2}^N)\|_0$
4	0.186536	1.36784	0.160306	1.40745
8	0.0502415	0.710559	0.0519997	0.798709
16	0.0126097	0.357623	0.0147521	0.416217
32	0.00314492	0.178817	0.00382829	0.209643
	$\rho_{u,h,0}$	$\rho_{u,h,1}$	$\rho_{\phi,h,0}$	$\rho_{\phi,h,1}$
8	3.71279	1.92502	3.08283	1.76215
16	3.98436	1.98689	3.5249	1.91898
32	4.00954	1.99994	3.85344	1.98536

Table 3: The numerical results at $T=1$ with $\Delta t=0.01$, $\nu=0.01$ and $\theta=4.0$ for different h.

$1/h$	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
2	0.333976	0.831983	0.592141	0.629023
4	0.0693539	0.431048	0.195919	0.358844
8	0.0685211	0.252043	0.0542478	0.187652
16	0.0775053	0.198443	0.0178825	0.0960285
32	0.0808675	0.186711	0.0140446	0.0505207

Table 4: Convergence order of $\mathcal{O}(h^r)$ $T=1$ with $\Delta t=0.01$, $\nu=0.01$ and $\theta=4.0$ for different h.

$1/h$	$\|u_h^N-u_{h/2}^N\|_0$	$\|\nabla(u_h^N-u_{h/2}^N)\|_0$	$\|\phi_h^N-\phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N-\phi_{h/2}^N)\|_0$
4	0.314933	3.25999	0.15549	1.40043
8	0.0831566	1.64711	0.0518011	0.797107
16	0.0200227	0.789583	0.014627	0.415866
32	0.00534704	0.397218	0.00381673	0.209583
	$\rho_{u,h,0}$	$\rho_{u,h,1}$	$\rho_{\phi,h,0}$	$\rho_{\phi,h,1}$
8	3.78723	1.97922	3.00167	1.7569
16	4.15311	2.08605	3.54148	1.91674
32	3.74463	1.98778	3.83233	1.98425

the numerical results and the exact solutions for $\nu=1.0$, 0.01 and 0.001, respectively. We can see that the errors diminish with the spacing step h changing small. Tables 2, 4 and 6 show the convergence orders with respect to the spacing step h for $\nu=1.0$, 0.01 and 0.001, respectively. We can see that $\rho_{u,h,0}$ and $\rho_{\phi,h,0}$ are nearly 4.0 and $\rho_{u,h,1}$, $\rho_{\phi,h,1}$ approach 2.0. It suggests that the convergence order in space for the L^2-norm of u_h and ϕ_h are $\mathcal{O}(h^2)$,

Table 5: The numerical results at $T=1$ with $\Delta t=0.01,\nu=0.001$ and $\theta=4.0$ for different h.

$1/h$	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
2	0.349949	0.925095	0.591913	0.629033
4	0.0746775	0.500333	0.195544	0.358865
8	0.0732168	0.291469	0.0536393	0.187717
16	0.0820637	0.223925	0.0177584	0.0961669
32	0.0854242	0.208026	0.0145202	0.0508133

Table 6: Convergence order of $\mathcal{O}(h^r)$ $T=1$ with $\Delta t=0.01$, $\nu=0.001$ and $\theta=4.0$ for different h.

$1/h$	$\|u_h^N-u_{h/2}^N\|_0$	$\|\nabla(u_h^N-u_{h/2}^N)\|_0$	$\|\phi_h^N-\phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N-\phi_{h/2}^N)\|_0$
4	0.33257	3.64765	0.155497	1.40032
8	0.0881533	1.89566	0.0519193	0.797389
16	0.0213933	0.924649	0.0146362	0.416026
32	0.0057549	0.467865	0.00381922	0.209667
	$\rho_{u,h,0}$	$\rho_{u,h,1}$	$\rho_{\phi,h,0}$	$\rho_{\phi,h,1}$
8	3.77263	1.92421	2.99498	1.75613
16	4.12061	2.05015	3.54733	1.91668
32	3.7174	1.97632	3.83224	1.98422

Table 7: The numerical results at $T=1$ with $h=1/32$, $\nu=1.0$ and $\theta=4.0$ for different Δt.

Δt	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
0.5	0.0600635	0.0684851	0.0231833	0.0493908
0.25	0.027584	0.0476142	0.00652645	0.0477285
0.125	0.0237243	0.0454769	0.00438339	0.0476608
0.0625	0.0208114	0.0436585	0.00369908	0.0476051
0.03125	0.0168278	0.0410535	0.00317619	0.0475391

Table 8: Convergence order of $\mathcal{O}(\Delta t^r)$ $T=1$ with $h=1/32$, $\nu=1.0$ and $\theta=4.0$ for different Δt.

Δt	$\|u_h^N-u_{h/2}^N\|_0$	$\|\nabla(u_h^N-u_{h/2}^N)\|_0$	$\|\phi_h^N-\phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N-\phi_{h/2}^N)\|_0$
0.25	0.178712	0.466978	0.0624593	0.465792
0.125	0.0430235	0.126154	0.0156514	0.117084
0.0625	0.00985868	0.0422007	0.00384549	0.0289859
0.03125	0.0079011	0.036303	0.00106173	0.0082662
	$\rho_{u,\Delta t,0}$	$\rho_{u,\Delta t,1}$	$\rho_{\phi,\Delta t,0}$	$\rho_{\phi,\Delta t,1}$
0.125	4.15383	3.70166	3.99066	3.97827
0.0625	4.36402	2.98937	4.07005	4.03934
0.03125	1.24776	1.16246	3.6219	3.50656

the convergence order in space for the H^1-norm of u_h and ϕ_h are $\mathcal{O}(h^1)$.

Then, we focus on the convergence orders with respect to the time step Δt, we study the errors with a fixed time step $h=1/32$ and varying time steps $\Delta t=0.5,0.25,0.125,0.0625$ and 0.03125, respectively. Tables 7, 9 and 11 present the errors between the numerical results and the exact solutions for $\nu=1.0$, 0.01 and 0.001, respectively. We can see that the

Table 9: The numerical results at $T=0.01$ with $h=1/32$, $\nu=0.01$ and $\theta=4.0$ for different Δt.

Δt	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
0.5	0.0815619	0.0991371	0.029804	0.0507619
0.25	0.0293946	0.063132	0.0065347	0.0477487
0.125	0.0237782	0.0579196	0.00417864	0.0476713
0.0625	0.0203077	0.0537641	0.00353974	0.0476047
0.03125	0.0160209	0.0492438	0.00306932	0.0475339

Table 10: Convergence order of $\mathcal{O}(\Delta t^r)$ $T=1$ with $h=1/32$, $\nu=1.0$ and $\theta=4.0$ for different Δt.

Δt	$\|u_h^N-u_{h/2}^N\|_0$	$\|\nabla(u_h^N-u_{h/2}^N)\|_0$	$\|\phi_h^N-\phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N-\phi_{h/2}^N)\|_0$
0.25	0.188199	0.414847	0.0645817	0.478491
0.125	0.0444021	0.120286	0.0159055	0.118791
0.0625	0.0109503	0.0573838	0.0037441	0.0284255
0.03125	0.00857969	0.0583418	0.00103058	0.00794661
	$\rho_{u,\Delta t,0}$	$\rho_{u,\Delta t,1}$	$\rho_{\phi,\Delta t,0}$	$\rho_{\phi,\Delta t,1}$
0.125	4.23852	3.44883	4.06033	4.028
0.0625	4.05487	2.09617	4.24816	4.17904
0.03125	1.27631	0.983579	3.63302	3.57706

Table 11: The numerical results at $T=0.01$ with $h=1/32$, $\nu=0.001$ and $\theta=4.0$ for different Δt.

Δt	$\frac{\|u(T)-u_h^N\|_0}{\|u\|_0}$	$\frac{\|\nabla(u(T)-u_h^N)\|_0}{\|u\|_0}$	$\frac{\|\phi(T)-\phi_h^N\|_0}{\|\phi(T)\|_0}$	$\frac{\|\nabla(\phi(T)-\phi_h^N)\|_0}{\|\nabla\phi(T)\|_0}$
0.5	0.0821624	0.0999722	0.0297175	0.0507326
0.25	0.0295736	0.0636198	0.00653324	0.0477496
0.125	0.0239263	0.0583415	0.00417921	0.0476718
0.0625	0.0204167	0.0541207	0.00353977	0.047605
0.03125	0.0160864	0.0495441	0.00306868	0.0475339

Table 12: Convergence order of $\mathcal{O}(\Delta t^r)$ $T=1$ with $h=1/32$,$\nu=0.001$ and $\theta=4.0$ for different Δt.

Δt	$\|u_h^N-u_{h/2}^N\|_0$	$\|\nabla(u_h^N-u_{h/2}^N)\|_0$	$\|\phi_h^N-\phi_{h/2}^N\|_0$	$\|\nabla(\phi_h^N-\phi_{h/2}^N)\|_0$
0.25	0.188326	0.413749	0.0645731	0.478325
0.125	0.0444182	0.120054	0.0159067	0.118797
0.0625	0.0110042	0.0578813	0.00374262	0.0284173
0.03125	0.00866925	0.0590004	0.00103091	0.00794814
	$\rho_{u,\Delta t,0}$	$\rho_{u,\Delta t,1}$	$\rho_{\phi,\Delta t,0}$	$\rho_{\phi,\Delta t,1}$
0.125	4.23984	3.44635	4.05949	4.02642
0.0625	4.03646	2.07414	4.25015	4.18043
0.03125	1.26934	0.981033	3.63041	3.57533

errors diminish with the time step Δt changing small. Tables 8, 10 and 12 show the convergence orders with respect to the time step Δt for $\nu=1.0$, 0.01 and 0.001, respectively. We can see that $\rho_{u,\Delta t,0}$, $\rho_{\phi,\Delta t,0}$, $\rho_{u,\Delta t,1}$ and $\rho_{\phi,\Delta t,1}$ approach 4.0. It suggests that the convergence order in time for the L^2-norm of u_h and ϕ_h and the H^1-norm of u_h and ϕ_h are super convergence, which is an interesting thing.

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References

[1] I. Babuška and G. N. Gatica, *A residual-based a posteriori error estimator for the Stokes-Darcy coupled problem,* SIAM J. Numer. Anal., 48(2) (2010), pp. 498–523.
[2] S. Badia and R. Codina, *Unified stabilized finite element formulations for the Stokes and the Darcy problems,* SIAM J. Numer. Anal., 47(3) (2009), pp. 1971–2000.
[3] L. Badea, M. Discacciati and A. Quarteroni, *Numerical analysis of the Navier-Stokes/Darcy coupling,* Numer. Math., 115(2) (2010), pp. 195–227.
[4] G. S. Beavers and D. D. Joseph, *Boundary conditions at a naturally permeable wall,* J. Fluid Mech., 30 (1967), pp. 197–207.
[5] J. Blasco and R. Codina, *Error estimates for an operator-splitting method for incompressible flows,* Appl. Numer. Math., 51 (2004), pp. 1–17.
[6] J. Blasco, R. Codina and A. Huerta, *A fractional-step method for the incompressible Navier-Stokes equations related to a predictor-multicorrector algorithm,* Int. J. Numer. Methods Fluids, 28 (1998), pp. 1391–1419.
[7] M. C. Cai, M. Mu and J. C. Xu, *Numerical solution to a mixed Navier-Stokes/Darcy model by the two-grid approach,* SIAM J. Numer. Anal., 47(5) (2009), pp. 3325–3338.
[8] J. Camaño, G. N. Gatica, R. Oyarzúa, R. Ruiz-Baier and P. Venegas, *New fully-mixed finite element methods for the Stokes-Darcy coupling,* Comput. Method. Appl. Mech. Eng., 295 (2015), pp. 362–395.
[9] Y. Z. Cao, M. Gunzburger, X. M. He and X. M. Wang, *Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition,* Numer. Math., 117(4) (2011), pp. 601–629.
[10] Y. Z. Cao, M. Gunzburger, X. L. Hu, F. Hua, X. M. Wang and W. D. Zhao, *Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions,* SIAM J. Numer. Anal., 47(6) (2010), pp. 4239–4256.
[11] A. J. Chorin, *Numerical solution of the Navier-Stokes equations,* Math. Comput., 22 (1968), pp. 745–762.
[12] W. B. Chen, M. Gunzburger, F. Hua and X. M. Wang, *A parallel Robin-Robin domain decomposition method for the Stokes-Darcy system,* SIAM J. Numer. Anal., 49(3) (2011), pp. 1064–1084.
[13] R. Codina, *Pressure stability in fractional step finite element methods for incompressible flows,* J. Comput. Phys., 170 (2001), pp. 112–140.
[14] R. Codina and S. Badia, *On some pressure segregation methods of fractional-step type for the finite element approximation of incompressible flow problems,* Comput. Method Appl. Mech. Eng., 195 (2006), pp. 2900–2918.
[15] M. Cui and N. N. Yan, *A posteriori error estimate for the Stokes-Darcy system,* Math. Method Appl. Sci., 34(9) (2011), pp. 1050–1064.
[16] M. Discacciati and A. Quarteroni, *Navier-Stokes/Darcy coupling: Modeling, analysis and numerical approximation,* Rev. Mat. Complut., 22(2) (2009), pp. 315–426.

[17] M. DISCACCIATI, A. QUARTERONI AND A. VALLI, *Robin-Robin domain decomposition methods for the Stokes-Darcy coupling,* SIAM J. Numer. Anal., 45(3) (2007), pp. 1246–1268.

[18] X. FENG, Y. HE AND P. HUANG, *A stabilized implicit fractional-step method for the time-dependent Navier–Stokes equations using equal-order pairs,* J. Math. Anal. Appl., 392 (2012), pp. 209–224.

[19] X. FENG, Y. HE AND D. LIU, *Convergence analysis of an implicit fractional-step method for the incompressible Navier-Stokes equations,* Appl. Math. Model, 35 (2011), pp. 5856–5871.

[20] E. FERNÁ-CARA AND M. MARÍN BELTRÁN, *The convergence of two numerical schemes for the Navier-Stokes equations,* Numer. Math., 55 (1989), pp. 33–60.

[21] V. GIRAULT AND P. A. RAVIART, Finite Element Method for Navier–Stokes Equations: Theory and Algorithms, Springer-Verlag, Berlin, Herdelberg, 1987.

[22] F. GUILLÉN-GONZÁLEZ AND M. V. REDONDO-NEBLE, *New error estimates for a viscosity-splitting scheme in time for the three-dimensional Navier-Stokes equations,* IMA J. Numer. Anal., 31 (2011), pp. 556–579.

[23] G. N. GATICA, R. OYARZUA AND F. J. SAYAS, *Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem,* Math. Comput., 80(276) (2011), pp. 1911–1948.

[24] G. N. GATICA, A. MÁRQUEZ, R. OYARZÚA AND R. REBOLLEDO, *Analysis of an augmented fully-mixed approach for the coupling of quasi-newtonian fluids and porous media,* Comput. Method Appl. Mech. Eng., 270 (2014), pp. 76–112.

[25] V. GIRAULT AND B. RIVIÉRE, *DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition,* SIAM J. Numer. Anal., 47(3) (2009), pp. 2052–2089.

[26] F. GUILLÉN-GONZÁLEZ AND M. V. REDONDO-NEBLE, *New error estimates for a viscosity-splitting scheme in time for the three-dimensional Navier-Stokes equations,* IMA J. Numer. Anal., 31 (2011), pp. 556–579.

[27] R. GLOWINSKI, T. W. PAN AND J. PERIAUX, *A fictitious domain method for external incompressible viscous flow modeled by Navier-Stokes equations,* Comput. Methods Appl. Mech. Eng., 112 (1994), pp. 133–148.

[28] F. HECHT, *New development in FreeFem++,* J. Numer. Math., 20 (2012), pp. 251–265.

[29] Y. HOU, *Optimal error estimates of a decoupled scheme based on two-grid finite element for mixed Stokes-Darcy model,* Appl. Math. Lett., 57 (2016), pp. 90–96

[30] G. KANSCHAT AND B. RIVIERE, *A strongly conservative finite element method for the coupling of Stokes and Darcy flow,* J. Comput. Phys., 29(17) (2010), pp. 5933–5943.

[31] W. L. LAYTON, H. TRAN AND X. XIONG, *Long time stability of four methods for splitting the evolutionary Stokes-Darcy problems into Stokes and Darcy subproblems,* J. Comput. Appl. Math., 236(13) (2012), pp. 3198–3217.

[32] A. MÁRQUEZ, S. MEDDAHI AND F.-J. SAYAS, *A decoupled preconditioning technique for a mixed Stokes-Darcy model,* J. Sci. Comput., 57 (2013), pp. 174–192.

[33] A. MÁRQUEZ, S. MEDDAHI AND F.-J. SAYAS, *Strong coupling of finite element methods for the Stokes-Darcy problem,* IMA J. Numer. Anal., 35(2) (2015), pp. 969–988.

[34] M. MORAITI, *On the quasistatic approximation in the Stokes-Darcy model of groundwater-surface water flows,* J. Math. Anal. Appl., 394 (2012), pp. 796–808.

[35] M. MU AND J. C. XU, *A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow,* SIAM J. Numer. Anal., 45(5) (2007), pp. 1801–1813.

[36] M. MU AND X. H. ZHU, *Decoupled schemes for a non-stationary mixed Stokes-Darcy model,* Math. Comput., 79(270) (2010), pp. 707–731.

[37] P. G. SAFFMAN, *On the boundary at the surface of a porous medium,* Studies Appl. Math., 50

(1971), pp. 93–101.

[38] L. Shan, H. Zheng and W. J. Layton, *A decoupling method with different subdomain time steps for the nonstationary Stokes-Darcy model,* Numer. Methods Partial Differential Equations, 29 (2013), pp. 549–583.

[39] Z. Si, Y. Wang and S. Li, *Decoupled modified characteristics finite element method for the time dependent Navier-Stokes/Darcy problem,* Math. Method Appl. Sci., 37 (2014), pp. 1392–1404.

[40] R. Temam, *Une méthode d'approximations de la solution des equations de Navier-Stokes,* Bull. Soc. Math. Fr., 98 (1968), pp. 115–152.

[41] D. Vassilev and I. Yotov, *Coupling Stokes-Darcy flow with transport,* SIAM J. Sci. Comput., 31(5) (2009), pp. 3661–3684.

[42] T. Zhang, D. Pedro and J. Yuan, *A large time stepping viscosity-splitting finite element method for the viscoelastic flow problem,* Adv. Comput. Math., 41 (2015), pp. 149–190.

[43] T. Zhang and J. Y. Yuan, *Two novel decoupling algorithms for the steady Stokes-Darcy model based on two-grid discretizations,* Discrete Cont. Dyn. B, 19 (2014), pp. 849–865.

[44] L. Zuo and Y. Hou, *A decoupling two-grid algorithm for the mixed Stokes-Darcy model with the Beavers-Joseph interface condition,* Numer. Methods Partial Differential Equations, 30(3) (2014), pp. 1066–1082