Explicit High Order One-Step Methods for Decoupled Forward Backward Stochastic Differential Equations

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Abstract. By using the Feynman-Kac formula and combining with Itô-Taylor expansion and finite difference approximation, we first develop an explicit third order one-step method for solving decoupled forward backward stochastic differential equations. Then based on the third order one, an explicit fourth order method is further proposed. Several numerical tests are also presented to illustrate the stability and high order accuracy of the proposed methods.

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1 Introduction

Let $T > 0$ be a deterministic terminal time and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ denote a filtered complete probability space with the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ of an m-dimensional Brownian motion $W = (W_t)_{0 \leq t \leq T}$. Consider the decoupled forward backward stochastic differential equations (FBSDEs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$

$$
\begin{cases}
X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \\
Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s,
\end{cases}
\tag{1.1}
$$

where $t \in [0, T]$, $X_0 \in \mathcal{F}_0$ is an initial condition; $b: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are, respectively, the drift and diffusion coefficients of stochastic differential equations

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(SDEs); $\varphi : \mathbb{R}^d \to \mathbb{R}^n$ is the terminal condition and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \to \mathbb{R}^n$ is the generator of backward stochastic differential equations (BSDEs). A triple (X_t, Y_t, Z_t) is called an L^2-adapted solution of the FBSDE (1.1) if it is $\mathcal{F}_t$-adapted, square integrable and satisfies (1.1).

Pardoux and Peng [22] first proved the existence and uniqueness of the adapted solution of the nonlinear BSDEs. Then by using the solutions of BSDEs, Peng [24] gave a probabilistic interpretation for quasilinear parabolic partial differential equations (PDEs). Since then, the study on FBSDEs has been extensively conducted due to its applications in research on PDEs [6, 17, 24], mathematical finance [11, 15, 20], stochastic optimal control [14, 23], and mean-field BSDEs [2, 3, 26, 28], to name a few. However, the analytic solutions of FBSDEs are seldom known. Hence, it is important and popular to solve FBSDEs with some numerical methods.

Up to now, there have been a considerable number of numerical methods for solving BSDEs [4, 5, 12, 13, 31–33, 36, 37] and decoupled FBSDEs [7–10, 18, 19, 25, 29, 30, 34, 35, 38, 39] in literatures. Nevertheless, except some of the multistep methods [30, 34, 37, 39], few of one-step methods can achieve high order convergence exceeding two. In [31], the authors constructed a class of third-order one-step methods for solving BSDEs whose driver f is independent of Z. This is the first attempt to study the third order one-step method for solving BSDEs. However, this one-step method can only solve BSDEs with f not taking Z as input, which causes significant limitations in application.

In this paper, we aim to design two high order one-step methods for solving general decoupled FBSDEs by extending the method given in [31]. Based on the Feynman-Kac formula, by combining with the Itô-Taylor expansions and the high order finite difference approximations, we first develop an explicit third order one-step method containing a parameter θ for FBSDEs. Then by taking $\theta = \frac{1}{2}$ and utilizing the prediction-correction method, we further propose a fourth order one-step method for FBSDEs. To attest the stability and high order accuracy of the proposed methods, we carry out some numerical experiments. All of the numerical results show that both methods are stable and high order accurate, and the fourth order method can achieve a fourth order accuracy when the weak order 4.0 Itô-Taylor scheme is used to solve SDEs.

The rest of the paper is organized as follows. In Section 2, some preliminaries on the Feynman-Kac formula, Itô-Taylor expansion, and finite difference approximation are presented, then based on which we propose an explicit third order one-step method and an explicit fourth order one-step method for decoupled FBSDEs in Section 3. In Section 4, some numerical tests are presented to show the stability and high order accuracy of the proposed methods. Finally, we give some conclusions in Section 5.

2 Preliminaries

2.1 The Feynman-Kac formula

Let $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ denote the solution of the FBSDE (1.1) with the forward SDE starting from the time-space point (t,x), i.e.,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r, \\ Y_s^{t,x} = \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \end{cases} \tag{2.1}$$

where the superscript ^{t,x} shows that the forward SDE starts at point (t,x). If no confusion is possible, this superscript is omitted.

Let $C_b^{l,k,k}$ denote the set of functions $\phi(t,x,y)$, which have uniformly bounded partial derivatives $\partial_t^{l_1}\phi$ and $\partial_x^{k_1}\partial_y^{k_2}\phi$ for $l_1 \leq l$ and $k_1 + k_2 \leq k$. Now we recall the following Feynman-Kac formula.

Lemma 2.1 (Feynman-Kac formula). *Let f be continuous with respect to s and uniformly Lipschitz continuous with respect to (x,y,z). Assume $\varphi \in C_b^{2+\alpha}$ for some $\alpha \in (0,1)$. Then the solution $(Y_s^{t,x}, Z_s^{t,x})$ of the BSDE in (2.1) can be represented as:*

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad Z_s^{t,x} = (u_x'\sigma)(s, X_s^{t,x}), \tag{2.2}$$

where $u(t,x)$ is the smooth solution of the following PDE

$$u_t'(t,x) + (u_x'b)(t,x) + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m}(u_{x_ix_j}''\sigma_{ik}\sigma_{jk})(t,x) + f(t,x,u(t,x),(u_x'\sigma)(t,x)) = 0 \tag{2.3}$$

with the terminal condition $u(T,x) = \varphi(x)$, where

$$u_x'(t,x) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_d} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_p}{\partial x_1} & \frac{\partial u_p}{\partial x_2} & \cdots & \frac{\partial \varphi_p}{\partial x_d} \end{pmatrix}, \quad u_t'(t,x) = \begin{pmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \vdots \\ \frac{\partial u_p}{\partial t} \end{pmatrix}, \quad u_{x_ix_j}''(t,x) = \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_i \partial x_j} \\ \frac{\partial^2 u_2}{\partial x_i \partial x_j} \\ \vdots \\ \frac{\partial^2 u_p}{\partial x_i \partial x_j} \end{pmatrix}.$$

Remark 2.1. For the solution of the PDE (2.3), we also have the following conclusion [12]: If $b, \sigma \in C_b^{1+k,3+2k}$, $f \in C_b^{1+k,3+2k,3+2k,3+2k}$ and $\varphi \in C_b^{3+2k+\alpha}$ for some $\alpha \in (0,1)$, then we have

$$u \in C_b^{1+k,2+2k}, \quad k = 0,1,\cdots.$$

2.2　The Itô-Taylor expansion

In this subsection, we recall the Itô-Taylor expansion which plays a fundamental role in constructing our high order one-step schemes. For simplicity, we consider $d=m=1$. To proceed, we introduce some definitions first.

i) Multi-index. Let $\alpha:=(j_1,j_2,\cdots,j_l)$ be a multi-index with $j_i\in\{0,1,\cdots,m\}$, $i=1,2,\cdots,l$. Let $l(\alpha)=l$ be the length of α, and $n(\alpha)$ be the number of the components of α which are equal to zero. Denote by v the multi-index of length zero, i.e., $l(v)=0$. Let $\mathcal{M}$ be the set of all multi-indices, i.e.,

$$\mathcal{M}=\{(j_1,j_2,\cdots,j_l):j_i\in\{0,1,\cdots,m\},\ i\in\{1,\cdots,l\},\ l=1,2,3,\cdots\}\cup\{v\}.$$

Given $\alpha\in\mathcal{M}$ with $l(\alpha)\geq 1$, $-\alpha$ and $\alpha-$ are two multi-indices obtained by deleting the first and the last component of α, respectively. In particular, we let α^+ be the multiple index after removing all the zero elements from α, and denote

$$\alpha=(\underbrace{0,0,\cdots,0}_{p})$$

by $\alpha=(0)_p$.

ii) Multiple Itô integrals. The multiple Itô integral operator I_α on adapted right continuous stochastic processes $\{f=f(t),\ t\geq 0\}$ with left limits, is defined by

$$I_\alpha[f(\cdot)]_{\rho,\tau}:=\begin{cases} f(\tau), & l=0, \\ \displaystyle\int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}ds, & l\geq 1 \quad\text{and}\quad j_l=0, \\ \displaystyle\int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}dW_s^{j_l}, & l\geq 1 \quad\text{and}\quad j_l\geq 1, \end{cases}$$

where ρ and τ are two stopping times satisfying $0\leq\rho(w)\leq\tau(w)\leq T$, a.s.. In particular, $I_\alpha[1]_{0,\tau}$ will be written as $I_{\alpha,\tau}$, and $I_\alpha[1]_{\rho,\tau}$ as $I_{\alpha,\rho,\tau}$.

iii) Itô coefficient functions. For each index $\alpha=(j_1,\cdots,j_l)$ and a function $f\in C^r(\mathbb{R}^+\times\mathbb{R}^d)$ with $r=l(\alpha)+n(\alpha)$, the coefficient function f_α is defined by

$$f_\alpha=\begin{cases} f, & l=0, \\ L^{j_1}f_{-\alpha}, & l\geq 1, \end{cases}$$

where the operators L^j, $j=0,1,\cdots,m$ are defined as

$$L^0 f(t,x)=\frac{\partial f}{\partial t}(t,x)+\sum_i b_i(t,x)\frac{\partial f}{\partial x_i}(t,x)+\frac{1}{2}\sum_{i,j}(\sigma\sigma^\top)_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i\partial x_j}(t,x), \tag{2.4a}$$

$$L^j f(t,x)=\sum_{k=1}^d\frac{\partial f}{\partial x^k}(t,x)\sigma_{kj}(t,x), \quad j=1,\cdots,m. \tag{2.4b}$$

iv) Hierarchical and remainder sets. We call a subset $\mathcal{A} \subset \mathcal{M}$ a hierarchical set if

$$\mathcal{A} \neq \varnothing, \quad \sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty \quad \text{and} \quad -\alpha \in \mathcal{A} \quad \text{for each} \quad \alpha \in \mathcal{A} \backslash \{v\},$$

and its remainder set $\mathcal{B}(\mathcal{A})$ is defined by

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \mathcal{A} : -\alpha \in \mathcal{A}\}.$$

With the above notations, we now state the Itô-Taylor expansion in the following lemma (see Chapter 5 in [16]).

Lemma 2.2 (Itô-Taylor expansion). *Let ρ and τ be two stopping times with $0 \leq \rho(w) \leq \tau(w) \leq T$, and $\mathcal{A} \subset \mathcal{M}$ be a hierarchical set. For a function $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, the following Itô-Taylor expansion holds:*

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} f_\alpha(\rho, X_\rho) I_{\alpha, \rho, \tau} + \sum_{\beta \in \mathcal{B}(\mathcal{A})} I_\beta [f_\beta(\cdot, X.)]_{\rho, \tau}, \tag{2.5}$$

providing that all of the derivatives of b, σ and f and all of the multiple Itô integrals appearing in (2.5) exist.

Moreover, for multiple stochastic integrals, the following lemma holds [16].

Lemma 2.3. *If $\alpha \in \mathcal{M} \backslash \{\varnothing\}$, $l(\alpha) \neq n(\alpha)$, and $t_0 \leq \rho \leq \tau \leq T < \infty$, then*

$$\mathbb{E}\left[I_\alpha[f(\cdot)]_{\rho, \tau} | \mathcal{F}_\rho\right] = 0, \quad w.p.1,$$

provided that the stochastic integral $I_\alpha[f(\cdot)]_{\rho, \tau}$ exists.

For multiple index $\alpha = (j_1, j_2, \ldots, j_l) \in \mathcal{M}$, let $L^\alpha = L^{j_1} \circ L^{j_2} \circ \cdots \circ L^{j_l}$, and define

$$\mathfrak{g}^\alpha := \{u : [0, T] \times \mathbb{R} \to \mathbb{R} | \ L^\alpha u \text{ exists and is continous}\},$$
$$\mathfrak{g}_b^\alpha := \{u : [0, T] \times \mathbb{R} \to \mathbb{R} | \ u \in \mathfrak{g}^\alpha \text{ and } L^\alpha u \text{ is bounded}\}.$$

Denote $u(t, X_t)$ and u_t^α by, respectively, u_t and $L^\alpha u_t$. Let r be a given small positive number, then we can derive the following theorem [16, 31].

Theorem 2.1. *If $p \geq 0$ and $u \in \mathfrak{g}_b^{(0)^{p+1}}$, then it holds that*

$$\mathbb{E}\left[u(t + r, X_{t+r}) | \mathcal{F}_t\right] = u_t + r u_t^{(0)} + \frac{r^2}{2!} u_t^{(0,0)} + \cdots + \frac{r^p}{p!} u_t^{(0)^p} + \mathcal{O}(r^{p+1}). \tag{2.6}$$

Proof. By using Lemma 2.3, it is easy to deduce (2.6). For more details, please refer to [31]. We omit it here. □

2.3 Finite difference approximations

In this subsection, we recall some high order finite difference approximations, which play a key role in proposing our high order one-step schemes.

For a given function $g = g(x)$, we define the following two difference quotient operators

$$D_h^0 g(x) = \frac{-g(x+2h) + 8g(x+h) - 8g(x-h) + g(x-2h)}{12h}, \tag{2.7a}$$

$$D_h^2 g(x) = \frac{-g(x+2h) + 16g(x+h) - 30g(x) + 16g(x-h) - g(x-2h)}{12h^2}, \tag{2.7b}$$

with a positive real number h. Using Taylor expansion, it is easy to deduce

$$D_h^0 g(x) - g'(x) = \mathcal{O}(h^4), \quad g \in C_b^5, \tag{2.8a}$$

$$D_h^2 g(x) - g''(x) = \mathcal{O}(h^4), \quad g \in C_b^6. \tag{2.8b}$$

We take D_h^0 for instance. Let $g \in C_b^5$, then by Taylor expansions

$$g(x+h) = g(x) + hg'(x) + \frac{h^2}{2}g''(x) + \frac{h^3}{3!}g'''(x) + \frac{h^4}{4!}g^{(4)}(x) + \frac{h^5}{5!}g^{(5)}(\xi_1), \tag{2.9a}$$

$$g(x-h) = g(x) - hg'(x) + \frac{h^2}{2}g''(x) - \frac{h^3}{3!}g'''(x) + \frac{h^4}{4!}g^{(4)}(x) - \frac{h^5}{5!}g^{(5)}(\xi_2), \tag{2.9b}$$

$$g(x+2h) = g(x) + 2hg'(x) + 2h^2 g''(x) + \frac{8h^3}{3!}g'''(x) + \frac{16h^4}{4!}g^{(4)}(x) + \frac{32h^5}{5!}g^{(5)}(\xi_3), \tag{2.9c}$$

$$g(x-2h) = g(x) - 2hg'(x) + 2h^2 g''(x) - \frac{8h^3}{3!}g'''(x) + \frac{16h^4}{4!}g^{(4)}(x) - \frac{32h^5}{5!}g^{(5)}(\xi_4), \tag{2.9d}$$

where $\xi_1 \in [x, x+h]$, $\xi_2 \in [x-h, x]$, $\xi_3 \in [x, x+2h]$ and $\xi_4 \in [x-2h, x]$. Then by (2.7) and (2.9), we get

$$D_h^0 g(x) - g'(x) = \frac{h^4}{180}\left(g^{(5)}(\xi_1) + g^{(5)}(\xi_2) - 4g^{(5)}(\xi_3) - 4g^{(5)}(\xi_4)\right) = \mathcal{O}(h^4). \tag{2.10}$$

The equations in (2.8) imply that $D_h^0 g(x)$ and $D_h^2 g(x)$ approximate, respectively, the first-order derivative $g'(x)$ and the second-order derivative $g''(x)$ with error $\mathcal{O}(h^4)$.

3 Explicit one-step schemes for FBSDEs

We introduce the following regular time partition over interval $[0, T]$

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

Then for $n = 0, 1, \cdots, N-1$, we define

$$\Delta t_n = t_{n+1} - t_n, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n}.$$

3.1 Numerical schemes for SDEs

In this subsection, we introduce some Itô-Taylor type schemes for solving SDEs [16].

Let X^n denote the approximation of the solution X_t of SDE in (1.1) at time $t = t_n$, and define $b^n = b(t_n, X^n)$ and $\sigma^n = \sigma(t_n, X^n)$. Then we present the following five Itô-Taylor schemes for SDEs.

1. The Euler Scheme:
$$X^{n+1} = X^n + b^n \Delta t_n + \sigma^n \Delta W_n. \tag{3.1}$$

2. The Milstein scheme:
$$X^{n+1} = X^n + b^n \Delta t_n + \sigma^n \Delta W_n + \frac{1}{2} \sigma^n \sigma_x^n \left((\Delta W_n)^2 - \Delta t_n \right). \tag{3.2}$$

3. The weak order 2.0 Itô-Taylor scheme:
$$\begin{aligned}
X^{n+1} = {} & X^n + b^n \Delta t_n + \sigma^n \Delta W_n + \frac{1}{2} \sigma^n \sigma_x^n \left((\Delta W_n)^2 - \Delta t_n \right) \\
& + \frac{1}{2} \left(b_t^n + b^n b_x^n + \frac{1}{2} (\sigma^n)^2 b_{xx}^n \right) (\Delta t_n)^2 \\
& + \frac{1}{2} \left(\sigma^n b_x^n + \sigma_t^n + b^n \sigma_x^n + \frac{1}{2} (\sigma^n)^2 \sigma_{xx}^n \right) \Delta t_n \Delta W_n.
\end{aligned} \tag{3.3}$$

4. The weak order 3.0 Itô-Taylor scheme:
$$\begin{aligned}
X^{n+1} = {} & X^n + b^n \Delta t_n + \sigma^n \Delta W_n + \frac{1}{2} L^1 \sigma^n \left((\Delta W_n)^2 - \Delta t_n \right) \\
& + L^1 b^n \xi_n + \frac{1}{2} L^0 b^n (\Delta t_n)^2 + L^0 \sigma^n (\Delta W_n \Delta t_n - \xi_n) \\
& + \frac{1}{6} \left(L^0 L^0 \sigma^n + L^0 L^1 b^n + L^1 L^0 b^n \right) \Delta W_n (\Delta t_n)^2 \\
& + \frac{1}{6} \left(L^1 L^1 b^n + L^1 L^0 \sigma^n + L^0 L^1 \sigma^n \right) \left((\Delta W_n)^2 - \Delta t_n \right) \Delta t_n \\
& + \frac{1}{6} L^0 L^0 b^n (\Delta t_n)^3 + \frac{1}{6} L^1 L^1 \sigma^n \left((\Delta W_n)^2 - 3\Delta t_n \right) \Delta W_n.
\end{aligned} \tag{3.4}$$

5. The weak order 4.0 Itô-Taylor scheme:
$$\begin{aligned}
X^{n+1} = {} & X^n + b^n \Delta t_n + \sigma^n \Delta W_n + \frac{1}{2} L^0 b^n (\Delta t_n)^2 + L^1 b^n \xi_n + L^0 \sigma^n (\Delta W_n \Delta t_n - \xi_n) \\
& + L^1 L^1 b^n \left(2 \Delta W_n \xi_n - \frac{5}{6} (\Delta W_n)^2 \Delta t_n - \frac{1}{6} (\Delta t_n)^2 \right) \\
& + \frac{1}{6} \left(L^0 L^0 \sigma^n + L^0 L^1 b^n + L^1 L^0 b^n \right) \Delta W_n (\Delta t_n)^2 \\
& + \frac{1}{6} L^0 L^0 b^n (\Delta t_n)^3 + \frac{1}{24} L^0 L^0 L^0 b^n (\Delta t_n)^4 \\
& + \frac{1}{24} L^1 L^1 L^1 b^n \Delta W_n \left((\Delta W_n)^2 - 3\Delta t_n \right) \Delta t_n
\end{aligned}$$

$$+\frac{1}{24}\left(L^1L^0L^0b^n+L^0L^1L^0b^n+L^0L^0L^1b^n+L^0L^0L^0\sigma^n\right)\Delta W_n(\Delta t_n)^3$$

$$+\frac{1}{24}\left(L^1L^1L^0b^n+L^0L^1L^1b^n+L^1L^0L^1b^n\right)\left((\Delta W_n)^2-\Delta t_n\right)(\Delta t_n)^2. \tag{3.5}$$

Here ξ_n and ΔW_n are correlated normal random variables satisfying

$$\xi_n\sim N\left(0,\frac{1}{3}(\Delta t_n)^3\right),\quad \mathbb{E}[\Delta W_n\xi_n]=\frac{1}{2}(\Delta t_n)^2.$$

3.2 Discretisation of FBSDEs

Let (X_t,Y_t,Z_t) be the unique solution of (1.1), then for $n=0,1,\cdots,N-1$, we get

$$Y_{t_n}=Y_{t_{n+1}}+\int_{t_n}^{t_{n+1}}f(s,X_s,Y_s,Z_s)ds-\int_{t_n}^{t_{n+1}}Z_sdW_s. \tag{3.6}$$

Taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_{t_n},X_{t_n}=x]$ on both sides of (3.6), we obtain

$$Y_{t_n}=\mathbb{E}_{t_n}^x[Y_{t_{n+1}}]+\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^x[f(s,X_s,Y_s,Z_s)]ds. \tag{3.7}$$

By the Feynman-Kac formula (2.2), we have

$$Y_t=u(t,X_t),\quad Z_t=(u_x'\sigma)(t,X_t), \tag{3.8}$$

where u is the smooth solution of (2.3). Then by (2.3), (2.4) and (3.8), we deduce

$$L^0u(t,X_t)=-f\left(t,X_t,u(t,X_t),(u_x'\sigma)(t,X_t)\right)=-f(t,X_t,Y_t,Z_t), \tag{3.9}$$

i.e., $u^{(0)}(t,X_t)=-f(t,X_t,Y_t,Z_t)$. Assume that $u\in\mathfrak{g}_b^{(0)5}$, and then by Theorem 2.1 and the fact $\mathbb{E}_{t_n}^x[\cdot]=\mathbb{E}_{t_n}^x\left[\mathbb{E}[\cdot|\mathcal{F}_{t_n}]\right]$, we get

$$\mathbb{E}_{t_n}^x\left[f(s,X_s,Y_s,Z_s)\right]=-\mathbb{E}_{t_n}^x\left[u^{(0)}(s,X_s)\right]=-\mathbb{E}_{t_n}^x\left[\mathbb{E}\left[u^{(0)}(s,X_s)|\mathcal{F}_{t_n}\right]\right]$$

$$=-\left(u_{t_n}^{(0)}+(s-t_n)u_{t_n}^{(0)2}+\frac{(s-t_n)^2}{2!}u_{t_n}^{(0)3}+\frac{(s-t_n)^3}{3!}u_{t_n}^{(0)4}+\mathcal{O}\left((s-t_n)^4\right)\right),$$

which leads to

$$\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^x\left[f(s,X_s,Y_s,Z_s)\right]ds$$

$$=-\left(\Delta t_nu_{t_n}^{(0)}+\frac{(\Delta t_n)^2}{2!}u_{t_n}^{(0)2}+\frac{(\Delta t_n)^3}{3!}u_{t_n}^{(0)3}+\frac{(\Delta t_n)^4}{4!}u_{t_n}^{(0)4}+\mathcal{O}\left((\Delta t_n)^5\right)\right). \tag{3.10}$$

Let A denote the approximation of $\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^x[f(s,X_s,Y_s,Z_s)]ds$, and we write

$$\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^x\left[f(s,X_s,Y_s,Z_s)\right]ds=A+R_{Y1}^n, \tag{3.11}$$

where R_{Y1}^n is the approximation error and

$$A = -\left(\theta \Delta t_n u_{t_n}^{(0)} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x\left[u_{t_{n+1}}^{(0)}\right] + (\Delta t_n)^2 \alpha u_{t_n}^{(0)_2} + (\Delta t_n)^2 \beta \mathbb{E}_{t_n}^x\left[u_{t_{n+1}}^{(0)_2}\right]\right), \quad (3.12)$$

with $\theta \in [0,1]$, α and β being parameters. Note that, by Theorem 2.1, we deduce

$$\begin{aligned}
A = & -\left(\theta \Delta t_n u_{t_n}^{(0)} + (1-\theta)\Delta t_n \left[u_{t_n}^{(0)} + \Delta t_n u_{t_n}^{(0)_2} + \frac{(\Delta t_n)^2}{2!} u_{t_n}^{(0)_3} + \frac{(\Delta t_n)^3}{3!} u_{t_n}^{(0)_4}\right]\right. \\
& + (\Delta t_n)^2 \alpha u_{t_n}^{(0)_2} + (\Delta t_n)^2 \beta \left[u_{t_n}^{(0)_2} + \Delta t_n u_{t_n}^{(0)_3} + \frac{(\Delta t_n)^2}{2!} u_{t_n}^{(0)_4}\right] + \mathcal{O}\left((\Delta t_n)^5\right)\right) \\
= & -\left(\Delta t_n u_{t_n}^{(0)} + (1-\theta+\alpha+\beta)(\Delta t_n)^2 u_{t_n}^{(0)_2} + \left(\frac{1-\theta}{2}+\beta\right)(\Delta t_n)^3 u_{t_n}^{(0)_3}\right. \\
& \left. + \left(\frac{1-\theta}{6}+\frac{\beta}{2}\right)(\Delta t_n)^4 u_{t_n}^{(0)_4} + \mathcal{O}\left((\Delta t_n)^5\right)\right).
\end{aligned} \quad (3.13)$$

Define the parameters α and β in A as

$$\alpha = \frac{3\theta-1}{6}, \quad \beta = \frac{3\theta-2}{6}, \quad (3.14)$$

and we obtain

$$A = -\left(\Delta t_n u_{t_n}^{(0)} + \frac{1}{2!}(\Delta t_n)^2 u_{t_n}^{(0)_2} + \frac{1}{3!}(\Delta t_n)^3 u_{t_n}^{(0)_3} + \frac{\theta}{12}(\Delta t_n)^4 u_{t_n}^{(0)_4} + \mathcal{O}\left((\Delta t_n)^5\right)\right). \quad (3.15)$$

Then by (3.10), (3.11) and (3.15), we deduce

$$\begin{aligned}
R_{Y1}^n = & \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f(s,X_s,Y_s,Z_s)]ds - A \\
= & \left(\frac{\theta}{12}-\frac{1}{24}\right)(\Delta t_n)^4 u_{t_n}^{(0)_4} + \mathcal{O}\left((\Delta t_n)^5\right),
\end{aligned}$$

which implies that

$$R_{Y1}^n = \begin{cases} \mathcal{O}((\Delta t)^4), & \theta \neq 0.5, \\ \mathcal{O}((\Delta t)^5), & \theta = 0.5. \end{cases} \quad (3.16)$$

By using (3.7), (3.11), (3.12) and (3.14), we arrive at

$$\begin{aligned}
Y_{t_n} = & \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] - \Delta t_n \left(\theta u_{t_n}^{(0)} + (1-\theta)\mathbb{E}_{t_n}^x[u_{t_{n+1}}^{(0)}]\right) \\
& - (\Delta t_n)^2 \left(\frac{3\theta-1}{6} u_{t_n}^{(0)_2} + \frac{3\theta-2}{6}\mathbb{E}_{t_n}^x[u_{t_{n+1}}^{(0)_2}]\right) + R_{Y1}^n.
\end{aligned} \quad (3.17)$$

To write $u_{t_n}^{(0)_2}$ in (3.17) in an explicit form, we apply the operator L^0 to (3.9) to get

$$
\begin{aligned}
-u_t^{(0)_2} =& L^0 f\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right) = \left(\frac{\partial}{\partial t} + b_t \frac{\partial}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2}{\partial x^2}\right) f\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right) \\
=& f'_{t,1} + f'_{t,3}\frac{\partial u_t}{\partial t} + f'_{t,4}\left(\frac{\partial \sigma_t}{\partial t}\frac{\partial u_t}{\partial x} + \sigma_t \frac{\partial^2 u_t}{\partial x \partial t}\right) \\
&+ b_t\left(f'_{t,2} + f'_{t,3}\frac{\partial u_t}{\partial x} + f'_{t,4}\left(\frac{\partial \sigma_t}{\partial x}\frac{\partial u_t}{\partial x} + \sigma_t \frac{\partial^2 u_t}{\partial x^2}\right)\right) \\
&+ \frac{1}{2}\sigma_t^2 \frac{\partial}{\partial x}\left(f'_{t,2} + f'_{t,3}\frac{\partial u_t}{\partial x} + f'_{t,4}\left(\frac{\partial \sigma_t}{\partial x}\frac{\partial u_t}{\partial x} + \sigma_t \frac{\partial^2 u_t}{\partial x^2}\right)\right),
\end{aligned}
\tag{3.18}
$$

where $b_t = b(t, X_t)$ and $\sigma_t = \sigma(t, X_t)$. Here and in the sequel, we define

$$
\begin{aligned}
f_t &= f\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right), & f'_{t,1} &= \frac{\partial f}{\partial t}\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right), \\
f'_{t,2} &= \frac{\partial f}{\partial x}\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right), & f'_{t,3} &= \frac{\partial f}{\partial y}\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right), \\
f'_{t,4} &= \frac{\partial f}{\partial z}\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right), & f''_{t,22} &= \frac{\partial^2 f}{\partial x^2}\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right),
\end{aligned}
$$

and define $b'_{t,i}$, $\sigma'_{t,i}$, $b''_{t,ij}$, $\sigma''_{t,ij}$, $(i,j=1,2)$, and $f''_{t,ij}$, $(i,j=2,3,4)$ in a similar way. By (3.9) and the definition of L^0, it holds that

$$
\frac{\partial u_t}{\partial t} = -\left(b_t \frac{\partial u_t}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 u_t}{\partial x^2} + f\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right)\right),
\tag{3.19}
$$

which leads to

$$
\begin{aligned}
\frac{\partial^2 u_t}{\partial x \partial t} =& -\frac{\partial}{\partial x}\left(b_t \frac{\partial u_t}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 u_t}{\partial x^2} + f\left(t, X_t, u_t, \sigma_t \frac{\partial u_t}{\partial x}\right)\right) \\
=& -\left(\frac{\partial b_t}{\partial x}\frac{\partial u_t}{\partial x} + b_t \frac{\partial^2 u_t}{\partial x^2} + \sigma_t \frac{\partial \sigma_t}{\partial x}\frac{\partial^2 u_t}{\partial x^2} + \frac{1}{2}\sigma_t^2 \frac{\partial^3 u_t}{\partial x^3}\right. \\
&\left. + f'_{t,2} + f'_{t,3}\frac{\partial u_t}{\partial x} + f'_{t,4}\left(\frac{\partial \sigma_t}{\partial x}\frac{\partial u_t}{\partial x} + \sigma_t \frac{\partial^2 u_t}{\partial x^2}\right)\right).
\end{aligned}
\tag{3.20}
$$

Inserting (3.19) and (3.20) into (3.18), we deduce

$$
\begin{aligned}
-u_t^{(0)_2} =& f'_{t,1} + f'_{t,2}(b_t - \sigma_t f'_{t,4}) - f'_{t,3}f_t + f'_{t,4}\left(\sigma'_{t,1} + b_t\sigma'_{t,2} + \frac{1}{2}\sigma_t^2 \sigma''_{t,22}\right)\frac{\partial u_t}{\partial x} \\
&- f'_{t,4}\sigma_t\left((b'_{t,2} + f'_{t,3} + f'_{t,4}\sigma'_{t,2})\frac{\partial u_t}{\partial x} + \sigma_t f'_{t,4}\frac{\partial^2 u_t}{\partial x^2}\right) \\
&+ \frac{1}{2}\sigma_t^2\left(f''_{t,22} + 2f''_{t,23}\frac{\partial u_t}{\partial x} + 2f''_{t,24}\left(\sigma'_{t,2}\frac{\partial u_t}{\partial x} + \sigma_t \frac{\partial^2 u_t}{\partial x^2}\right) + f''_{t,33}\left(\frac{\partial u_t}{\partial x}\right)^2\right.
\end{aligned}
$$

$$+2f''_{t,34}\frac{\partial u_t}{\partial x}\left(\sigma'_{t,2}\frac{\partial u_t}{\partial x}+\sigma_t\frac{\partial^2 u_t}{\partial x^2}\right)+f''_{t,44}\left(\sigma'_{t,2}\frac{\partial u_t}{\partial x}+\sigma_t\frac{\partial^2 u_t}{\partial x^2}\right)^2\right)$$

$$=F\left(t,X_t,u_t,\frac{\partial u_t}{\partial x},\frac{\partial^2 u_t}{\partial x^2}\right),\tag{3.21}$$

where $F(t,x,y,z,\gamma):[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Let

$$F_t=F\left(t,X_t,u_t,\frac{\partial u_t}{\partial x},\frac{\partial^2 u_t}{\partial x^2}\right),$$

then by (3.9), (3.17) and (3.21), we have

$$Y_{t_n}=\mathbb{E}^x_{t_n}[Y_{t_{n+1}}]+\theta\Delta t_n f_{t_n}+(1-\theta)\Delta t_n\mathbb{E}^x_{t_n}[f_{t_{n+1}}]$$
$$+\frac{(3\theta-1)(\Delta t_n)^2}{6}F_{t_n}+\frac{(3\theta-2)(\Delta t_n)^2}{6}\mathbb{E}^x_{t_n}[F_{t_{n+1}}]+R^n_{Y1}.\tag{3.22}$$

By (2.7) and (3.8), the solution Z_t can be represented as

$$Z_t=\sigma_t\frac{\partial u_t}{\partial x}=\sigma_t D^0_h u_t+\mathcal{O}(h^4),\tag{3.23}$$

where $D^0_h u_t=D^0_h Y_t(X_t)$. Then we define

$$f_{t,h}=f\left(t,X_t,u_t,\sigma_t D^0_h u_t\right),\tag{3.24a}$$
$$f'_{t,h,i}=f'_{t,i}\left(t,X_t,u_t,\sigma_t D^0_h u_t\right),\qquad(i=1,2,3,4),\tag{3.24b}$$
$$f''_{t,h,ij}=f''_{t,ij}\left(t,X_t,u_t,\sigma_t D^0_h u_t\right),\qquad(i,j=2,3,4),\tag{3.24c}$$

and denote by

$$F_{t,h}=f'_{t,h,1}+f'_{t,h,2}(b_t-\sigma_t f'_{t,h,4})-f'_{t,h,3}f_{t,h}+f'_{t,h,4}\left(\sigma'_{t,1}+b_t\sigma'_{t,2}+\frac{1}{2}\sigma^2_t\sigma''_{t,22}\right)D^0_h u_t$$
$$-f'_{t,h,4}\sigma_t\left((b'_{t,2}+f'_{t,h,3}+f'_{t,h,4}\sigma'_{t,2})D^0_h u_t+\sigma_t f'_{t,h,4}D^2_h u_t\right)$$
$$+\frac{1}{2}\sigma^2_t\left(f''_{t,h,22}+2f''_{t,h,23}D^0_h u_t+2f''_{t,h,24}(\sigma'_{t,2}D^0_h u_t+\sigma_t D^2_h u_t)+f''_{t,h,33}(D^0_h u_t)^2\right.$$
$$\left.+2f''_{t,h,34}D^0_h u_t(\sigma'_{t,2}D^0_h u_t+\sigma_t D^2_h u_t)+f''_{t,h,44}(\sigma'_{t,2}D^0_h u_t+\sigma_t D^2_h u_t)^2\right),\tag{3.25}$$

where $D^2_h u_t=D^2_h Y_t(X_t)$. Then by (3.22), (3.24) and (3.25), we get

$$Y_{t_n}=\mathbb{E}^x_{t_n}[Y_{t_{n+1}}]+\theta\Delta t_n f_{t_n,h}+(1-\theta)\Delta t_n\mathbb{E}^x_{t_n}[f_{t_{n+1},h}]$$
$$+\frac{(3\theta-1)(\Delta t_n)^2}{6}F_{t_n,h}+\frac{(3\theta-2)(\Delta t_n)^2}{6}\mathbb{E}^x_{t_n}[F_{t_{n+1},h}]+R^n_{Y1}+R^n_{Y2},\tag{3.26}$$

where

$$
\begin{aligned}
R_{Y2}^n = {} & \theta \Delta t_n \left(f_{t_n} - f_{t_n,h} \right) + (1-\theta) \Delta t_n \mathbb{E}_{t_n}^x \left[f_{t_{n+1}} - f_{t_{n+1},h} \right] \\
& + \frac{(3\theta-1)(\Delta t_n)^2}{6} \left(F_{t_n} - F_{t_n,h} \right) + \frac{(3\theta-2)(\Delta t_n)^2}{6} \mathbb{E}_{t_n}^x \left[F_{t_{n+1}} - F_{t_{n+1},h} \right].
\end{aligned}
\tag{3.27}
$$

To solve Y_{t_n} in an explicit way, we first use Euler scheme to approximate Y_{t_n} in $f_{t_n,h}$ and $F_{t_n,h}$, which yields

$$
Y_{t_n} = \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^x [f_{t_{n+1},h}] + R_{y1}^n = \bar{Y}_{t_n} + R_{y1}^n,
\tag{3.28}
$$

where R_{y1}^n is the approximation error and

$$
\bar{Y}_{t_n} = \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^x [f_{t_{n+1},h}].
\tag{3.29}
$$

Moreover, we let

$$
\bar{f}_{t_n,h} = f \left(t_n, x, \bar{Y}_{t_n}(x), D_h^0 \bar{Y}_{t_n}(x) \sigma(t_n,x) \right),
\tag{3.30}
$$

and further approximate Y_{t_n} in $f_{t_n,h}$ and $F_{t_n,h}$ by Crank-Nicolson scheme

$$
Y_{t_n} = \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \frac{1}{2} \Delta t_n \bar{f}_{t_n,h} + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^x [f_{t_{n+1},h}] + R_{y2}^n = \bar{\bar{Y}}_{t_n} + R_{y2}^n,
\tag{3.31}
$$

where R_{y2}^n is the approximation error and

$$
\bar{\bar{Y}}_{t_n} = \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \frac{1}{2} \Delta t_n \bar{f}_{t_n,h} + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^x [f_{t_{n+1},h}].
\tag{3.32}
$$

Denote by

$$
\bar{\bar{f}}_{t_n,h} = f \left(t_n, x, \bar{\bar{Y}}_{t_n}(x), D_h^0 \bar{\bar{Y}}_{t_n}(x) \sigma(t_n,x) \right),
\tag{3.33a}
$$

$$
\bar{\bar{F}}_{t_n,h} = F_{t_n,h} \left(t_n, x, \bar{\bar{Y}}_{t_n}(x), D_h^0 \bar{\bar{Y}}_{t_n}(x), D_h^2 \bar{\bar{Y}}_{t_n}(x) \right).
\tag{3.33b}
$$

Then by (3.26) and (3.33), we get the first reference equation for solving Y_t

$$
\begin{aligned}
Y_{t_n} = {} & \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \theta \Delta t_n \bar{\bar{f}}_{t_n,h} + (1-\theta) \Delta t_n \mathbb{E}_{t_n}^x [f_{t_{n+1},h}] \\
& + \frac{(3\theta-1)(\Delta t_n)^2}{6} \bar{\bar{F}}_{t_n,h} + \frac{(3\theta-2)(\Delta t_n)^2}{6} \mathbb{E}_{t_n}^x [F_{t_{n+1},h}] + R_{y3}^n,
\end{aligned}
\tag{3.34}
$$

where $R_{y3}^n = R_{Y1}^n + R_{Y2}^n + R_{Y3}^n$ with

$$
R_{Y3}^n = \theta \Delta t_n \left(f_{t_n,h} - \bar{\bar{f}}_{t_n,h} \right) + \frac{(3\theta-1)(\Delta t_n)^2}{6} \left(F_{t_n,h} - \bar{\bar{F}}_{t_n,h} \right).
\tag{3.35}
$$

Now based on (3.34), we turn to constructing the second reference equation for Y_t. To this end, we take $\theta = \frac{1}{2}$ in (3.26) to obtain

$$
\begin{aligned}
Y_{t_n} = & \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \frac{1}{2}\Delta t_n f_{t_n,h} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x \left[f_{t_{n+1},h} \right] \\
& + \frac{1}{12}(\Delta t_n)^2 F_{t_n,h} - \frac{1}{12}(\Delta t_n)^2 \mathbb{E}_{t_n}^x \left[F_{t_{n+1},h} \right] + \hat{R}_{Y1}^n + \hat{R}_{Y2}^n,
\end{aligned} \tag{3.36}
$$

where $\hat{R}_{Y1}^n$ and $\hat{R}_{Y2}^n$ are the truncation error terms R_{Y1}^n and R_{Y2}^n with $\theta = \frac{1}{2}$. By removing the truncation error term R_{y3}^n in (3.34), we have

$$
\begin{aligned}
\hat{Y}_{t_n} = & \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \theta \Delta t_n \bar{\bar{f}}_{t_n,h} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x \left[f_{t_{n+1},h} \right] \\
& + \frac{(3\theta-1)(\Delta t_n)^2}{6} \bar{\bar{F}}_{t_n,h} + \frac{(3\theta-2)(\Delta t_n)^2}{6} \mathbb{E}_{t_n}^x \left[F_{t_{n+1},h} \right].
\end{aligned} \tag{3.37}
$$

Let

$$
\hat{f}_{t_n,h} = f\left(t_n, x, \hat{Y}_{t_n}(x), D_h^0 \hat{Y}_{t_n}(x)\sigma(t_n,x)\right), \tag{3.38a}
$$

$$
\hat{F}_{t_n,h} = F_{t_n,h}\left(t_n, x, \hat{Y}_{t_n}(x), D_h^0 \hat{Y}_{t_n}(x), D_h^2 \hat{Y}_{t_n}(x)\right). \tag{3.38b}
$$

Then by (3.36) and (3.38), we get the second reference equation for solving Y_t

$$
\begin{aligned}
Y_{t_n} = & \mathbb{E}_{t_n}^x [Y_{t_{n+1}}] + \frac{1}{2}\Delta t_n \hat{f}_{t_n,h} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x \left[f_{t_{n+1},h} \right] \\
& + \frac{1}{12}(\Delta t_n)^2 \hat{F}_{t_n,h} - \frac{1}{12}(\Delta t_n)^2 \mathbb{E}_{t_n}^x \left[F_{t_{n+1},h} \right] + R_{y4}^n,
\end{aligned} \tag{3.39}
$$

where $R_{y4}^n = \hat{R}_{Y1}^n + \hat{R}_{Y2}^n + \hat{R}_{Y3}^n$ with

$$
\hat{R}_{Y3}^n = \frac{1}{2}\Delta t_n \left(f_{t_n,h} - \hat{f}_{t_n,h} \right) + \frac{1}{12}(\Delta t_n)^2 \left(F_{t_n,h} - \hat{F}_{t_n,h} \right). \tag{3.40}
$$

3.3   The explicit one-step schemes for FBSDEs

Let (X^n, Y^n, Z^n) denote the numerical approximation of the solution (X_t, Y_t, Z_t) of the FB-SDE (1.1) at time level $t = t_n$, $(n = N, \cdots, 0)$. For simplicity, we define $f^{n,h}$ and $F^{n,h}$ by

$$
\begin{aligned}
f^{n,h} &= f\left(t_n, X^n, Y^n(X^n), D_h^0 Y^n(X^n)\sigma(t_n, X^n)\right), \\
F^{n,h} &= F\left(t_n, X^n, Y^n(X^n), D_h^0 Y^n(X^n), D_h^2 Y^n(X^n)\right).
\end{aligned}
$$

Based on the reference equations (3.23), (3.29), (3.32) and (3.34), we first propose the following explicit third order one-step scheme for solving the FBSDE (1.1).

Scheme 3.1. Given random variable Y^N, for $n = N-1, \cdots, 0$, we solve the random variables $Y^n = Y^n(X^n)$ and $Z^n = Z^n(X^n)$ by

$$\bar{Y}^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}],$$

$$\bar{\bar{Y}}^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \frac{1}{2}\Delta t_n \bar{f}^{n,h} + \frac{1}{2}\Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}],$$

$$Y^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \theta \Delta t_n \bar{\bar{f}}^{n,h} + (1-\theta)\Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}]$$
$$+ \frac{(3\theta-1)(\Delta t_n)^2}{6}\bar{\bar{F}}^{n,h} + \frac{(3\theta-2)(\Delta t_n)^2}{6}\mathbb{E}^x_{t_n}[F^{n+1,h}],$$

$$Z^n = D^0_h Y^n \sigma(t_n, X^n),$$

where $\theta \in [0,1]$ and

$$\bar{f}^{n,h} = f\big(t_n, X^n, \bar{Y}^n(X^n), D^0_h \bar{Y}^n(X^n)\sigma(t_n, X^n)\big),$$

$$\bar{\bar{f}}^{n,h} = f\big(t_n, X^n, \bar{\bar{Y}}^n(X^n), D^0_h \bar{\bar{Y}}^n(X^n)\sigma(t_n, X^n)\big),$$

$$\bar{\bar{F}}^{n,h} = F\big(t_n, X^n, \bar{\bar{Y}}^n(X^n), D^0_h \bar{\bar{Y}}^n(X^n), D^2_h \bar{\bar{Y}}^n(X^n)\big),$$

with X^n being solved by Itô-Taylor schemes.

Furthermore, based on the reference equations (3.23), (3.29), (3.32), (3.37) and (3.39), we construct the following explicit fourth order one-step scheme for the FBSDE (1.1).

Scheme 3.2. Given random variable Y^N, for $n = N-1, \cdots, 0$, we solve the random variables $Y^n = Y^n(X^n)$ and $Z^n = Z^n(X^n)$ by

$$\bar{Y}^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}],$$

$$\bar{\bar{Y}}^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \frac{1}{2}\Delta t_n \bar{f}^{n,h} + \frac{1}{2}\Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}],$$

$$\hat{Y}^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \theta \Delta t_n \bar{\bar{f}}^{n,h} + (1-\theta)\Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}]$$
$$+ \frac{(3\theta-1)(\Delta t_n)^2}{6}\bar{F}^{n,h} + \frac{(3\theta-2)(\Delta t_n)^2}{6}\mathbb{E}^x_{t_n}[F^{n+1,h}],$$

$$Y^n = \mathbb{E}^x_{t_n}[Y^{n+1}] + \frac{1}{2}\Delta t_n \hat{f}^{n,h} + \frac{1}{2}\Delta t_n \mathbb{E}^x_{t_n}[f^{n+1,h}]$$
$$+ \frac{1}{12}(\Delta t_n)^2 \hat{F}^{n,h} - \frac{1}{12}(\Delta t_n)^2 \mathbb{E}^x_{t_n}[F^{n+1,h}],$$

$$Z^n = D^0_h Y^n \sigma(t_n, X^n),$$

where $\theta \in [0,1]$ and

$$\bar{f}^{n,h} = f\big(t_n, X^n, \bar{Y}^n(X^n), D^0_h \bar{Y}^n(X^n)\sigma(t_n, X^n)\big),$$

$$\bar{\bar{f}}^{n,h} = f\big(t_n, X^n, \bar{\bar{Y}}^n_2(X^n), D^0_h \bar{\bar{Y}}^n_2(X^n)\sigma(t_n, X^n)\big),$$

$$\bar{F}^{n,h} = F\big(t_n, X^n, \bar{\bar{Y}}^n_2(X^n), D^0_h \bar{\bar{Y}}^n_2(X^n), D^2_h \bar{\bar{Y}}^n_2(X^n)\big),$$

$$\hat{f}^{n,h} = f\big(t_n, X^n, \hat{Y}^n(X^n), D^0_h \hat{Y}^n(X^n)\sigma(t_n, X^n)\big),$$

$$\hat{F}^{n,h} = F\big(t_n, X^n, \hat{Y}^n(X^n), D^0_h \hat{Y}^n(X^n), D^2_h \hat{Y}^n(X^n)\big),$$

with X^n being solved by Itô-Taylor schemes.

3.4 Error estimates

Now we turn to analyzing the accuracy and convergence of Schemes 3.1 and 3.2. To this end, we give the estimations of the truncation error terms R_{y3}^n and R_{y4}^n in the reference equations (3.34) and (3.39), respectively.

Assumption 3.1. The functions b, σ, f and φ are bounded and smooth enough with bounded derivatives.

To proceed, we first estimate the truncation error terms R_{y1}^n and R_{y2}^n.

Theorem 3.1. *Under Assumption* 3.1, *we have the following local estimates for* $n=0,1,\cdots,N-1$:

$$\left|R_{y1}^n\right| \le C\left((\Delta t_n)^2 + h^4 \Delta t_n\right), \tag{3.41a}$$

$$\left|R_{y2}^n\right| \le C\left((\Delta t_n)^3 + h^4 \Delta t_n\right), \tag{3.41b}$$

where C is a positive constant depending on T and the upper bounds of derivatives of b, σ, f and φ.

Proof. By (3.7) and (3.28), we have

$$R_{y1}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s] ds - \Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1},h}]$$
$$= \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^x[f_s] - \mathbb{E}_{t_n}^x[f_{t_{n+1}}]\right) ds + \Delta t_n \left(\mathbb{E}_{t_n}^x[f_{t_{n+1}} - f_{t_{n+1},h}]\right). \tag{3.42}$$

By Itô-Taylor expansion, it is easy to deduce (see [38])

$$\left|\int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^x[f_s] - \mathbb{E}_{t_n}^x[f_{t_{n+1}}]\right) ds\right| \le C(\Delta t_n)^2. \tag{3.43}$$

Using the accuracy of D_h^0 in (2.7), we get

$$\left|f_t - f_{t,h}\right| \le C\left|Y_t(X_t) - D_h^0 Y_t(X_t)\right| \le Ch^4. \tag{3.44}$$

Eqs. (3.42)–(3.44) lead to the inequality (3.41a). By (3.7) and (3.31), we have

$$R_{y2}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s] ds - \frac{1}{2} \Delta t_n \bar{f}_{t_n,h} - \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1},h}]$$
$$= \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^x[f_s] - \frac{1}{2} f_{t_n} - \frac{1}{2} \mathbb{E}_{t_n}^x[f_{t_{n+1}}]\right) ds$$
$$+ \frac{1}{2} \Delta t_n \left(f_{t_n} - f_{t_n,h}\right) + \frac{1}{2} \Delta t_n \left(f_{t_n,h} - \bar{f}_{t_n,h}\right)$$
$$+ \frac{1}{2} \Delta t_n \left(\mathbb{E}_{t_n}^x[f_{t_{n+1}} - f_{t_{n+1},h}]\right). \tag{3.45}$$

By Itô-Taylor expansion, it is easy to obtain ([38])

$$\left| \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^x [f_s] - \frac{1}{2} f_{t_n} - \frac{1}{2} \mathbb{E}_{t_n}^x [f_{t_{n+1}}] \right) ds \right| \leq C(\Delta t_n)^3. \tag{3.46}$$

Since $R_{y1}^n = Y_{t_n} - \bar{Y}_{t_n}$, then by (2.7) and (3.41a), we deduce

$$\begin{aligned} \left| f_{t_n,h} - \bar{f}_{t_n,h} \right| &\leq C \left(\left| Y_{t_n} - \bar{Y}_{t_n} \right| + \left| D_h^0 Y_{t_n} - D_h^0 \bar{Y}_{t_n} \right| \right) \\ &\leq C \left((\Delta t_n)^2 + h^4 \right). \end{aligned} \tag{3.47}$$

Using (3.44)–(3.47), we get the inequality (3.41b).　　□

For the term $R_{y3}^n = R_{Y1}^n + R_{Y2}^n + R_{Y3}^n$ and $R_{y4}^n = \hat{R}_{Y1}^n + \hat{R}_{Y2}^n + \hat{R}_{Y3}^n$, we have the following theorem.

Theorem 3.2. *Under Assumption 3.1, we have the following local estimates for $n = 0, 1, \cdots, N-1$:*

$$\left| R_{y3}^n \right| \leq C \left((\Delta t_n)^4 + h^4 \Delta t_n \right), \tag{3.48a}$$

$$\left| R_{y4}^n \right| \leq C \left((\Delta t_n)^5 + h^4 \Delta t_n \right), \tag{3.48b}$$

where C is a positive constant depending on T and the upper bounds of derivatives of b, σ, f and φ.

Proof. Utilizing (3.16) and (3.27), it is easy to deduce

$$|R_{Y1}^n| \leq C(\Delta t_n)^4, \quad |R_{Y2}^n| \leq C(h^4 \Delta t_n). \tag{3.49}$$

Moreover, $R_{y2}^n = Y_{t_n} - \bar{\bar{Y}}_{t_n}$, then by (2.7) and (3.41b), we have

$$\left| f_{t_n,h} - \bar{\bar{f}}_{t_n,h} \right| \leq C \left(\left| Y_{t_n} - \bar{\bar{Y}}_{t_n} \right| + \left| D_h^0 Y_{t_n} - D_h^0 \bar{\bar{Y}}_{t_n} \right| \right) \leq C \left((\Delta t_n)^3 + h^4 \right),$$

$$\left| F_{t_n,h} - \bar{\bar{F}}_{t_n,h} \right| \leq C \left(\left| Y_{t_n} - \bar{\bar{Y}}_{t_n} \right| + \left| D_h^0 Y_{t_n} - D_h^0 \bar{\bar{Y}}_{t_n} \right| + \left| D_h^2 Y_{t_n} - D_h^2 \bar{\bar{Y}}_{t_n} \right| \right) \leq C \left((\Delta t_n)^3 + h^4 \right),$$

which imply that

$$\begin{aligned} |R_{Y3}^n| &= \left| \theta \Delta t_n \left(f_{t_n,h} - \bar{\bar{f}}_{t_n,h} \right) + \frac{(3\theta - 1)(\Delta t_n)^2}{6} \left(F_{t_n,h} - \bar{\bar{F}}_{t_n,h} \right) \right| \\ &\leq C \left((\Delta t_n)^4 + h^4 \Delta t_n \right). \end{aligned} \tag{3.50}$$

Eqs. (3.49) and (3.50) lead to the inequality (3.48a).

Note that $\hat{R}_{Y1}^n$ and $\hat{R}_{Y2}^n$ are given by taking $\theta = \frac{1}{2}$ in R_{Y1}^n and R_{Y2}^n. Then by (3.16) and (3.27), we get

$$\left| \hat{R}_{Y1}^n \right| \leq C \left((\Delta t_n)^5 \right), \quad \left| \hat{R}_{Y2}^n \right| \leq C \left(h^4 \Delta t_n \right). \tag{3.51}$$

By (3.34) and (3.37), we have $R_{y3}^n = Y_{t_n} - \hat{Y}_{t_n}$. Then by (2.7) and (3.48a), we deduce

$$
\begin{aligned}
\left| f_{t_n,h} - \hat{f}_{t_n,h} \right| &\leq C \left(\left| Y_{t_n} - \hat{Y}_{t_n} \right| + \left| D_h^0 Y_{t_n} - D_h^0 \hat{Y}_{t_n} \right| \right) \leq C \left((\Delta t_n)^4 + h^4 \right), \\
\left| F_{t_n,h} - \hat{F}_{t_n,h} \right| &\leq C \left(\left| Y_{t_n} - \hat{Y}_{t_n} \right| + \left| D_h^0 Y_{t_n} - D_h^0 \hat{Y}_{t_n} \right| + \left| D_h^2 Y_{t_n} - D_h^2 \hat{Y}_{t_n} \right| \right) \leq C \left((\Delta t_n)^4 + h^4 \right),
\end{aligned}
$$

which imply that

$$
\begin{aligned}
\left| \hat{R}_{Y3}^n \right| &= \left| \frac{1}{2} \Delta t_n \left(f_{t_n,h} - \hat{f}_{t_n,h} \right) + \frac{1}{12} (\Delta t_n)^2 \left(F_{t_n,h} - \hat{F}_{t_n,h} \right) \right| \\
&\leq C \left((\Delta t_n)^5 + h^4 \Delta t_n \right).
\end{aligned}
\tag{3.52}
$$

Using (3.51) and (3.52), we obtain the inequality (3.48b). $\qquad\square$

By Theorem 3.2, we have

$$
\begin{cases}
R_{y3}^n = \mathcal{O}\left((\Delta t_n)^4 \right), & \text{if } h = \mathcal{O}\left((\Delta t_n)^{\frac{3}{4}} \right), \\
R_{y4}^n = \mathcal{O}\left((\Delta t_n)^5 \right), & \text{if } h = \mathcal{O}\left(\Delta t_n \right).
\end{cases}
\tag{3.53}
$$

Hence, we can expect that

1. Scheme 3.1 is convergent with third order when we choose $h = (\Delta t_n)^{\frac{3}{4}}$.

2. Scheme 3.2 is convergent with fourth order when we choose $h = \Delta t_n$.

4 Numerical tests

In this section, we present some numerical experiments to illustrate the stability and accuracy of the proposed schemes. For simplicity, we adopt the uniform time partition, and the time partition number N is given by $N = \frac{T}{\Delta t}$. For some $h > 0$, the space partition $\mathcal{S}_h$ is chosen as

$$
\mathcal{S}_h = \mathcal{S}_{1,h} \times \mathcal{S}_{2,h} \times \cdots \times \mathcal{S}_{d,h},
\tag{4.1}
$$

where $\mathcal{S}_{j,h}$ is the partition of the one-dimensional real axis $\mathbb{R}$

$$
\mathcal{S}_{j,h} = \{ x_i^j : x_i^j = ih, \ i = 0, \pm 1, \cdots, \pm \infty \}
$$

for $j = 1, 2, \cdots, d$.

4.1   The approximation of conditional expectation

By using the distribution of the Brownian motion and the Gauss-Hermite quadrature rule, we show how to approximate the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$.

Without loss of generality, we take $\mathbb{E}_{t_n}^x[Y^{n+1}]$ for example and use the Euler scheme to solve X_t. Then we get

$$
\begin{aligned}
\mathbb{E}_{t_n}^{X^n}[Y^{n+1}] &= \mathbb{E}_{t_n}^{X^n}\left[Y^{n+1}\left(X^{n+1}\right)\right] \\
&= \mathbb{E}\left[Y^{n+1}\left(X^n + b^n \Delta t_n + \sigma^n \Delta W_n\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} Y^{n+1}\left(X^n + b^n \Delta t_n + \sigma^n \sqrt{\Delta t_n}s\right)e^{-\frac{s^2}{2}}ds \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} Y^{n+1}\left(X^n + b^n \Delta t_n + \sigma^n \sqrt{2\Delta t_n}p\right)e^{-p^2}dp \\
&\approx \sum_{i=1}^{M} Y^{n+1}\left(X^n + b^n \Delta t_n + \sigma^n \sqrt{2\Delta t_n}a_i\right)w_i,
\end{aligned}
\tag{4.2}
$$

where $\{a_i\}_{i=1}^M$ are the roots of the Hermite polynomial of degree M and $\{w_i\}_{i=1}^M$ are the corresponding weights [1].

In general,

$$
X_i^n = X^n + b^n \Delta t_n + \sigma^n \sqrt{2\Delta t_n}a_i
$$

are not on the grid points. Hence, the interpolation methods are needed. Let I_h denote the local interpolation operator such that $I_h g(x) = g(x)$ for $x \in S_h$, and we obtain

$$
\mathbb{E}_{t_n}^{X^n}[Y^{n+1}] \approx \sum_{i=1}^{M} I_h Y^{n+1}\left(X^n + b^n \Delta t_n + \sigma^n \sqrt{2\Delta t_n}a_i\right)w_i.
$$

In all our examples, we shall set the number of Gauss-Hermite quadrature points $M = 8$ such that the spacial approximated error caused by the Gauss-Hermite quadrature rule can be neglected.

Remark 4.1. Note that Gauss-Hermite quadrature rule and interpolation methods are used to approximate the conditional expectations in our schemes, which will cause the exponential growth in computation when solving high dimensional FBSDEs.

To solve this problem, in our future study, we shall focus on building an efficient sparse grid spatial discretization, using the sparse grid Gaussian-Hermite quadrature rule to approximate the conditional expectations and adopting an efficient spectral method to deal with the associated high dimensional interpolations.

4.2   Some numerical tests

In our tests, we set the terminal time $T = 1.0$. We also denote by $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ the absolute errors between the exact solution (Y_t, Z_t) of the FBSDE (1.1) at time $t = 0$ and the

numerical solutions (Y^n, Z^n) of Schemes 3.1 and 3.2 at $n=0$. Moreover, we will choose $h=(\Delta t)^{\frac{3}{4}}$ in Scheme 3.1 and $h=\Delta t$ in Scheme 3.2, respectively.

The convergence rate (CR) with respect to the time step Δt is obtained by using linear least square fitting to the numerical errors.

Example 4.1. Consider the following pure BSDE

$$\begin{cases} -dY_t = \dfrac{1}{2}\big(\exp(t^2) - 4tY_t - 3\exp(t^2 - Y_t\exp(-t^2)) - Z_t^2\exp(-t^2)\big)dt - Z_t dW_t, \\ Y_T = \exp(T^2)\ln\big(3 + \sin(W_T)\big). \end{cases} \quad (4.3)$$

It can be checked that the analytic solution yields

$$Y_t = \exp(t^2)\ln\big(3 + \sin(W_t)\big), \quad Z_t = \exp(t^2)\frac{\cos(W_t)}{3 + \sin(W_t)}.$$

We use Schemes 3.1 and 3.2 to solve the BSDE (4.3). In Tables 1 and 2, we have listed the numerical errors $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ and the convergence rates of Schemes 3.1 and 3.2, respectively.

From the numerical results in Tables 1 and 2, we can conclude that Schemes 3.1 and 3.2 are stable and accurate for solving the pure BSDEs for different values of the parameter θ. Moreover, Scheme 3.1 is convergent with order three and Scheme 3.2 with order four, which are consistent with theoretical estimates in Theorem 3.2.

Table 1: Errors and convergence rates of Scheme 3.1 for Example 4.1.

	$\theta=0$		$\theta=\frac{1}{4}$		$\theta=\frac{1}{3}$	
N	$|Y_0-Y^0|$	$|Z_0-Z^0|$	$|Y_0-Y^0|$	$|Z_0-Z^0|$	$|Y_0-Y^0|$	$|Z_0-Z^0|$
8	4.183E-03	1.992E-04	2.117E-03	7.696E-05	1.428E-03	3.267E-05
16	5.079E-04	2.126E-05	2.534E-04	7.879E-06	1.685E-04	2.770E-06
32	6.250E-05	2.422E-06	3.099E-05	9.009E-07	2.048E-05	2.946E-07
64	7.748E-06	2.877E-07	3.831E-06	1.113E-07	2.523E-06	3.749E-08
128	9.646E-07	3.505E-08	4.765E-07	1.441E-08	3.135E-07	5.315E-09
CR	3.020	3.115	3.028	3.091	3.037	3.138

	$\theta=\frac{1}{2}$		$\theta=\frac{2}{3}$		$\theta=1$	
N	$|Y_0-Y^0|$	$|Z_0-Z^0|$	$|Y_0-Y^0|$	$|Z_0-Z^0|$	$|Y_0-Y^0|$	$|Z_0-Z^0|$
8	5.044E-05	6.670E-05	1.328E-03	1.549E-04	4.081E-03	3.307E-04
16	1.532E-06	9.185E-06	1.714E-04	1.940E-05	5.110E-04	3.972E-05
32	6.027E-07	1.207E-06	2.164E-05	2.418E-06	6.371E-05	4.837E-06
64	9.885E-08	1.542E-07	2.717E-06	3.019E-07	7.949E-06	5.967E-07
128	1.376E-08	1.948E-08	3.400E-07	3.768E-08	9.923E-07	7.405E-08
CR	2.763	2.938	2.984	3.002	3.002	3.031

Table 2: Errors and convergence rates of Scheme 3.2 for Example 4.1.

N	$\theta=0$		$\theta=\frac{1}{4}$		$\theta=\frac{1}{3}$	
	$\|Y_0-Y^0\|$	$\|Z_0-Z^0\|$	$\|Y_0-Y^0\|$	$\|Z_0-Z^0\|$	$\|Y_0-Y^0\|$	$\|Z_0-Z^0\|$
8	1.530E-04	3.503E-05	4.823E-05	1.334E-05	1.273E-05	2.273E-06
16	9.738E-06	2.037E-06	3.005E-06	7.484E-07	7.278E-07	1.010E-07
32	6.102E-07	1.218E-07	1.860E-07	4.362E-08	4.272E-08	6.662E-09
64	3.950E-08	7.652E-09	1.292E-08	2.841E-09	3.962E-09	5.230E-10
128	2.480E-09	4.802E-10	8.170E-10	1.820E-10	2.570E-10	4.835E-11
CR	3.977	4.037	3.956	4.036	3.871	3.864

N	$\theta=\frac{1}{2}$		$\theta=\frac{2}{3}$		$\theta=1$	
	$\|Y_0-Y^0\|$	$\|Z_0-Z^0\|$	$\|Y_0-Y^0\|$	$\|Z_0-Z^0\|$	$\|Y_0-Y^0\|$	$\|Z_0-Z^0\|$
8	5.653E-05	8.353E-06	1.264E-04	2.282E-05	2.660E-04	5.174E-05
16	3.727E-06	5.403E-07	8.216E-06	1.399E-06	1.719E-05	3.118E-06
32	2.382E-07	3.457E-08	5.210E-07	8.669E-08	1.087E-06	1.909E-07
64	1.366E-08	1.969E-09	3.138E-08	5.176E-09	6.683E-08	1.159E-08
128	8.461E-10	1.163E-10	1.955E-09	3.151E-10	4.171E-09	7.588E-10
CR	4.015	4.037	3.999	4.037	3.993	4.019

Example 4.2. Consider the following decoupled FBSDE

$$\begin{cases} dX_t = \sin(t+X_t)dt + \dfrac{1}{10}\cos(t+X_t)dW_t, \\ -dY_t = \left(\dfrac{1}{20}Y_tZ_t - \cos(t+X_t)\big(1+\sin(t+X_t)\big) \right)dt - Z_tdW_t, \end{cases} \quad (4.4)$$

with the initial value $X_0=0$ and the terminal condition $Y_T=\sin(T+X_T)$. It can be checked that the analytic solution yields

$$Y_t = \sin(t+X_t), \quad Z_t = \dfrac{1}{10}\cos^2(t+X_t).$$

We use the Schemes 3.1 and 3.2 to solve the FBSDE (4.4). For simplicity, we only test our schemes with $\theta=0.5$ and $\theta=1.0$. The Euler scheme (3.1), the Milstein scheme (3.2), the weak order-2.0 Itô-Taylor scheme (3.3), the weak order-3.0 Itô-Taylor scheme (3.4), and the weak order-4.0 Itô-Taylor scheme (3.5) are used to solve SDEs, respectively.

In Tables 3 and 4, we have listed the numerical errors $|Y_0-Y^0|$ and $|Z_0-Z^0|$ and the convergence rates of Schemes 3.1 and 3.2, respectively.

The errors and the convergence rates in Tables 3 and 4 show that Schemes 3.1 and 3.2 are stable and accurate for solving the decoupled FBSDE (4.4), and their accuracy depends on the Itô-Taylor schemes used for SDEs. From Tables 3 and 4, we can draw the following conclusions.

1. Schemes 3.1 and 3.2 are of order one accurate if the Euler scheme or Milstein scheme are used to solve SDEs.

Table 3: Errors and convergence rates of Scheme 3.1 for Example 4.2.

SDE Sch	N	4	8	16	32	64	CR
Euler	$\|Y_0-Y^0\|$	5.624E-02	2.825E-02	1.422E-02	7.527E-03	3.697E-03	0.976
	$\|Z_0-Z^0\|$	3.282E-03	8.105E-04	8.826E-05	2.820E-04	1.362E-04	1.070
Milstein	$\|Y_0-Y^0\|$	5.624E-02	2.785E-02	1.486E-02	7.457E-03	3.828E-03	0.965
	$\|Z_0-Z^0\|$	3.540E-03	1.391E-03	4.292E-04	2.783E-04	1.538E-04	1.137
Weak-2.0	$\|Y_0-Y^0\|$	1.813E-03	7.707E-04	1.312E-04	3.660E-05	8.481E-06	1.988
	$\|Z_0-Z^0\|$	3.468E-04	1.034E-04	2.452E-05	4.029E-06	1.078E-06	2.134
Weak-3.0	$\|Y_0-Y^0\|$	2.386E-03	2.961E-04	3.579E-05	4.373E-06	5.407E-07	3.030
	$\|Z_0-Z^0\|$	4.218E-05	1.826E-06	5.438E-07	7.210E-08	8.981E-09	2.906
Weak-4.0	$\|Y_0-Y^0\|$	1.001E-03	6.047E-05	3.954E-06	5.424E-07	1.689E-07	3.187
	$\|Z_0-Z^0\|$	1.370E-04	1.631E-05	1.470E-06	1.128E-07	6.114E-08	2.944

Table 4: Errors and convergence rates of Scheme 3.2 for Example 4.2.

SDE Sch	N	4	8	16	32	64	CR
Euler	$\|Y_0-Y^0\|$	5.624E-02	2.785E-02	1.486E-02	7.457E-03	3.828E-03	0.965
	$\|Z_0-Z^0\|$	3.541E-03	1.391E-03	4.292E-04	2.783E-04	1.538E-04	1.137
Milstein	$\|Y_0-Y^0\|$	5.624E-02	3.027E-02	1.450E-02	7.707E-03	3.796E-03	0.975
	$\|Z_0-Z^0\|$	3.781E-03	7.372E-04	3.148E-04	3.194E-04	1.469E-04	1.058
Weak-2.0	$\|Y_0-Y^0\|$	1.817E-03	7.697E-04	1.311E-04	3.658E-05	8.478E-06	1.988
	$\|Z_0-Z^0\|$	3.448E-04	1.032E-04	2.449E-05	4.026E-06	1.078E-06	2.132
Weak-3.0	$\|Y_0-Y^0\|$	2.391E-03	2.971E-04	3.595E-05	4.394E-06	5.433E-07	3.029
	$\|Z_0-Z^0\|$	4.455E-05	1.565E-06	5.222E-07	7.002E-08	8.750E-09	2.911
Weak-4.0	$\|Y_0-Y^0\|$	5.171E-04	3.282E-05	1.933E-06	6.110E-09	6.423E-08	3.834
	$\|Z_0-Z^0\|$	3.984E-05	6.188E-07	2.153E-08	1.863E-09	3.602E-10	4.189

2. Schemes 3.1 and 3.2 are of order two accurate if the weak order 2.0 Itô-Taylor scheme is used to solve SDEs.

3. Scheme 3.1 is of order three accurate if the weak order 3.0 or weak order 4.0 Itô-Taylor schemes are used to solve SDEs.

4. Scheme 3.2 is of order 3.0 accurate if the weak order 3.0 Itô-Taylor scheme is used to solve SDEs.

5. Scheme 3.2 is of order 4.0 accurate if the weak order 4.0 Itô-Taylor scheme is used to solve SDEs.

All of the conclusions above verify again the theoretical estimates in Theorem 3.2.

Example 4.3. Consider the following decoupled FBSDE

$$\begin{cases} dX_t = \dfrac{\exp(t+X_t)}{1+2\exp(t+X_t)}dt + \dfrac{\exp(t+X_t)}{10(1+\exp(t+X_t))}dW_t, \\ -dY_t = \left(-\dfrac{2Y_t}{1+2\exp(t+X_t)} - \dfrac{1}{20}\left(\dfrac{Y_t Z_t}{1+\exp(t+X_t)} - Y_t^2 Z_t \right) \right)dt - Z_t dW_t, \end{cases} \tag{4.5}$$

with the initial value $X_0 = 0$ and the terminal condition

$$Y_T = \frac{\exp(T+X_T)}{1+\exp(T+X_T)}.$$

It can be checked that the analytic solution yields

$$Y_t = \frac{\exp(t+X_t)}{1+\exp(t+X_t)}, \quad Z_t = \frac{(\exp(t+X_t))^2}{10(1+\exp(t+X_t))^3}.$$

We use the Schemes 3.1 and 3.2 to solve the FBSDE (4.5). For simplicity, we test our schemes with $\theta = 0.5$ and $\theta = 1.0$. The Euler scheme (3.1), the Milstein scheme (3.2), the weak order-2.0 Itô-Taylor scheme (3.3), the weak order-3.0 Itô-Taylor scheme (3.4) and the weak order-4.0 Itô-Taylor scheme (3.5) are used to solve SDEs, respectively.

In Tables 5 and 6, we have listed the numerical errors $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ and the convergence rates of Schemes 3.1 and 3.2, respectively.

Table 5: Errors and convergence rates of Scheme 3.1 for Example 4.3.

SDE Sch	N	5	10	15	20	25	CR
Euler	$\|Y_0 - Y^0\|$	3.471E-03	1.922E-03	1.098E-03	8.267E-04	7.615E-04	0.997
	$\|Z_0 - Z^0\|$	1.961E-04	9.468E-05	7.337E-05	5.385E-05	3.737E-05	0.979
Milstein	$\|Y_0 - Y^0\|$	3.471E-03	1.922E-03	1.098E-03	8.267E-04	7.615E-04	0.997
	$\|Z_0 - Z^0\|$	1.898E-04	8.595E-05	6.562E-05	4.743E-05	3.182E-05	1.052
Weak-2.0	$\|Y_0 - Y^0\|$	2.122E-04	6.237E-05	2.352E-05	1.324E-05	9.616E-06	1.977
	$\|Z_0 - Z^0\|$	2.596E-05	8.512E-06	3.086E-06	1.677E-06	1.098E-06	2.007
Weak-3.0	$\|Y_0 - Y^0\|$	9.242E-06	2.403E-06	9.018E-07	2.198E-08	2.524E-07	3.039
	$\|Z_0 - Z^0\|$	2.644E-06	3.444E-07	1.172E-07	2.188E-08	2.891E-08	3.034
Weak-4.0	$\|Y_0 - Y^0\|$	6.571E-06	2.541E-06	9.137E-07	4.323E-07	1.910E-08	3.096
	$\|Z_0 - Z^0\|$	5.199E-06	5.768E-07	1.954E-07	9.057E-08	3.689E-08	2.999

The errors and the convergence rates in Tables 5 and 6 imply that Schemes 3.1 and 3.2 are stable and accurate for solving the decoupled FBSDE (4.5), and their accuracy depends on the Taylor schemes used to solve SDEs. Based on the results listed in Tables 5–6, we can draw the same conclusions as the ones in Example 4.2.

Table 6: Errors and convergence rates of Scheme 3.2 for Example 4.3.

SDE Sch	N	5	10	15	20	25	CR
Euler	$\lvert Y_0 - Y^0 \rvert$	3.369E-03	1.912E-03	1.272E-03	7.795E-04	6.139E-04	1.068
	$\lvert Z_0 - Z^0 \rvert$	1.945E-04	9.129E-05	5.758E-05	4.925E-05	3.957E-05	0.990
Milstein	$\lvert Y_0 - Y^0 \rvert$	3.320E-03	1.531E-03	1.271E-03	9.505E-04	5.869E-04	0.989
	$\lvert Z_0 - Z^0 \rvert$	1.963E-04	1.078E-04	5.719E-05	4.297E-05	4.079E-05	1.047
Weak-2.0	$\lvert Y_0 - Y^0 \rvert$	2.251E-04	6.365E-05	2.387E-05	1.338E-05	9.686E-06	2.008
	$\lvert Z_0 - Z^0 \rvert$	2.744E-05	8.689E-06	3.137E-06	1.699E-06	1.109E-06	2.035
Weak-3.0	$\lvert Y_0 - Y^0 \rvert$	2.065E-05	4.193E-07	1.519E-07	4.054E-07	5.311E-08	3.241
	$\lvert Z_0 - Z^0 \rvert$	6.285E-06	3.496E-07	1.252E-07	7.973E-08	3.258E-08	3.152
Weak-4.0	$\lvert Y_0 - Y^0 \rvert$	1.268E-05	7.862E-07	1.696E-07	6.179E-08	3.008E-08	3.774
	$\lvert Z_0 - Z^0 \rvert$	1.036E-06	5.486E-08	1.165E-08	3.629E-09	2.227E-09	3.891

5　Conclusions

In this work, we developed an explicit third order one-step scheme and an explicit fourth order one-step scheme for decoupled FBSDEs. Numerical tests show that the one-step schemes are efficient and are of high order rates of convergence. In our future work, we shall consider to combine the sparse grid method to deal with high dimensional coupled FBSDEs.

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