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New Energy Analysis of Yee Scheme for Metamaterial Maxwell's Equations on Non-Uniform Rectangular Meshes

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Abstract. In this paper, several new energy identities of metamaterial Maxwell's equations with the perfectly electric conducting (PEC) boundary condition are proposed and proved. These new energy identities are different from the Poynting theorem. By using these new energy identities, it is proved that the Yee scheme on non-uniform rectangular meshes is stable in the discrete L^2 and H^1 norms when the Courant-Friedrichs-Lewy (CFL) condition is satisfied. Numerical experiments in two-dimension (2D) and 3D are carried out and confirm our analysis, and the superconvergence in the discrete H^1 norm is found.

AMS subject classifications: 65M06, 65M12, 35L15, 78A48

Key words: Metamaterial Maxwell's equations, Yee scheme, non-uniform rectangular meshes, energy identities, stability.

1 Introduction

Metamaterials are artificial composite materials designed to exhibit exotic electromagnetic properties. The metamaterial with negative refraction index was first proposed by Veselago in 1968 [26] and constructed by Smith in 2000 [23,24], which has brought a new revolution in electromagnetic and material science. Since 2000, there are numerous reference sources on the study of metamaterials and their potential applications, such as, design of invisiblity cloak [8,22], sub-wavelength imaging [1,28], construction of perfect lens [25]. Matical side there has recently been increased interest in the understanding of the mathematical properties of metamaterial Maxwell's equations relevant to numerical analysis. For example, finite-difference time-domain (FDTD) methods [10,15–17], finite element methods [11,13,29,30], and the monograph [12].

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Due to its efficiency and robustness, FDTD or Yee scheme, firstly introduced by Yee [31] in 1966, is still one of the most popular numerical methods in computational electromagnetics. The FDTD scheme uses central-difference approximations to the space and time partial derivatives at a fully staggered grid, and is second-order accurate in both time and space and easy to implement. For this aspect of theoretical study on the finite difference method for solving time-dependent Maxwell's equations, there are already excellent work in mathematical literature including FDTD scheme and related numerical methods, such as, the alternating direction implicit FDTD (ADI-FDTD) [20, 32], the energy conserved splitting FDTD (EC-S-FDTD) [5, 20], the splitting FDTD (S-FDTD) [2], the splitting multi-symplectic method [7] etc. Among the above-mentioned methods, the stability and error estimates in the L^2 norm have been studied by the energy method. In 2011, Gao and Zhang [4] were firstly studied the important stability and convergence analysis in the H^1 norm of the FDTD scheme with PEC boundary condition on uniform meshes, and extended the similar results to other relevant numerical methods [3, 18].

Recently, the theoretical analysis of the Yee scheme on non-uniform meshes have attracted much attention. The rigorous error analysis of the Yee scheme on non-uniform rectangular meshes can be traced back 1994 by Monk and Süli [19]. They used the special structure of local errors to prove that the Yee scheme still has second-order convergent on a non-uniform mesh although the local truncation error is only of the first order. Remis [21] studied the stability condition of the Yee scheme for solving the Maxwell's equations in lossless medium on non-uniform meshes, by the eigenvalues of the FDTD iteration matrix. In 2016, Li and Shields [14] extended Monk and Süli's technique to give the superconvergence analysis of Yee scheme for solving Maxwell's equations in metamaterials on non-uniform meshes, and extended to an implicit scheme [27]. In these work, the energy method was used to study the stability and error estimates in the L^2 norm. However, no results is available for the import stability and convergence analysis in H^1 norm of the Yee scheme for metamaterial Maxwell's equations on non-uniform rectangular meshes.

Encouraged by the nice properties of the Yee scheme for Maxwell's equations in simple media on uniform meshes [4], in this paper, we study the stability and convergence of the Yee scheme for metamaterial Maxwell's equations on non-uniform rectangular meshes by a new energy method. This new method is motivated by the new energy identities of metamaterial Maxwell's equations established in this paper and is different from the usual one in L^2 norm (cf. [14,27]). By making use of this new energy method, we prove that the Yee scheme with the PEC boundary condition on non-uniform rectangular meshes is stable in the discrete H^1 norm when the CFL condition is satisfied. Numerical results are also presented to confirm the theoretical analysis. Moreover, the superconvergence phenomena are proved for solving metamaterial Maxwell's equations by the Yee scheme on non-uniform rectangular and cubic meshes. To our best knowledge, this is the first result for the important stability analysis in the discrete H^1 norm of the Yee scheme for solving metamaterial Maxwell's equations on non-uniform rectangular meshes.

The rest of the paper is organized as follows. In Section 2, the new energy identities of metamaterial Maxwell's equations in H^1 norm are proposed and the fully discrete Yee scheme on non-uniform rectangular meshes with some notations are given. We present detail analysis of the discrete stability in L^2 and H^1 norms of Yee scheme on non-uniform rectangular meshes in Section 3. In Section 4, numerical simulations results are demonstrated, and the superconvergence in the discrete H^1 norm is found. Besides, the classical example of backward wave propagation in metamaterials was shown. We conclude the paper in Section 5.

2 Energy identities and the Yee scheme of metamaterial Maxwell's equations

In this section, we first describe the problem of two-dimensional metamaterial Maxwell's equations with the PEC boundary condition in this paper. Then, we derive several new energy identities of metamaterial Maxwell's equations, which will be found helpful in study the Yee scheme in H^1 norm on non-uniform rectangular meshes.

2.1 The metamaterial model

Consider the following 2D Maxwell's equations in metamaterials [9]

$$\begin{cases} \epsilon_0 \dfrac{\partial E_x}{\partial t} = \dfrac{\partial H_z}{\partial y} - J_x, & (2.1) \\ \epsilon_0 \dfrac{\partial E_y}{\partial t} = -\dfrac{\partial H_z}{\partial x} - J_y, & (2.2) \\ \mu_0 \dfrac{\partial H_z}{\partial t} = \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} - K_z, & (2.3) \\ \dfrac{1}{\epsilon_0 \omega_{pe}^2} \dfrac{\partial J_x}{\partial t} + \dfrac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} J_x = E_x, & (2.4) \\ \dfrac{1}{\epsilon_0 \omega_{pe}^2} \dfrac{\partial J_y}{\partial t} + \dfrac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} J_y = E_y, & (2.5) \\ \dfrac{1}{\mu_0 \omega_{pm}^2} \dfrac{\partial K_z}{\partial t} + \dfrac{\Gamma_m}{\mu_0 \omega_{pm}^2} K_z = H_z. & (2.6) \end{cases}$$

The PEC boundary condition is given by

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{2.7}$$

and the initial conditions

$$\mathbf{E}(\mathbf{x},\mathbf{0}) = \mathbf{E}^{\mathbf{0}}(\mathbf{x}), \quad H_z(\mathbf{x},0) = H_z^0(\mathbf{x}), \quad \mathbf{J}(\mathbf{x},\mathbf{0}) = \mathbf{J}^{\mathbf{0}}(\mathbf{x}), \quad K_z(\mathbf{x},0) = K_z^0(\mathbf{x}), \tag{2.8}$$

where $\mathbf{E}=(E_x,E_y)^T$, H_z, $\mathbf{J}=(J_x,J_y)^T$ and K_z denote the electric field, magnetic field, induced electric currents and induced magnetic currents vector, respectively. ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability, ω_{pe}, ω_{pm}, Γ_e and Γ_m, respectively, are electric plasma frequency, magnetic plasma frequency, electric damping frequency and magnetic damping frequency. $\mathbf{n}$ represents the unit exterior normal vector to the boundary of the spatial domains $\Omega=[a,b]\times[c,d]$. The initial conditions $\mathbf{E}^0$, H_z^0, $\mathbf{J}^0$, K_z^0 denote some given functions.

2.2 Energy identities of Maxwell's equations in metamaterial

The following Lemma 2.1 which is the well-known Poynting theorem, its proof can be seen in [14].

Lemma 2.1. *Let* $\mathbf{E}(t)=(E_x(t,x,y),E_y(t,x,y))$, $H_z(t)=H_z(t,x,y)$, $\mathbf{J}(t)=(J_x(t,x,y),J_y(t,x,y))$, $K_z(t)=K_z(t,x,y)$ *be the solution to the problem* (2.1)-(2.8). *Then for any* $t\in(0,T]$,

$$\begin{aligned}&\|\mathbf{E}(t)\|^2+\|H_z(t)\|^2+\|\mathbf{J}(t)\|^2+\|K_z(t)\|^2+2\int_0^T\Gamma_e\|\mathbf{J}(\tau)\|^2+\Gamma_m\|K_z(\tau)\|^2d\tau\\ =&\|\mathbf{E}(0)\|^2+\|H_z(0)\|^2+\|\mathbf{J}(0)\|^2+\|K_z(0)\|^2,\end{aligned}\tag{2.9}$$

where $\|\cdot\|$ *denotes the* L^2 *norm with the weight. For example, in* (2.9),

$$\begin{aligned}&\|\mathbf{E}(t)\|^2=\|E_x(t)\|^2+\|E_y(t)\|^2, &&\|E_x(t)\|^2=\int_a^b\int_c^d\epsilon_0|E_x(x,y,t)|^2dydx,\\ &\|\mathbf{J}(t)\|^2=\|J_x(t)\|^2+\|J_y(t)\|^2, &&\|J_x(t)\|^2=\int_a^b\int_c^d\frac{1}{\epsilon_0\omega_{pe}^2}|J_x(x,y,t)|^2dydx.\end{aligned}$$

Using similar techniques as in [4], we can obtain the following new energy identities.

Theorem 2.1. *Let the* $\mathbf{E}(t)$, $H_z(t)$, $\mathbf{J}(t)$, $K_z(t)$ *be the solution to the problem* (2.1)-(2.8), *and possesses the following regularity property:*

$$\begin{aligned}&E_x,E_y,H_z\in C^1([0,T],C^2(\bar{\Omega})),\\ &J_x,J_y,K_z\in C^1([0,T],C^1(\bar{\Omega})).\end{aligned}$$

Then for $u=x,y$,

$$\begin{aligned}&\Big\|\frac{\partial\mathbf{E}(t)}{\partial u}\Big\|^2+\Big\|\frac{\partial H_z(t)}{\partial u}\Big\|^2+\Big\|\frac{\partial\mathbf{J}(t)}{\partial u}\Big\|^2+\Big\|\frac{\partial K_z(t)}{\partial u}\Big\|^2+2\int_0^t\Gamma_e\Big\|\frac{\partial\mathbf{J}(\tau)}{\partial u}\Big\|^2+\Gamma_m\Big\|\frac{\partial K_z(\tau)}{\partial u}\Big\|^2d\tau\\ =&\Big\|\frac{\partial\mathbf{E}(0)}{\partial u}\Big\|^2+\Big\|\frac{\partial H_z(0)}{\partial u}\Big\|^2+\Big\|\frac{\partial\mathbf{J}(0)}{\partial u}\Big\|^2+\Big\|\frac{\partial K_z(0)}{\partial u}\Big\|^2,\end{aligned}\tag{2.10a}$$

$$\begin{aligned}&\|\mathbf{E}(t)\|_1^2+\|H_z(t)\|_1^2+\|\mathbf{J}(t)\|_1^2+\|K_z(t)\|_1^2+2\int_0^t\Gamma_e\|\mathbf{J}(\tau)\|_1^2+\Gamma_m\|K_z(\tau)\|_1^2d\tau\\ =&\|\mathbf{E}(0)\|_1^2+\|H_z(0)\|_1^2+\|\mathbf{J}(0)\|_1^2+\|K_z(0)\|_1^2,\end{aligned}\tag{2.10b}$$

where $\|\cdot\|_1$ *is the* $H^1(\Omega)$ *norm, defined by*

$$\|f\|_1^2=\|f\|^2+\Big\|\frac{\partial f}{\partial x}\Big\|^2+\Big\|\frac{\partial f}{\partial y}\Big\|^2.$$

Proof. We only prove (2.10a) with $u=x$. The other cases with $u=y$ can be obtained similarly.

Differentiating each of Eqs. (2.1)-(2.6) with respect to x, we have

$$\begin{cases}\epsilon_0\dfrac{\partial^2 E_x}{\partial x\partial t}=\dfrac{\partial^2 H_z}{\partial x\partial y}-\dfrac{\partial J_x}{\partial x}, & (2.11)\\ \epsilon_0\dfrac{\partial^2 E_y}{\partial x\partial t}=-\dfrac{\partial^2 H_z}{\partial^2 x}-\dfrac{\partial J_y}{\partial x}, & (2.12)\\ \mu_0\dfrac{\partial^2 H_z}{\partial x\partial t}=\dfrac{\partial^2 E_y}{\partial^2 x}-\dfrac{\partial^2 E_x}{\partial x\partial y}-\dfrac{\partial K_z}{\partial x}, & (2.13)\\ \dfrac{1}{\epsilon_0\omega_{pe}^2}\dfrac{\partial^2 J_x}{\partial x\partial t}+\dfrac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\dfrac{\partial J_x}{\partial x}=\dfrac{\partial E_x}{\partial x}, & (2.14)\\ \dfrac{1}{\epsilon_0\omega_{pe}^2}\dfrac{\partial^2 J_y}{\partial x\partial t}+\dfrac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\dfrac{\partial J_y}{\partial x}=\dfrac{\partial E_y}{\partial x}, & (2.15)\\ \dfrac{1}{\mu_0\omega_{pm}^2}\dfrac{\partial^2 K_z}{\partial x\partial t}+\dfrac{\Gamma_m}{\mu_0\omega_{pm}^2}\dfrac{\partial K_z}{\partial x}=\dfrac{\partial H_z}{\partial x}. & (2.16)\end{cases}$$

Multiplying (2.11) by $\frac{\partial E_x}{\partial x}$, (2.12) by $\frac{\partial E_y}{\partial x}$, (2.13) by $\frac{\partial H_z}{\partial x}$, (2.14) by $\frac{\partial J_x}{\partial x}$, (2.15) by $\frac{\partial J_y}{\partial x}$, (2.16) by $\frac{\partial K_z}{\partial x}$, then summing the six product equations up over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\Big\|\frac{\partial E_x}{\partial x}\Big\|^2+\Big\|\frac{\partial E_y}{\partial x}\Big\|^2+\Big\|\frac{\partial H_z}{\partial x}\Big\|^2+\Big\|\frac{\partial J_x}{\partial x}\Big\|^2+\Big\|\frac{\partial J_y}{\partial x}\Big\|^2+\Big\|\frac{\partial K_z}{\partial x}\Big\|^2\Big)\\ &\quad+\Gamma_e\Big(\Big\|\frac{\partial J_x(\tau)}{\partial x}\Big\|^2+\Big\|\frac{\partial J_y(\tau)}{\partial x}\Big\|^2\Big)+\Gamma_m\Big\|\frac{\partial K_z(\tau)}{\partial x}\Big\|^2=r(t),\end{aligned} \tag{2.17}$$

where

$$r(t)=\int_a^b\int_c^d\Big(\frac{\partial^2 H_z}{\partial x\partial y}\frac{\partial E_x}{\partial x}-\frac{\partial^2 H_z}{\partial^2 x}\frac{\partial E_y}{\partial x}+\frac{\partial^2 E_y}{\partial^2 x}\frac{\partial H_z}{\partial x}-\frac{\partial^2 E_x}{\partial x\partial y}\frac{\partial H_z}{\partial x}\Big). \tag{2.18}$$

From the PEC boundary conditions (2.7) and the domain $\Omega=[a,b]\times[c,d]$, it follows that

$$E_x(x,c)=E_x(x,d)=0,\quad E_y(a,y)=E_y(b,y)=0, \tag{2.19}$$

thus

$$\frac{\partial E_x(x,c)}{\partial x}=\frac{\partial E_x(x,d)}{\partial x}=0, \tag{2.20a}$$

$$\frac{\partial E_y(a,y)}{\partial y}=\frac{\partial E_y(b,y)}{\partial y}=0. \tag{2.20b}$$

Using integration by parts and Eq. (2.20a), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\Big\|\frac{\partial E_x}{\partial x}\Big\|^2+\Big\|\frac{\partial E_y}{\partial x}\Big\|^2+\Big\|\frac{\partial H_z}{\partial x}\Big\|^2+\Big\|\frac{\partial J_x}{\partial x}\Big\|^2+\Big\|\frac{\partial J_y}{\partial x}\Big\|^2+\Big\|\frac{\partial K_z}{\partial x}\Big\|^2\Big)\\&\quad+\Gamma_e\Big\|\frac{\partial \mathbf{J}(\tau)}{\partial u}\Big\|^2+\Gamma_m\Big\|\frac{\partial K_z(\tau)}{\partial u}\Big\|^2=r(t),\end{aligned}\tag{2.21}$$

where

$$r(t)=\int_c^d\frac{\partial E_y}{\partial x}(b,y,t)\frac{\partial H_z}{\partial x}(b,y,t)-\frac{\partial E_y}{\partial x}(a,y,t)\frac{\partial H_z}{\partial x}(a,y,t)dy.$$

Let $\hat{x}=a$ or b, from (2.2) and the PEC boundary condition (2.7), we have

$$\begin{aligned}&\Big(\frac{\partial E_y}{\partial x}\frac{\partial H_z}{\partial x})(\hat{x},y,t\Big)=\lim_{x\to\hat{x}}\Big(\frac{\partial E_y}{\partial x}\frac{\partial H_z}{\partial x}\Big)(x,y,t)\\&=-\lim_{x\to\hat{x}}\frac{\partial E_y}{\partial x}(x,y,t)\Big(\epsilon_0\frac{\partial E_y}{\partial t}+J_y\Big)(x,y,t)=0.\end{aligned}\tag{2.22}$$

Thus, $r(t)=0$.

Then, by integrating (2.21) with respect to time over $[0,t]$, $(0\le t\le T)$, we get

$$\begin{aligned}&\Big\|\frac{\partial \mathbf{E}(t)}{\partial x}\Big\|^2+\Big\|\frac{\partial H_z(t)}{\partial x}\Big\|^2+\Big\|\frac{\partial \mathbf{J}(t)}{\partial x}\Big\|^2+\Big\|\frac{\partial K_z(t)}{\partial x}\Big\|^2+2\int_0^t\Gamma_e\Big\|\frac{\partial \mathbf{J}(\tau)}{\partial x}\Big\|^2+\Gamma_m\Big\|\frac{\partial K_z(\tau)}{\partial x}\Big\|^2d\tau\\&=\Big\|\frac{\partial \mathbf{E}(0)}{\partial x}\Big\|^2+\Big\|\frac{\partial H_z(0)}{\partial x}\Big\|^2+\Big\|\frac{\partial \mathbf{J}(0)}{\partial x}\Big\|^2+\Big\|\frac{\partial K_z(0)}{\partial x}\Big\|^2.\end{aligned}\tag{2.23}$$

We complete the proof of (2.10a), with $u=x$. Combining (2.9) and (2.10a), Eq. (2.10b) holds. □

2.3 The fully discrete Yee scheme on non-uniform rectangular meshes

The rectangular domain $[a,b]\times[c,d]$ is partitioned by a non-uniform rectangular grid as follows:

$$a=x_0<x_1<\cdots<x_I=b,\quad c=y_0<y_1<\cdots<y_J=d.$$

Denote

$$x_{i+\frac{1}{2}}=\frac{x_i+x_{i+1}}{2},\quad i=0,\cdots,I-1,\quad y_{j+\frac{1}{2}}=\frac{y_j+y_{j+1}}{2},\quad j=0,\cdots,J-1.$$

For convenience, denote the following mesh step sizes h_i, k_j, $h_{i+\frac{1}{2}}$, $k_{j+\frac{1}{2}}$

$$\begin{aligned}&h_i=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}, && h_{i+\frac{1}{2}}=x_{i+1}-x_i,\\&k_j=y_{j+\frac{1}{2}}-y_{j-\frac{1}{2}}, && k_{j+\frac{1}{2}}=y_{j+1}-y_j,\\&h_{\min}=\min\{h_i,h_{i+\frac{1}{2}}\}, && k_{\min}=\min\{k_j,k_{j+\frac{1}{2}}\}.\end{aligned}$$

Denote Δt as the time step size, $N_t\Delta t=T$. For $\alpha=i,i+\frac{1}{2}$, $\beta=j,j+\frac{1}{2}$ and $n=m,m+\frac{1}{2}$, where $i=0,1,\cdots,I-1$, $j=0,1,\cdots,J-1$, $m=0,1,\cdots,N_t$ with positive integers I, J and N_t, then for a grid function $f^n_{\alpha,\beta}=f(x_\alpha,y_\beta,t^n)$, define

$$\delta_x f^n_{\alpha,\beta}=\frac{f^n_{\alpha+\frac{1}{2},\beta}-f^n_{\alpha-\frac{1}{2},\beta}}{h_\alpha},\quad \delta_y f^n_{\alpha,\beta}=\frac{f^n_{\alpha,\beta+\frac{1}{2}}-f^n_{\alpha,\beta-\frac{1}{2}}}{k_\beta}.$$

With the above preparation, we can have the following fully-discrete Yee scheme:

Given initial approximations ${E_x}^0_{i+\frac{1}{2},j}$, ${E_y}^0_{i,j+\frac{1}{2}}$, ${H_z}^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$, ${J_x}^{\frac{1}{2}}_{i+\frac{1}{2},j}$, ${J_y}^{\frac{1}{2}}_{i,j+\frac{1}{2}}$, ${K_z}^1_{i+\frac{1}{2},j+\frac{1}{2}}$, for any $0\le n\le N_t-1$, solve ${E_x}^{N_t}_{i+\frac{1}{2},j}$, ${E_y}^{N_t}_{i,j+\frac{1}{2}}$, ${H_z}^{N_t+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$, ${J_x}^{N_t+\frac{1}{2}}_{i+\frac{1}{2},j}$, ${J_y}^{N_t+\frac{1}{2}}_{i,j+\frac{1}{2}}$, ${K_z}^{N_t+1}_{i+\frac{1}{2},j+\frac{1}{2}}$ from:

$$\left\{\begin{aligned}
&\epsilon_0\frac{E_x^{n+1}-E_x^n}{\Delta t}=\delta_y H_z^{n+\frac{1}{2}}-J_x^{n+\frac{1}{2}}|_{i+\frac{1}{2},j}, &(2.24)\\
&\epsilon_0\frac{E_y^{n+1}-E_y^n}{\Delta t}=-\delta_x H_z^{n+\frac{1}{2}}-J_y^{n+\frac{1}{2}}|_{i,j+\frac{1}{2}}, &(2.25)\\
&\mu_0\frac{H_z^{n+\frac{3}{2}}-H_z^{n+\frac{1}{2}}}{\Delta t}=\delta_y E_x^{n+1}-\delta_x E_y^{n+1}-K_z^{n+1}|_{i+\frac{1}{2},j+\frac{1}{2}}, &(2.26)\\
&\frac{1}{\epsilon_0\omega_{pe}^2}\frac{J_x^{n+\frac{3}{2}}-J_x^{n+\frac{1}{2}}}{\Delta t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{J_x^{n+\frac{3}{2}}+J_x^{n+\frac{1}{2}}}{2}=E_x^{n+1}|_{i+\frac{1}{2},j}, &(2.27)\\
&\frac{1}{\epsilon_0\omega_{pe}^2}\frac{J_y^{n+\frac{3}{2}}-J_y^{n+\frac{1}{2}}}{\Delta t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{J_y^{n+\frac{3}{2}}+J_y^{n+\frac{1}{2}}}{2}=E_y^{n+1}|_{i,j+\frac{1}{2}}, &(2.28)\\
&\frac{1}{\mu_0\omega_{pm}^2}\frac{K_z^{n+2}-K_y^{n+1}}{\Delta t}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{K_z^{n+2}+K_z^{n+1}}{2}=H_z^{n+\frac{3}{2}}|_{i+\frac{1}{2},j+\frac{1}{2}}. &(2.29)
\end{aligned}\right.$$

The PEC boundary condition

$${E_x}^n_{i+\frac{1}{2},0}={E_x}^n_{i+\frac{1}{2},J}={E_y}^n_{0,j+\frac{1}{2}}={E_y}^n_{I,j+\frac{1}{2}}=0,\quad n=0,1,\cdots,N_t. \tag{2.30}$$

The initial conditions which are obtained by imposing the initial conditions (2.8) at $t=0$,

$${E_x}^0_{i+\frac{1}{2},j},\quad {E_y}^0_{i,j+\frac{1}{2}},\quad {H_z}^{\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}},\quad {J_x}^{\frac{1}{2}}_{i+\frac{1}{2},j},\quad {J_y}^{\frac{1}{2}}_{i,j+\frac{1}{2}},\quad {K_z}^1_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{2.31}$$

Remark 2.1. From Eqs. (2.4), (2.5) and the PEC boundary condition (2.30), we can easily get

$${J_x}^{n+\frac{1}{2}}_{i+\frac{1}{2},0}={J_x}^{n+\frac{1}{2}}_{i+\frac{1}{2},J}={J_y}^{n+\frac{1}{2}}_{0,j+\frac{1}{2}}={J_y}^{n+\frac{1}{2}}_{I,j+\frac{1}{2}}=0,\quad n=0,1,\cdots,N_t. \tag{2.32}$$

3 Stability of the Yee scheme on non-uniform rectangular meshes

In this section, we investigate the stability of the Yee scheme on non-uniform rectangular meshes in the discrete L^2 and H^1 norms, by the similar techniques as in the last section.

3.1 Discrete mesh-dependent energy norms

We define the following discrete mesh-dependent energy norms. For a grid function $V_{\alpha,\beta}$, where $\alpha = i$ or $i+\frac{1}{2}$, $\beta = j$ or $j+\frac{1}{2}$,

$$\|V_x\|_{E_x}^2 = \sum_{i=0}^{I-1}\sum_{j=1}^{J-1} \epsilon_0 (V_{x_{i+\frac{1}{2},j}})^2 h_{i+\frac{1}{2}} k_j, \qquad \|V_y\|_{E_y}^2 = \sum_{i=1}^{I-1}\sum_{j=0}^{J-1} \epsilon_0 (V_{y_{i,j+\frac{1}{2}}})^2 h_i k_{j+\frac{1}{2}},$$

$$\|V_x\|_{J_x}^2 = \sum_{i=0}^{I-1}\sum_{j=1}^{J-1} \frac{1}{\epsilon_0 \omega_{pe}^2} (V_{x_{i+\frac{1}{2},j}})^2 h_{i+\frac{1}{2}} k_j, \qquad \|V_z\|_{K_z}^2 = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \frac{1}{\mu_0 \omega_{pm}^2} (V_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2 h_{i+\frac{1}{2}} k_{j+\frac{1}{2}},$$

$$\|V_z\|_{H_z}^2 = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \mu_0 (V_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2 h_{i+\frac{1}{2}} k_{j+\frac{1}{2}}, \qquad \|\delta_x V_x\|_{\delta_x E_x}^2 = \sum_{i=1}^{I-1}\sum_{j=1}^{J-1} \epsilon_0 (\delta_x V_{x_{i,j}})^2 h_i k_j,$$

$$\|\delta_x V_y\|_{\delta_x E_y}^2 = \sum_{i=1}^{I-2}\sum_{j=0}^{J-1} \epsilon_0 (\delta_x V_{y_{i+\frac{1}{2},j+\frac{1}{2}}})^2 h_{i+\frac{1}{2}} k_{j+\frac{1}{2}}, \qquad \|\delta_x V_z\|_{\delta_x H_z}^2 = \sum_{i=1}^{I-1}\sum_{j=0}^{J-1} \mu_0 (\delta_x V_{z_{i,j+\frac{1}{2}}})^2 h_i k_{j+\frac{1}{2}},$$

$$|V|_x^2 = \sum_{i=0}^{I-1} \epsilon_0 \Big((V_{i,1})^2 \frac{h_i}{k_{\frac{1}{2}}} + (V_{i,J-1})^2 \frac{h_i}{k_{J-\frac{1}{2}}} \Big), \qquad |V|_y^2 = \sum_{j=0}^{J-1} \epsilon_0 \Big((V_{1,j})^2 \frac{k_j}{h_{\frac{1}{2}}} + (V_{I-1,j})^2 \frac{k_j}{h_{I-\frac{1}{2}}} \Big).$$

The norms $\|\cdot\|_{J_y}$, $\|\cdot\|_{\delta_x J_x}$, $\|\cdot\|_{\delta_x J_y}$, $\|\cdot\|_{\delta_x K_z}$, $\|\cdot\|_{\delta_y E_x}$, $\|\cdot\|_{\delta_y E_y}$, $\|\cdot\|_{\delta_y H_z}$, $\|\cdot\|_{\delta_y J_x}$, $\|\cdot\|_{\delta_y J_y}$, $\|\cdot\|_{\delta_y K_z}$ can be similarly defined by changing the indices i,j. In these norms, the subscripts mean that the sum is taken over the sets of the spatial indices.

3.2 Stability of the Yee scheme in the discrete H^1 semi-norm

The following lemma will be used in the following analysis. It can be easily proved by using summation by parts and the PEC boundary condition (2.30).

Lemma 3.1. *Let $E_{x_{i+\frac{1}{2},j}}^n$, $E_{y_{i,j+\frac{1}{2}}}^n$, $H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^m$ be the solution to the fully discrete Yee scheme with the PEC boundary condition* (2.30)*, then for $m = n+\frac{1}{2}$ or $n-\frac{1}{2}$ with $n \geq 1$, we have*

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1} \delta_x \delta_y E_x^n \cdot \delta_x H_z^m |_{i,j+\frac{1}{2}} h_i k_{j+\frac{1}{2}} = -\sum_{i=1}^{I-1}\sum_{j=1}^{J-1} \delta_x E_x^n \cdot \delta_x \delta_y H_z^m |_{i,j} h_i k_j,$$

$$
\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\delta_x\delta_x E_y^n\cdot\delta_x H_z^m|_{i,j+\frac12}h_ik_{j+\frac12}\\
=&-\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\delta_x E_y^n\cdot\delta_x\delta_x H_z^m|_{i+\frac12,j+\frac12}h_{i+\frac12}k_{j+\frac12}\\
&-\sum_{j=0}^{J-1}E_y^n\cdot\delta_x H_z^m|_{1,j+\frac12}\frac{k_{j+\frac12}}{h_{\frac12}}-\sum_{j=0}^{J-1}E_y^n\cdot\delta_x H_z^m|_{I-1,j+\frac12}\frac{k_{j+\frac12}}{h_{I-\frac12}},\\
&\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\delta_y\delta_y E_x^n\cdot\delta_y H_z^m|_{i+\frac12,j}h_{i+\frac12}k_j\\
=&-\sum_{i=0}^{I-1}\sum_{j=1}^{J-2}\delta_y E_x^n\cdot\delta_y\delta_y H_z^m|_{i+\frac12,j+\frac12}h_{i+\frac12}k_{j+\frac12}\\
&-\sum_{i=0}^{I-1}E_x^n\cdot\delta_y H_z^m|_{i+\frac12,1}\frac{h_{i+\frac12}}{k_{\frac12}}-\sum_{i=0}^{I-1}E_x^n\cdot\delta_y H_z^m|_{i+\frac12,J-1}\frac{h_{i+\frac12}}{k_{J-\frac12}},\\
&\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\delta_y\delta_x E_y^n\cdot\delta_y H_z^m|_{i+\frac12,j}h_{i+\frac12}k_j=-\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\delta_y E_y^n\cdot\delta_x\delta_y H_z^m|_{i,j}h_ik_j.
\end{aligned}
$$

Theorem 3.1. *For $n\geq 0$, let ${E_x}^n_{i+\frac12,j}$, ${E_y}^n_{i,j+\frac12}$, ${H_z}^{n+\frac12}_{i+\frac12,j+\frac12}$, ${J_x}^{n+\frac12}_{i+\frac12,j}$, ${J_y}^{n+\frac12}_{i,j+\frac12}$, ${K_z}^{n+1}_{i+\frac12,j+\frac12}$ be the solution to the fully discrete Yee scheme* (2.24)-(2.29) *with the PEC boundary condition* (2.30)*, and possesses the following regularity property:*

$$
\begin{aligned}
&E_x,E_y,H_z\in C^1([0,T],C^2(\bar\Omega)),\\
&J_x,J_y,K_z\in C^1([0,T],C^1(\bar\Omega)).
\end{aligned}
$$

If the CFL condition

$$
\Delta t<\min\left\{\frac{\sqrt{\mu_0\epsilon_0}}{2}\sqrt{\frac{h_{\min}^2k_{\min}^2}{h_{\min}^2+k_{\min}^2}},\frac{\sqrt2}{\sqrt3\omega_{pe}},\frac{1}{\omega_{pm}}\right\}
$$

is satisfied, then the following estimate holds

$$
\begin{aligned}
&|(E_x^n,E_y^n)|_1^2+|(J_x^{n+\frac12},J_y^{n+\frac12})|_1^2+|H_z^{n+\frac12}|_1^2+|K_z^{n+1}|_1^2\\
\leq& C\Big\{|(E_x^0,E_y^0)|_1^2+|(J_x^{\frac12},J_y^{\frac12})|_1^2+|H_z^{\frac12}|_1^2+|K_z^1|_1^2\Big\}
\end{aligned}
\tag{3.1}
$$

for $n\geq 1$ where C is a constant independent of n and for $n\geq 0$, the discrete H^1 semi-norm is defined as

$$
|(E_x^n,E_y^n)|_1^2=\|\delta_xE_x^n\|_{\delta_xE_x}^2+\Big\|\delta_xE_y^n\Big\|_{\delta_xE_y}^2+\|\delta_yE_x^n\|_{\delta_yE_x}^2+\Big\|\delta_yE_y^n\Big\|_{\delta_yE_y}^2+|E_x^n|_x^2+|E_y^n|_y^2,
$$

$$|(J_x^{n+\frac{1}{2}},J_y^{n+\frac{1}{2}})|_1^2=\left\|\delta_x J_x^{n+\frac{1}{2}}\right\|_{\delta_x J_x}^2+\left\|\delta_x J_y^{n+\frac{1}{2}}\right\|_{\delta_x J_y}^2+\left\|\delta_y J_x^{n+\frac{1}{2}}\right\|_{\delta_y J_x}^2+\left\|\delta_y J_y^{n+\frac{1}{2}}\right\|_{\delta_y J_y}^2,$$

$$|H_z^{n+\frac{1}{2}}|_1^2=\left\|\delta_x H_z^{n+\frac{1}{2}}\right\|_{\delta_x H_z}^2+\left\|\delta_y H_z^{n+\frac{1}{2}}\right\|_{\delta_y H_z}^2,\quad |K_z^{n+1}|_1^2=\left\|\delta_x K_z^{n+1}\right\|_{\delta_x K_z}^2+\left\|\delta_y K_z^{n+1}\right\|_{\delta_y K_z}^2.$$

Proof. Applying the difference operator δ_x to the equations in the fully discrete Yee scheme(2.24)-(2.29) gives the δ_x-Yee scheme:

$$\epsilon_0\frac{\delta_x E_x^{n+1}-\delta_x E_x^n}{\Delta t}=\delta_x\delta_y H_z^{n+\frac{1}{2}}-\delta_x J_x^{n+\frac{1}{2}}|_{i,j}, \tag{3.2a}$$

$$\epsilon_0\frac{\delta_x E_y^{n+1}-\delta_x E_y^n}{\Delta t}=-\delta_x\delta_x H_z^{n+\frac{1}{2}}-\delta_x J_y^{n+\frac{1}{2}}|_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.2b}$$

$$\mu_0\frac{\delta_x H_z^{n+\frac{3}{2}}-\delta_x H_z^{n+\frac{1}{2}}}{\Delta t}=\delta_x\delta_y E_x^{n+1}-\delta_x\delta_x E_y^{n+1}-\delta_x K_z^{n+1}|_{i,j+\frac{1}{2}}, \tag{3.2c}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\delta_x J_x^{n+\frac{3}{2}}-\delta_x J_x^{n+\frac{1}{2}}}{\Delta t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{\delta_x J_x^{n+\frac{3}{2}}+\delta_x J_x^{n+\frac{1}{2}}}{2}=\delta_x E_x^{n+1}|_{i,j}, \tag{3.2d}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\delta_x J_y^{n+\frac{3}{2}}-\delta_x J_y^{n+\frac{1}{2}}}{\Delta t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\frac{\delta_x J_y^{n+\frac{3}{2}}+\delta_x J_y^{n+\frac{1}{2}}}{2}=\delta_x E_y^{n+1}|_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.2e}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\frac{\delta_x K_z^{n+2}-\delta_x K_y^{n+1}}{\Delta t}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\frac{\delta_x K_z^{n+2}+\delta_x K_z^{n+1}}{2}=\delta_x H_z^{n+\frac{3}{2}}|_{i,j+\frac{1}{2}}. \tag{3.2f}$$

Note that the subscript i,j in each equation and its range are changed after the application by δ_x.

Multiplying both sides of Eqs. (3.2a)-(3.2f) by $\Delta t(\delta_x E_x^{n+1}+\delta_x E_x^n)|_{i,j}h_ik_j$, $\Delta t(\delta_x E_y^{n+1}+\delta_x E_y^n)|_{i+\frac{1}{2},j}h_{i+\frac{1}{2}}k_j$, $\Delta t(\delta_x H_z^{n+\frac{3}{2}}+\delta_x H_z^{n+\frac{1}{2}})|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}$, $(\delta_x J_x^{n+\frac{3}{2}}+\delta_x J_x^{n+\frac{1}{2}})|_{i,j}h_ik_j$, $(\delta_x J_y^{n+\frac{3}{2}}+\delta_x J_y^{n+\frac{1}{2}})|_{i+\frac{1}{2},j}h_{i+\frac{1}{2}}k_j$, $(\delta_x K_z^{n+2}+\delta_x K_z^{n+1})|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}$, respectively, and summing them up over i, j in their valid ranges. By adding the six equations together, we obtain the sum of the left hand side (LHS) as

$$\begin{aligned}\text{LHS}=&\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon_0\Big((\delta_x E_x^{n+1})^2-(\delta_x E_x^n)^2\Big)\Big|_{i,j}h_ik_j\\&+\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\epsilon_0\Big((\delta_x E_y^{n+1})^2-(\delta_x E_y^n)^2\Big)\Big|_{i+\frac{1}{2},j+\frac{1}{2}}h_{i+\frac{1}{2}}k_{j+\frac{1}{2}}\\&+\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\epsilon_0\Big((\delta_x H_z^{n+\frac{3}{2}})^2-(\delta_x H_z^{n+\Delta t})^2\Big)\Big|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\Big\{\frac{1}{\epsilon_0\omega_{pe}^2}\Big((\delta_x J_x^{n+\frac{3}{2}})^2-(\delta_x J_x^{n+\frac{1}{2}})^2\Big)+\frac{\Delta t\Gamma_e}{2}(\delta_x J_x^{n+\frac{3}{2}}+\delta_x J_x^{n+\frac{1}{2}})^2\Big\}\Big|_{i,j}h_ik_j\\
&+\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\Big\{\frac{1}{\epsilon_0\omega_{pe}^2}\Big((\delta_x J_y^{n+\frac{3}{2}})^2-(\delta_x J_y^{n+\frac{1}{2}})^2\Big)+\frac{\Delta t\Gamma_e}{2}(\delta_x J_y^{n+\frac{3}{2}}+\delta_x J_y^{n+\frac{1}{2}})^2\Big\}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}h_{i+\frac{1}{2}}k_{j+\frac{1}{2}}\\
&+\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\Big\{\frac{1}{\mu_0\omega_{pm}^2}\Big((\delta_x K_z^{n+2})^2-(\delta_x K_z^{n+1})^2\Big)+\frac{\Delta t\Gamma_m}{2}(\delta_x K_z^{n+2}+\delta_x K_z^{n+1})^2\Big\}\Big|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}\\
=&\|\delta_x E_x^{n+1}\|^2_{\delta_x E_x}-\|\delta_x E_x^{n}\|^2_{\delta_x E_x}+\|\delta_x E_y^{n+1}\|^2_{\delta_x E_y}-\|\delta_x E_y^{n}\|^2_{\delta_x E_y}+\|\delta_x H_z^{n+\frac{3}{2}}\|^2_{\delta_x H_z}-\|\delta_x H_z^{n+\frac{1}{2}}\|^2_{\delta_x H_z}\\
&+\|\delta_x J_x^{n+\frac{3}{2}}\|^2_{\delta_x J_x}-\|\delta_x J_x^{n+\frac{1}{2}}\|^2_{\delta_x J_x}+\|\delta_x J_y^{n+\frac{3}{2}}\|^2_{\delta_x J_y}-\|\delta_x J_y^{n+\frac{1}{2}}\|^2_{\delta_x J_y}+\|\delta_x K_z^{n+2}\|^2_{\delta_x K_z}-\|\delta_x K_z^{n+1}\|^2_{\delta_x K_z}\\
&+\frac{\Delta t}{2}\Big[\Gamma_e\Big(\|\delta_x J_x^{n+\frac{3}{2}}+\delta_x J_x^{n+\frac{1}{2}}\|^2_{\delta_x J_x}+\|\delta_x J_y^{n+\frac{3}{2}}+\delta_x J_y^{n+\frac{1}{2}}\|^2_{\delta_x J_y}\Big)+\Gamma_m\|\delta_x K_z^{n+2}+\delta_x K_z^{n+1}\|^2_{\delta_x K_z}\Big].
\end{aligned}
$$

The second equal term sign comes from the definite of mesh-dependent energy norms.

The corresponding right hand side (RHS)

$$
\begin{aligned}
\text{RHS}=&\Delta t\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\Big(\delta_x\delta_y H_z^{n+\frac{1}{2}}(\delta_x E_x^{n+1}+\delta_x E_x^{n})-\delta_x J_x^{n+\frac{1}{2}}\delta_x E_x^{n}+\delta_x J_x^{n+\frac{3}{2}}\delta_x E_x^{n+1}\Big)\Big|_{i,j}h_ik_j\\
&+\Delta t\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\Big(-\delta_x\delta_x H_z^{n+\frac{1}{2}}(\delta_x E_y^{n+1}+\delta_x E_y^{n})\\
&-\delta_x J_y^{n+\frac{1}{2}}\delta_x E_y^{n}+\delta_x J_y^{n+\frac{3}{2}}\delta_x E_y^{n+1}\Big)\Big|_{i+\frac{1}{2},j+\frac{1}{2}}h_{i+\frac{1}{2}}k_{j+\frac{1}{2}}\\
&+\Delta t\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\Big((\delta_x\delta_y E_x^{n+1}-\delta_x\delta_x E_y^{n+1})(\delta_x H_z^{n+\frac{3}{2}}+\delta_x H_z^{n+\frac{1}{2}})\\
&-\delta_x K_z^{n+1}\delta_x H_z^{n+\frac{1}{2}}+K_z^{n+2}\delta_x H_z^{n+\frac{3}{2}}\Big)\Big|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}\\
=&\Delta t\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\delta_x\delta_y H_z^{n+\frac{1}{2}}\delta_x E_x^{n}|_{i,j}h_ik_j+\Delta t\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\delta_x\delta_y E_x^{n+1}\delta_x H_z^{n+\frac{3}{2}}|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}\\
&+\Delta t\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}(-\delta_x\delta_x H_z^{n+\frac{1}{2}}\delta_x E_y^{n})|_{i+\frac{1}{2},j+\frac{1}{2}}h_{i+\frac{1}{2}}k_{j+\frac{1}{2}}\\
&+\Delta t\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(-\delta_x\delta_x E_y^{n+1}\delta_x H_z^{n+\frac{3}{2}})|_{i,j+\frac{1}{2}}h_ik_{j+\frac{1}{2}}\\
&-\sum_{j=0}^{J-1}E_y^n\cdot\delta_x H_z^{n+\frac{1}{2}}|_{1,j+\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}-\sum_{j=0}^{J-1}E_y^n\cdot\delta_x H_z^{n+\frac{1}{2}}|_{I-1,j+\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{I-\frac{1}{2}}}
\end{aligned}
$$

$$+\Delta t\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\Big(\delta_x J_x^{n+\frac12}\delta_x E_x^n+\delta_x J_x^{n+\frac32}\delta_x E_x^{n+1}\Big)\Big|_{i,j}h_ik_j$$
$$+\Delta t\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\Big(-\delta_x J_y^{n+\frac12}\delta_x E_y^n+\delta_x J_y^{n+\frac32}\delta_x E_y^{n+1}\Big)\Big|_{i+\frac12,j+\frac12}h_{i+\frac12}k_{j+\frac12}$$
$$+\Delta t\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\Big(-\delta_x K_z^{n+1}\delta_x H_z^{n+\frac12}+K_z^{n+2}\delta_x H_z^{n+\frac32}\Big)\Big|_{i,j+\frac12}h_ik_{j+\frac12}.$$

By make use of the Lemma 3.1, the second equal term sign holds.

Summing up LHS and RHS from $n=0$ to $N-1$, and using Lemma 3.1, we have

$$\begin{aligned}
&\|\delta_x E_x^N\|^2_{\delta_x E_x}-\|\delta_x E_x^0\|^2_{\delta_x E_x}+\|\delta_x E_y^N\|^2_{\delta_x E_y}-\|\delta_x E_y^0\|^2_{\delta_x E_y}+\|\delta_x H_z^{N+\frac12}\|^2_{\delta_x H_z}-\|\delta_x H_z^{\frac12}\|^2_{\delta_x H_z}\\
&\quad+\|\delta_x J_x^{N+\frac12}\|^2_{\delta_x J_x}-\|\delta_x J_x^{\frac12}\|^2_{\delta_x J_x}+\|\delta_x J_y^{N+\frac12}\|^2_{\delta_x J_y}-\|\delta_x J_y^{\frac12}\|^2_{\delta_x J_y}+\|\delta_x K_z^{N+1}\|^2_{\delta_x K_z}-\|\delta_x K_z^{1}\|^2_{\delta_x K_z}\\
&\quad+\frac{\Delta t}{2}\sum_{n=0}^{N-1}\bigg[\Gamma_e\Big(\|\delta_x J_x^{n+\frac32}+\delta_x J_x^{n+\frac12}\|^2_{\delta_x J_x}+\|\delta_x J_y^{n+\frac32}+\delta_x J_y^{n+\frac12}\|^2_{\delta_x J_y}\Big)\\
&\quad+\Gamma_m\|\delta_x K_z^{n+2}+\delta_x K_z^{n+1}\|^2_{\delta_x K_z}\bigg]\\
&=T_1+T_2+T_3+T_4,
\end{aligned}\tag{3.3}$$

where

$$T_1=\Delta t\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\delta_x E_x^0\cdot\delta_x\delta_y H_z^{\frac12})|_{i,j}h_ik_j-\Delta t\sum_{i=1}^{I-2}\sum_{j=1}^{J-1}(\delta_x E_y^0\cdot\delta_x\delta_x H_z^{\frac12})|_{i+\frac12,j+\frac12}h_{i+\frac12}k_{j+\frac12},$$
$$T_2=\Delta t\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x\delta_y E_x^N-\delta_x\delta_x E_y^N)\cdot\delta_x H_z^{N+\frac12}|_{i+\frac12,j+\frac12}h_{i+\frac12}k_{j+\frac12},$$
$$\begin{aligned}T_3=&\Delta t\sum_{j=0}^{J-1}\Big(E_y^N\cdot\delta_x H_z^{N-\frac12}+\sum_{n=1}^{N-1}E_y^n\cdot\delta_x(H_z^{n+\frac12}+H_z^{n-\frac12})\Big)\Big|_{1,j+\frac12}\frac{k_{j+\frac12}}{h_{\frac12}}\\
&+\Delta t\sum_{j=0}^{J-1}\Big(E_y^N\cdot\delta_x H_z^{N-\frac12}+\sum_{n=1}^{N-1}E_y^n\cdot\delta_x(H_z^{n+\frac12}+H_z^{n-\frac12})\Big)\Big|_{I-1,j+\frac12}\frac{k_{j+\frac12}}{h_{I-\frac12}},\end{aligned}$$
$$\begin{aligned}T_4=&\Delta t\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\delta_x J_x^{N+\frac12}\cdot\delta_x E_x^N-\delta_x J_x^{\frac12}\cdot\delta_x E_x^0)|_{i,j}h_ik_j\\
&+\Delta t\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}(\delta_x J_y^{N+\frac12}\cdot\delta_x E_y^N-\delta_x J_y^{\frac12}\cdot\delta_x E_y^0)|_{i+\frac12,j+\frac12}h_{i+\frac12}k_{j+\frac12}\\
&+\Delta t\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x K_z^{N+1}\cdot\delta_x H_z^{N+\frac12}-\delta_x K_z^{1}\cdot\delta_x H_z^{\frac12})|_{i,j+\frac12}h_ik_{j+\frac12}.\end{aligned}$$

To estimate T_1, by the Cauchy-Schwartz inequality and the definition of norms $\|\cdot\|_{\delta_x E_x}$ and $\|\cdot\|_{\delta_x E_y}$, we have

$$T_1 \leq \frac{1}{2}\left[\|\delta_x E_x^0\|^2_{\delta_x E_x} + \|\delta_x E_y^0\|^2_{\delta_x E_y} + \left(\frac{\Delta t}{\epsilon_0}\right)^2 \left(\|\delta_x \delta_y H_z^{\frac{1}{2}}\|^2_{\delta_x E_x} + \|\delta_x \delta_x H_z^{\frac{1}{2}}\|^2_{\delta_x E_y}\right)\right]. \tag{3.4}$$

Eq. (3.2c) can be rewritten as

$$\delta_x H_z^{N+\frac{1}{2}} - \frac{\Delta t}{\mu}(\delta_x \delta_y E_x^N - \delta_x \delta_x E_y^N) = \delta_x H_z^{N-\frac{1}{2}} - \frac{\Delta t}{\mu}\delta_x K_z^N|_{i,j+\frac{1}{2}}.$$

Squaring both sides of the above equation, multiplying both sides of the above equation by $\mu_0 h_i k_{j+\frac{1}{2}}$.

$$T_2 = \frac{1}{2}\left[\|\delta_x H_z^{N+\frac{1}{2}}\|^2_{\delta_x H_z} + \left(\frac{\Delta t}{\mu}\right)^2 \|\delta_x \delta_y E_x^N - \delta_x \delta_x E_y^N\|^2_{\delta_x H_z} - \left\|\delta_x H_z^{N-\frac{1}{2}} - \frac{\Delta t}{\mu}\delta_x K_z^N\right\|^2_{\delta_x H_z}\right]. \tag{3.5}$$

Then, we have the following estimate

$$T_2 \leq \frac{1}{2}\left[\|\delta_x H_z^{N+\frac{1}{2}}\|^2_{\delta_x H_z} + \left(\frac{\Delta t}{\mu}\right)^2 \|\delta_x \delta_y E_x^N - \delta_x \delta_x E_y^N\|^2_{\delta_x H_z}\right]. \tag{3.6}$$

To estimate T_3, take $i = i'$, from (2.25) we can easily have

$$E_y^n - \frac{\Delta t}{\epsilon_0}\delta_x H_z^{n+\frac{1}{2}} = E_y^{n+1} + \frac{\Delta t}{\epsilon_0} J_y^{n+\frac{1}{2}}\Big|_{i',j+\frac{1}{2}}, \tag{3.7a}$$

$$E_y^n + \frac{\Delta t}{\epsilon_0}\delta_x H_z^{n-\frac{1}{2}} = E_y^{n-1} - \frac{\Delta t}{\epsilon_0} J_y^{n-\frac{1}{2}}\Big|_{i',j+\frac{1}{2}}, \tag{3.7b}$$

$$E_y^N + \frac{\Delta t}{\epsilon_0}\delta_x H_z^{N-\frac{1}{2}} = E_y^{N-1} - \frac{\Delta t}{\epsilon_0} J_y^{N-\frac{1}{2}}\Big|_{i',j+\frac{1}{2}}, \tag{3.7c}$$

where $i'=1, I-1$. From these three equations and by a similar argument to that in deriving (3.6), we have

$$\begin{aligned}\Delta t E_y^n \cdot \delta_x (H_z^{n+\frac{1}{2}} + H_z^{n-\frac{1}{2}}) = \frac{1}{2}\Bigg[&\epsilon_0\left(E_y^{n-1} - \frac{\Delta t}{\epsilon_0} J_y^{n-\frac{1}{2}}\right)^2 - \epsilon_0\left(E_y^{n+1} + \frac{\Delta t}{\epsilon_0} J_y^{n+\frac{1}{2}}\right)^2 \\ &+ \frac{(\Delta t)^2}{\epsilon_0}\left((\delta_x H_z^{n+\frac{1}{2}})^2 - (\delta_x H_z^{n-\frac{1}{2}})^2\right)\Bigg],\end{aligned} \tag{3.8a}$$

$$\Delta t E_y^N \cdot \delta_x H_z^{N-\frac{1}{2}} = \frac{1}{2}\left[\epsilon_0\left(E_y^{N-1} - \frac{\Delta t}{\epsilon_0} J_y^{N-\frac{1}{2}}\right)^2 - \epsilon_0 (E_y^N)^2 - \frac{(\Delta t)^2}{\epsilon_0}(\delta_x H_z^{N-\frac{1}{2}})^2\right]. \tag{3.8b}$$

Substituting these two equations into the expression of T_3 and noting the definition of $|\cdot|_y$, it is found that

$$\begin{aligned}T_3=\frac{1}{2}\Bigg[&|E_y^0|_y^2+|E_y^1|_y^2-2|E_y^N|_y^2-\frac{(\Delta t)^2}{\epsilon_0^2}|\delta_x H_z^{\frac{1}{2}}|_y^2+\frac{(\Delta t)^2}{\epsilon_0^2}|J_y^{\frac{1}{2}}|_y^2\\&-2\Delta t\sum_{j=0}^{J-1}\sum_{n=1}^{N}(E_y^{n-1}+E_y^n)\cdot J_y^{n-\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}-2\Delta t\sum_{j=0}^{J-1}\sum_{n=1}^{N}(E_y^{n-1}+E_y^n)\cdot J_y^{n-\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{I-\frac{1}{2}}}\\&+2\Delta t\sum_{j=0}^{J-1}E_y^1\cdot J_y^{\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}+2\Delta t\sum_{j=0}^{J-1}E_y^1\cdot J_y^{\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{I-\frac{1}{2}}}\Bigg].\end{aligned}\tag{3.9}$$

From Eq. (2.28), we have

$$\Big(1+\frac{\Delta t\Gamma_e}{2}\Big)J_y^{n+\frac{3}{2}}+\Big(-1+\frac{\Delta t\Gamma_e}{2}\Big)J_y^{n+\frac{1}{2}}=\Delta t\epsilon_0\omega_{pe}^2E_y^{n+1}|_{i,j+\frac{1}{2}}.$$

Multiplying each side of the above equation by E_y^{n+1}, choose a constant

$$C=\max\Big\{1+\frac{\Delta t\Gamma_e}{2},-1+\frac{\Delta t\Gamma_e}{2}\Big\},\quad (C>0).$$

Then, we can obtain that

$$(J_y^{n+\frac{3}{2}}+J_y^{n+\frac{1}{2}})\cdot E_y^{n+1}\geq\frac{\Delta t\epsilon_0\omega_{pe}^2}{C}(E_y^{n+1})^2|_{i,j+\frac{1}{2}}.\tag{3.10}$$

Summing up Eq. (3.10) from $n=0$ to $N-1$ and noting the definition of $|\cdot|_y$, we obtain

$$\sum_{j=0}^{J-1}\Big(\sum_{n=1}^{N}(E_y^{n-1}+E_y^n)\cdot J_y^{n-\frac{1}{2}}+E_y^N\cdot J_y^{N+\frac{1}{2}}-E_y^0\cdot J_y^{\frac{1}{2}}\Big)\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}\geq\frac{\Delta t\omega_{pe}^2}{C}\sum_{n=0}^{N-1}|E_y^{n+1}|_y^2.\tag{3.11}$$

Combining Eq. (3.11) and (3.9), by the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}T_3\leq\frac{1}{2}\Bigg[&2|E_y^0|_y^2+2|E_y^1|_y^2-2|E_y^N|_y^2+3\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|J_y^{\frac{1}{2}}|_y^2\\&+2\Delta t\sum_{j=0}^{J-1}E_y^N\cdot J_y^{N+\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}+2\Delta t\sum_{j=0}^{J-1}E_y^N\cdot J_y^{N+\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{I-\frac{1}{2}}}\Bigg].\end{aligned}$$

By Remark (2.32), we known that

$$
\begin{aligned}
&2\Delta t\sum_{j=0}^{J-1}E_y^N J_y^{N+\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}\le\sum_{j=0}^{J-1}\epsilon_0(E_y^N)^2\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}+\frac{\Delta t}{\epsilon_0}\sum_{j=0}^{J-1}(J_y^{N+\frac{1}{2}}-0)^2\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}\\
&=\sum_{j=0}^{J-1}\epsilon_0(E_y^N)^2\frac{k_{j+\frac{1}{2}}}{h_{\frac{1}{2}}}+\frac{\Delta t}{\epsilon_0}\sum_{j=0}^{J-1}(\delta_x J_y^{N+\frac{1}{2}})^2 k_{j+\frac{1}{2}}h_{\frac{1}{2}},\\
&2\Delta t\sum_{j=0}^{J-1}E_y^N J_y^{N+\frac{1}{2}}\frac{k_{j+\frac{1}{2}}}{h_{I-\frac{1}{2}}}\le\sum_{j=0}^{J-1}\epsilon_0(E_y^N)^2\frac{k_{j+\frac{1}{2}}}{h_{I-\frac{1}{2}}}+\frac{\Delta t}{\epsilon_0}\sum_{j=0}^{J-1}(\delta_x J_y^{N+\frac{1}{2}})^2 k_{j+\frac{1}{2}}h_{I-\frac{1}{2}}.
\end{aligned}
$$

From above equation, we obtain that

$$
T_3\le\frac{1}{2}\left[2|E_y^0|_y^2+2|E_y^1|_y^2-|E_y^N|_y^2+3\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|J_y^{\frac{1}{2}}|_y^2+(\Delta t\omega_{pe})^2\|\delta_x J_y^{N+\frac{1}{2}}\|_{\delta_x J_y}^2\right]. \tag{3.12}
$$

To estimate T_4, using the Cauchy-Schwartz inequality, it follows that

$$
\begin{aligned}
T_4\le&\frac{1}{4}\|\delta_x E_x^N\|_{\delta_x E_x}^2+(\Delta t\omega_{pe})^2\|\delta_x J_x^{N+\frac{1}{2}}\|_{\delta_x J_x}^2+\frac{1}{4}\|\delta_x E_x^0\|_{\delta_x E_x}^2+(\Delta t\omega_{pe})^2\|\delta_x J_x^{\frac{1}{2}}\|_{\delta_x J_x}^2\\
&+\frac{1}{4}\|\delta_x E_y^N\|_{\delta_x E_y}^2+(\Delta t\omega_{pe})^2\|\delta_x J_y^{N+\frac{1}{2}}\|_{\delta_x J_y}^2+\frac{1}{4}\|\delta_x E_y^0\|_{\delta_x E_y}^2+(\Delta t\omega_{pe})^2\|\delta_x J_y^{\frac{1}{2}}\|_{\delta_x J_y}^2\\
&+\frac{1}{4}\|\delta_x H_z^{N+\frac{1}{2}}\|_{\delta_x H_z}^2+(\Delta t\omega_{pm})^2\|\delta_x K_z^{N+1}\|_{\delta_x K_z}^2\\
&+\frac{1}{4}\|\delta_x H_z^{\frac{1}{2}}\|_{\delta_x H_z}^2+(\Delta t\omega_{pm})^2\|\delta_x K_z^{1}\|_{\delta_x K_z}^2.
\end{aligned}\tag{3.13}
$$

Combining (3.4), (3.6), (3.12), (3.13) with (3.3) gives

$$
\begin{aligned}
&\frac{3}{4}\Big(\|\delta_x E_x^N\|_{\delta_x E_x}^2+\|\delta_x E_y^N\|_{\delta_x E_y}^2\Big)+\frac{1}{4}\|\delta_x H_z^{N+\frac{1}{2}}\|_{\delta_x H_z}^2\\
&\quad+\Big(1-(\Delta t\omega_{pe})^2\Big)\Big(\|\delta_x J_x^{N+\frac{1}{2}}\|_{\delta_x J_x}^2+\|\delta_x J_y^{N+\frac{1}{2}}\|_{\delta_x J_y}^2\Big)+\Big(1-(\Delta t\omega_{pm})^2\Big)\|\delta_x K_z^{N+1}\|_{\delta_x K_z}^2\\
\le&\frac{7}{4}\Big(\|\delta_x E_x^0\|_{\delta_x E_x}^2+\|\delta_x E_y^0\|_{\delta_x E_y}^2\Big)+\frac{5}{4}\|\delta_x H_z^{\frac{1}{2}}\|_{\delta_x H_z}^2\\
&\quad+\Big(1+(\Delta t\omega_{pe})^2\Big)\Big(\|\delta_x J_x^{\frac{1}{2}}\|_{\delta_x J_x}^2+\|\delta_x J_y^{\frac{1}{2}}\|_{\delta_x J_y}^2\Big)+\Big(1+(\Delta t\omega_{pm})^2\Big)\|\delta_x K_z^{1}\|_{\delta_x K_z}^2\\
&\quad+G_1+G_2,
\end{aligned}\tag{3.14}
$$

where

$$
G_1=\Big(\frac{\Delta t}{\sqrt{2}\epsilon_0}\Big)^2\left[\|\delta_x\delta_y H_z^{\frac{1}{2}}\|_{\delta_x E_x}^2+\|\delta_x\delta_x H_z^{\frac{1}{2}}\|_{\delta_x E_y}^2\right]+\Big(\frac{\Delta t}{\sqrt{2}\mu}\Big)^2\|\delta_x\delta_y E_x^N-\delta_x\delta_x E_y^N\|_{\delta_x H_z}^2, \tag{3.15a}
$$

$$
G_2=\frac{1}{2}\left[2|E_y^0|_y^2+2|E_y^1|_y^2-|E_y^N|_y^2+3\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|J_y^{\frac{1}{2}}|_y^2+(\Delta t\omega_{pe})^2\|\delta_x J_y^{N+\frac{1}{2}}\|_{\delta_x J_y}^2\right]. \tag{3.15b}
$$

Similarly, we obtain, by replacing δ_x with δ_y, that

$$\begin{aligned}
&\frac{3}{4}\Big(\|\delta_y E_x^N\|^2_{\delta_y E_x}+\|\delta_y E_y^N\|^2_{\delta_y E_y}\Big)+\frac{1}{4}\|\delta_y H_z^{N+\frac{1}{2}}\|^2_{\delta_y H_z}\\
&\quad+\Big(1-(\Delta t\omega_{pe})^2\Big)\Big(\|\delta_y J_x^{N+\frac{1}{2}}\|^2_{\delta_y J_x}+\|\delta_y J_y^{N+\frac{1}{2}}\|^2_{\delta_y J_y}\Big)+\Big(1-(\Delta t\omega_{pm})^2\Big)\|\delta_y K_z^{N+1}\|^2_{\delta_y K_z}\\
\le&\frac{7}{4}\Big(\|\delta_y E_x^0\|^2_{\delta_y E_x}+\|\delta_y E_y^0\|^2_{\delta_y E_y}\Big)+\frac{5}{4}\|\delta_y H_z^{\frac{1}{2}}\|^2_{\delta_y H_z}\\
&\quad+\Big(1+(\Delta t\omega_{pe})^2\Big)\Big(\|\delta_y J_x^{\frac{1}{2}}\|^2_{\delta_y J_x}+\|\delta_y J_y^{\frac{1}{2}}\|^2_{\delta_y J_y}\Big)+\Big(1+(\Delta t\omega_{pm})^2\Big)\|\delta_y K_z^1\|^2_{\delta_y K_z}\\
&\quad+\hat{G}_1+\hat{G}_2,
\end{aligned}\tag{3.16}$$

where

$$\hat{G}_1=\Big(\frac{\Delta t}{\sqrt{2}\epsilon_0}\Big)^2\Big[\|\delta_y\delta_y H_z^{\frac{1}{2}}\|^2_{\delta_y E_x}+\|\delta_y\delta_x H_z^{\frac{1}{2}}\|^2_{\delta_y E_y}\Big]+\Big(\frac{\Delta t}{\sqrt{2}\mu}\Big)^2\|\delta_y\delta_y E_x^N-\delta_y\delta_x E_y^N\|^2_{\delta_y H_z},\tag{3.17a}$$

$$\hat{G}_2=\frac{1}{2}\Big[2|E_x^0|_x^2+2|E_x^1|_x^2-|E_x^N|_x^2+3\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|J_x^{\frac{1}{2}}|_x^2+(\Delta t\omega_{pe})^2\|\delta_y J_x^{N+\frac{1}{2}}\|^2_{\delta_y J_x}\Big].\tag{3.17b}$$

For the first term of G_1 and $\hat{G}_1$, by the definition of norms $\|\cdot\|_{\delta_x E_x}$ and $\|\cdot\|_{\delta_y H_z}$, using the Cauchy-Schwartz inequality, it can be shown that

$$\begin{aligned}
\|\delta_x\delta_y H_z^{\frac{1}{2}}\|^2_{\delta_x E_x}=&\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon_0\left[\frac{\delta_y H^{\frac{1}{2}}_{z_{i+\frac{1}{2},j}}-\delta_y H^{\frac{1}{2}}_{z_{i-\frac{1}{2},j}}}{h_i}\right]^2 h_i k_j\\
\le&\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon_0\left[\frac{2}{h_i h_{i+\frac{1}{2}}}(\delta_y H^{\frac{1}{2}}_{z_{i+\frac{1}{2},j}})^2 h_{i+\frac{1}{2}}k_j+\frac{2}{h_i h_{i-\frac{1}{2}}}(\delta_y H^{\frac{1}{2}}_{z_{i-\frac{1}{2},j}})^2 h_{i-\frac{1}{2}}k_j\right]\\
\le&\frac{4\epsilon_0}{\mu_0 h^2_{\min}}\|\delta_y H_z^{\frac{1}{2}}\|^2_{\delta_y H_z},
\end{aligned}\tag{3.18a}$$

$$\|\delta_x\delta_x H_z^{\frac{1}{2}}\|^2_{\delta_x E_y}\le\frac{4\epsilon_0}{\mu_0 h^2_{\min}}\|\delta_x H_z^{\frac{1}{2}}\|^2_{\delta_x H_z},\tag{3.18b}$$

$$\|\delta_y\delta_x H_z^{\frac{1}{2}}\|^2_{\delta_y E_y}\le\frac{4\epsilon_0}{\mu_0 k^2_{\min}}\|\delta_x H_z^{\frac{1}{2}}\|^2_{\delta_x H_z},\tag{3.18c}$$

$$\|\delta_y\delta_y H_z^{\frac{1}{2}}\|^2_{\delta_y E_x}\le\frac{4\epsilon_0}{\mu_0 k^2_{\min}}\|\delta_y H_z^{\frac{1}{2}}\|^2_{\delta_y H_z}.\tag{3.18d}$$

For the second term of G_1 and $\hat{G}_1$.

$$\|\delta_x(\delta_y E_x^N-\delta_x E_y^N)\|^2_{\delta_x H_z}$$

$$
\begin{aligned}
&=\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu_0(\delta_x\delta_y E_x^N)^2 h_i k_{j+\frac{1}{2}}+\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu_0(\delta_x\delta_x E_y^N)^2 h_i k_{j+\frac{1}{2}}\\
&\quad -2\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu_0\delta_x\delta_y E_x^N\cdot\delta_x\delta_x E_y^N h_i k_{j+\frac{1}{2}}\\
&=:A_1+A_2+A_3.
\end{aligned}
$$

By the Cauchy-Schwartz inequality and the definition of the norms $\|\cdot\|_{\delta_y E_x}$, $\|\cdot\|_{\delta_x E_y}$, $|\cdot|_x$ and $|\cdot|_y$, we have

$$
\begin{aligned}
A_1&=\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu_0\left[\frac{\delta_y E^N_{x_{i+\frac{1}{2},j+\frac{1}{2}}}-\delta_y E^N_{x_{i-\frac{1}{2},j+\frac{1}{2}}}}{h_i}\right]^2 h_i k_{j+\frac{1}{2}}\\
&\le\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu_0\left[\frac{2}{h_i h_{i+\frac{1}{2}}}(\delta_y E^N_{x_{i+\frac{1}{2},j+\frac{1}{2}}})^2 h_{i+\frac{1}{2}}k_{j+\frac{1}{2}}+\frac{2}{h_i h_{i-\frac{1}{2}}}(\delta_y E^N_{x_{i-\frac{1}{2},j+\frac{1}{2}}})^2 h_{i-\frac{1}{2}}k_{j+\frac{1}{2}}\right]\\
&\le\frac{4\mu_0}{\epsilon_0 h_{\min}^2}(\|\delta_y E_x^N\|^2_{\delta_y E_x}+|E_x^N|_x^2),
\end{aligned}
\tag{3.19a}
$$

$$
A_2\le\frac{4\mu_0}{\epsilon_0 h_{\min}^2}(\|\delta_x E_y^N\|^2_{\delta_x E_y}+|E_y^N|_y^2).
\tag{3.19b}
$$

To estimate A_3, we have

$$
\begin{aligned}
A_3=&2\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\mu_0\delta_x E_x^N\cdot\delta_x\delta_x\delta_y E_y^N h_i k_j\\
=&2\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu_0\delta_x E_x^N\cdot\delta_x\delta_x\delta_y E_y^N h_i k_j\\
=&-2\sum_{i=1}^{I-2}\sum_{j=1}^{J-1}\mu_0\delta_x\delta_x E^N_{x_{i+\frac{1}{2},j}}\cdot\delta_x\delta_y E^N_{y_{i+\frac{1}{2},j}}h_{i+\frac{1}{2}}k_j\\
&+2\mu_0\sum_{j=1}^{J-1}\left[\delta_x E^N_{x_{1,j}}\cdot\delta_x\delta_y E^N_{y_{\frac{1}{2},j}}\frac{k_j}{h_{\frac{1}{2}}}-\delta_x E^N_{x_{I-1,j}}\cdot\delta_x\delta_y E^N_{y_{I-\frac{1}{2},j}}\frac{k_j}{h_{I-\frac{1}{2}}}\right].
\end{aligned}
$$

The first and third equal sign come from Lemma 3.1, the second equal sign comes from the PEC boundary condition (2.30).

Using the similar technique as in (3.19a), and the PEC boundary condition (2.30), we have

$$
A_3\le\frac{4\mu_0}{\epsilon_0 h_{\min}^2}\Big(\|\delta_x E_x^N\|^2_{\delta_x E_x}+\|\delta_y E_y^N\|^2_{\delta_y E_y}\Big).
\tag{3.20}
$$

Combining the (3.19a), (3.19b) and (3.20) give

$$\begin{aligned}&\|\delta_x(\delta_y E_x^N-\delta_x E_y^N)\|^2_{\delta_x H_z}\\ \le&\frac{4\mu}{\epsilon_0 h^2_{\min}}\Big(\|\delta_y E_x^N\|^2_{\delta_y E_x}+|E_x^N|^2_x+\|\delta_x E_y^N\|^2_{\delta_x E_y}\\ &\quad+|E_y^N|^2_y+\|\delta_x E_x^N\|^2_{\delta_x E_x}+\|\delta_y E_y^N\|^2_{\delta_y E_y}\Big). \end{aligned}\tag{3.21}$$

Similarly, we have

$$\begin{aligned}&\|\delta_y(\delta_y E_x^N-\delta_x E_y^N)\|^2_{\delta_y H_z}\\ \le&\frac{4\mu}{\epsilon_0 k^2_{\min}}\Big(\|\delta_y E_x^N\|^2_{\delta_y E_x}+|E_x^N|^2_x+\|\delta_x E_y^N\|^2_{\delta_x E_y}\\ &\quad+|E_y^N|^2_y+\|\delta_x E_x^N\|^2_{\delta_x E_x}+\|\delta_y E_y^N\|^2_{\delta_y E_y}\Big). \end{aligned}\tag{3.22}$$

Combining (3.18a)-(3.18d), (3.21), (3.22) with (3.15a) and (3.17a), we have

$$\begin{aligned}G_1+\hat{G}_1\le&\frac{2(\Delta t)^2}{\mu_0\epsilon_0}\Big(\frac{1}{h^2_{\min}}+\frac{1}{k^2_{\min}}\Big)\Big(\|\delta_x H_z^{\frac12}\|^2_{\delta_x H_z}+\|\delta_y H_z^{\frac12}\|^2_{\delta_y H_z}\\ &+\|\delta_y E_x^N\|^2_{\delta_y E_x}+|E_x^N|^2_x+\|\delta_x E_y^N\|^2_{\delta_x E_y}\\ &+|E_y^N|^2_y+\|\delta_x E_x^N\|^2_{\delta_x E_x}+\|\delta_y E_y^N\|^2_{\delta_y E_y}\Big). \end{aligned}\tag{3.23}$$

The key of estimate G_2 and $\hat{G}_2$ is to eliminate E_x^1 and E_y^1. Take $j=j'$ in (2.24), $i=i'$ in (2.25) and let $m=0$ in both equations to get

$$\begin{aligned}E_x^1&=E_x^0+\frac{\Delta t}{\epsilon_0}\delta_y H_z^{\frac12}-\frac{\Delta t}{\epsilon_0}J_x^{\frac12}\Big|_{i+\frac12,j'},\\ E_y^1&=E_y^0-\frac{\Delta t}{\epsilon_0}\delta_x H_z^{\frac12}-\frac{\Delta t}{\epsilon_0}J_y^{\frac12}\Big|_{i',j+\frac12}.\end{aligned}$$

where $i'=1,I-1$ and $j'=1,J-1$. Thus, together with the definitions of $|\cdot|_x$ and $|\cdot|_y$ implies that

$$\begin{aligned}|E_x^1|^2_x&\le 2|E_x^0|^2_x+4\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|\delta_y H_z^{\frac12}|^2_x+4\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|J_x^{\frac12}|^2_x,\\ |E_y^1|^2_y&\le 2|E_y^0|^2_y+4\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|\delta_x H_z^{\frac12}|^2_y+4\Big(\frac{\Delta t}{\epsilon_0}\Big)^2|J_y^{\frac12}|^2_y.\end{aligned}$$

Then, we have

$$\begin{aligned}G_2+\hat{G}_2\le&\frac{1}{2}\Bigg[6\Big(|E_x^0|_x^2+|E_y^0|_y^2\Big)-|E_x^N|_x^2-|E_y^N|_y^2+11\Big(\frac{\Delta t}{\epsilon_0}\Big)^2\Big(|J_y^{\frac{1}{2}}|_y^2+|J_x^{\frac{1}{2}}|_x^2\Big)\\&+(\Delta t\omega_{pe})^2\Big(\|\delta_y J_x^{N+\frac{1}{2}}\|_{\delta_y J_x}^2+\|\delta_x J_y^{N+\frac{1}{2}}\|_{\delta_x J_y}^2\Big)\\&+8\Big(\frac{\Delta t}{\epsilon_0}\Big)^2\Big(|\delta_x H_z^{\frac{1}{2}}|_y^2+|\delta_y H_z^{\frac{1}{2}}|_x^2\Big)\Bigg].\end{aligned}\tag{3.24}$$

Denote

$$C_{cfl}=\frac{(\Delta t)^2}{\mu_0\epsilon_0}\Big(\frac{1}{h_{\min}^2}+\frac{1}{k_{\min}^2}\Big),$$

combining (3.14), (3.16), (3.23) and (3.24) gives

$$\begin{aligned}&\Big(\frac{3}{4}-2C_{cfl}\Big)\Big(\Big\|\delta_x E_x^N\Big\|_{\delta_x E_x}^2+\Big\|\delta_x E_y^N\Big\|_{\delta_x E_y}^2+\Big\|\delta_y E_x^N\Big\|_{\delta_y E_x}^2+\Big\|\delta_y E_y^N\Big\|_{\delta_y E_y}^2\Big)\\&\quad+\Big(\frac{1}{2}-2C_{cfl}\Big)\Big(|E_x^N|_x^2+|E_y^N|_y^2\Big)+\frac{1}{4}\Big(\Big\|\delta_x H_z^{N+\frac{1}{2}}\Big\|_{\delta_x H_z}^2+\Big\|\delta_y H_z^{N+\frac{1}{2}}\Big\|_{\delta_y H_z}^2\Big)\\&\quad+\Big(1-(\Delta t\omega_{pe})^2\Big)\Big(\Big\|\delta_x J_x^{N+\frac{1}{2}}\Big\|_{\delta_x J_x}^2+\Big\|\delta_y J_y^{N+\frac{1}{2}}\Big\|_{\delta_y J_y}^2\Big)\\&\quad+\Big(1-\frac{3}{2}(\Delta t\omega_{pe})^2\Big)\Big(\Big\|\delta_x J_y^{N+\frac{1}{2}}\Big\|_{\delta_x J_y}^2+\Big\|\delta_y J_x^{N+\frac{1}{2}}\Big\|_{\delta_y J_x}^2\Big)\\&\quad+\Big(1-(\Delta t\omega_{pm})^2\Big)\Big(\Big\|\delta_x K_z^{N+1}\Big\|_{\delta_x K_z}^2+\Big\|\delta_y K_z^{N+1}\Big\|_{\delta_y K_z}^2\Big)\\&\le\frac{7}{4}\Big(\Big\|\delta_x E_x^0\Big\|_{\delta_x E_x}^2+\Big\|\delta_x E_y^0\Big\|_{\delta_x E_y}^2+\Big\|\delta_y E_x^0\Big\|_{\delta_y E_x}^2+\Big\|\delta_y E_y^0\Big\|_{\delta_y E_y}^2\Big)\\&\quad+3\Big(\Big|E_x^0\Big|_x^2+\Big|E_y^0\Big|_y^2\Big)+\Big(\frac{5}{4}+2C_{cfl}\Big)\Big(\Big\|\delta_x H_z^{\frac{1}{2}}\Big\|_{\delta_x H_z}^2+\Big\|\delta_y H_z^{\frac{1}{2}}\Big\|_{\delta_y H_z}^2\Big)\\&\quad+\Big(1+(\Delta t\omega_{pe})^2\Big)\Big(\Big\|\delta_x J_x^{\frac{1}{2}}\Big\|_{\delta_x J_x}^2+\Big\|\delta_x J_y^{\frac{1}{2}}\Big\|_{\delta_x J_y}^2+\Big\|\delta_y J_x^{\frac{1}{2}}\Big\|_{\delta_y J_x}^2+\Big\|\delta_y J_y^{\frac{1}{2}}\Big\|_{\delta_y J_y}^2\Big)\\&\quad+\Big(1+(\Delta t\omega_{pm})^2\Big)\Big(\Big\|\delta_x K_z^1\Big\|_{\delta_x K_z}^2+\Big\|\delta_y K_z^1\Big\|_{\delta_y K_z}^2\Big)\\&\quad+\frac{11(\Delta t)^2}{2\epsilon_0^2}\Big(|J_x^{\frac{1}{2}}|_x^2+|J_y^{\frac{1}{2}}|_y^2\Big)+\frac{4(\Delta t)^2}{\epsilon_0^2}\Big(|\delta_x H_z^{\frac{1}{2}}|_y^2+|\delta_y H_z^{\frac{1}{2}}|_x^2\Big)\\&\quad+\Big(1+(\Delta t\omega_{pe})^2\Big)\Big(\|\delta_x K_z^1\|_{\delta_x K_z}^2+\|\delta_y K_z^1\|_{\delta_y K_z}^2\Big),\end{aligned}\tag{3.25}$$

which implies the required estimate (3.1) if

$$\Delta t<\min\left\{\frac{\sqrt{\mu_0\epsilon_0}}{2}\sqrt{\frac{h_{\min}^2k_{\min}^2}{h_{\min}^2+k_{\min}^2}},\frac{\sqrt{2}}{\sqrt{3}\omega_{pe}},\frac{1}{\omega_{pm}}\right\}.$$

We can conclude the proof of the Theorem 3.1. □

3.3 Stability of the Yee scheme in the discrete L^2 and H^1 norm

Replacing δ_x by the identity operator I and arguing similarly to that in the proof of Theorem 3.2. We can prove the following result.

Theorem 3.2. *For $n\geq 0$, let ${E_x}^n_{i+\frac{1}{2},j}$, ${E_y}^n_{i,j+\frac{1}{2}}$, ${H_z}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$, ${J_x}^{n+\frac{1}{2}}_{i+\frac{1}{2},j}$, ${J_y}^{n+\frac{1}{2}}_{i,j+\frac{1}{2}}$, ${K_z}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}$ be the solution to the fully discrete Yee scheme* (2.24)-(2.29) *with the PEC boundary condition* (2.30), *and possesses the following regularity property:*

$$\begin{aligned}&E_x,E_y,H_z\in C^1([0,T],C^2(\bar{\Omega})),\\&J_x,J_y,K_z\in C^1([0,T],C^1(\bar{\Omega})).\end{aligned}$$

If the time step size constraint

$$\Delta t < \min\left\{\frac{\sqrt{\mu_0\epsilon_0}}{2}\sqrt{\frac{h_{\min}^2k_{\min}^2}{h_{\min}^2+k_{\min}^2}},\frac{\sqrt{2}}{\sqrt{3}\omega_{pe}},\frac{1}{\omega_{pm}}\right\}$$

is satisfied, then the following estimate holds:

$$\begin{aligned}&\|E_x^n\|_{E_x}^2+\|E_y^n\|_{E_y}^2+\|J_x^{n+\frac{1}{2}}\|_{J_x}^2+\|J_y^{n+\frac{1}{2}}\|_{J_y}^2+\|H_z^{n+\frac{1}{2}}\|_{H_z}^2+\|K_z^{n+1}\|_{K_z}^2\\ \leq&C\left\{\|E_x^0\|_{E_x}^2+\|E_y^0\|_{E_y}^2+\|J_x^{\frac{1}{2}}\|_{J_x}^2+\|J_y^{\frac{1}{2}}\|_{J_y}^2+\|H_z^{\frac{1}{2}}\|_{H_z}^2+\|K_z^1\|_{K_z}^2\right\}.\end{aligned}\tag{3.26}$$

Combining Theorems 3.1 and Theorems 3.2, we can obtain the stability result in the discrete H^1 norm.

Theorem 3.3. *For $n\geq 0$, let ${E_x}^n_{i+\frac{1}{2},j}$, ${E_y}^n_{i,j+\frac{1}{2}}$, ${H_z}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$, ${J_x}^{n+\frac{1}{2}}_{i+\frac{1}{2},j}$, ${J_y}^{n+\frac{1}{2}}_{i,j+\frac{1}{2}}$, ${K_z}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}$ be the solution to the fully discrete Yee scheme* (2.24)-(2.29) *with the PEC boundary condition* (2.30), *and possesses the following regularity property:*

$$\begin{aligned}&E_x,E_y,H_z\in C^1([0,T],C^2(\bar{\Omega})),\\&J_x,J_y,K_z\in C^1([0,T],C^1(\bar{\Omega})).\end{aligned}$$

If the time step size constraint

$$\Delta t<\min\left\{\frac{\sqrt{\mu_0\epsilon_0}}{2}\sqrt{\frac{h_{\min}^2k_{\min}^2}{h_{\min}^2+k_{\min}^2}},\frac{\sqrt{2}}{\sqrt{3}\omega_{pe}},\frac{1}{\omega_{pm}}\right\}$$

is satisfied, then the following estimate holds:

$$\|\boldsymbol{E}^n\|_1^2+\|\boldsymbol{J}^{n+\frac{1}{2}}\|_1^2+\|H_z^{n+\frac{1}{2}}\|_1^2+\|K_z^{n+1}\|_1^2\leq C\left\{\|\boldsymbol{E}^0\|_1^2+\|\boldsymbol{J}^{\frac{1}{2}}\|_1^2+\|H_z^{\frac{1}{2}}\|_1^2+\|K_z^1\|_1^2\right\},\tag{3.27}$$

where, for $n \geq 0$

$$\|\boldsymbol{E}^n\|_1^2 = \|E_x^n\|_{E_x}^2 + \|E_y^n\|_{E_y}^2 + |(E_x^n, E_y^n)|_1^2, \quad \|\boldsymbol{J}^{n+\frac{1}{2}}\|_1^2 = \|J_x^{n+\frac{1}{2}}\|_{J_x}^2 + \|J_y^{n+\frac{1}{2}}\|_{J_y}^2 + |(J_x^{n+\frac{1}{2}}, J_y^{n+\frac{1}{2}})|_1^2,$$
$$\|H_z^{n+\frac{1}{2}}\|_1^2 = \|H_z^{n+\frac{1}{2}}\|_{H_z}^2 + |H_z^{n+\frac{1}{2}}|_1^2, \qquad \|K_z^{n+1}\|_1^2 = \|K_z^{n+1}\|_{H_z}^2 + |K_z^{n+1}|_1^2.$$

This indicates that the fully discrete Yee scheme is conditionally stable in the discrete H^1 norm when the time step size constraint

$$\Delta t < \min\left\{ \frac{\sqrt{\mu_0\epsilon_0}}{2}\sqrt{\frac{h_{\min}^2 k_{\min}^2}{h_{\min}^2 + k_{\min}^2}}, \frac{\sqrt{2}}{\sqrt{3}\omega_{pe}}, \frac{1}{\omega_{pm}} \right\}$$

is satisfied.

4 Numerical results and discussion

In this section, some numerical experiments using the Yee scheme on non-uniform meshes have been carried out. We use Examples 4.1 and 4.2 to demonstrate the new energy stability, and investigate the convergence rates in discrete L^2 and H^1 norms on non-uniform rectangular and cubic meshes, respectively. In Example 4.3, we will numerically simulate the electromagnetic wave propagation in the metamaterial to show the backward wave propagation phenomenon. All our tests were carried out using MATLAB 2017b running on Dell Inspiron 7420 laptop with 12GB of RAM and 2.60GHz CPU.

In 2D case, denote the values of the exact solution of metamaterial Maxwell's equations (2.1)-(2.8) at the staggered points by $E_x(t^n)_{i+\frac{1}{2},j}$, $E_y(t^n)_{i,j+\frac{1}{2}}$, $H_z(t^{n+\frac{1}{2}})_{i+\frac{1}{2},j+\frac{1}{2}}$, $J_x(t^{n+\frac{1}{2}})_{i+\frac{1}{2},j}$, $J_y(t^{n+\frac{1}{2}})_{i,j+\frac{1}{2}}$ and $K_z(t^{n+1})_{i+\frac{1}{2},j+\frac{1}{2}}$ define the error of Yee scheme as follows

$$\mathcal{E}x^n_{i+\frac{1}{2},j} = E_x(t^n)_{i+\frac{1}{2},j} - E x^n_{i+\frac{1}{2},j}, \qquad \mathcal{E}y^n_{i,j+\frac{1}{2}} = E_y(t^n)_{i,j+\frac{1}{2}} - E y^n_{i,j+\frac{1}{2}},$$
$$\mathcal{J}x^{n+\frac{1}{2}}_{i+\frac{1}{2},j} = J_x(t^{n+\frac{1}{2}})_{i+\frac{1}{2},j} - J x^{n+\frac{1}{2}}_{i+\frac{1}{2},j}, \qquad \mathcal{J}y^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} = J_y(t^{n+\frac{1}{2}})_{i,j+\frac{1}{2}} - J y^{n+\frac{1}{2}}_{i,j+\frac{1}{2}},$$
$$\mathcal{H}z^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = H_z(t^{n+\frac{1}{2}})_{i+\frac{1}{2},j+\frac{1}{2}} - H z^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}, \qquad \mathcal{K}z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = K_z(t^{n+1})_{i+\frac{1}{2},j+\frac{1}{2}} - K z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}},$$

Denote E^n_{H1s2D} and E^n_{L22D} as the energy in the H^1 semi-norm and in the L^2 norm, respectively.

$$\begin{aligned} E^n_{H1s2D} = & \|\delta_x E_x^n\|^2_{\delta_x E_x} + \left\|\delta_x E_y^n\right\|^2_{\delta_x E_y} + \|\delta_y E_x^n\|^2_{\delta_y E_x} + \left\|\delta_y E_y^n\right\|^2_{\delta_y E_y} \\ & + \left\|\delta_x J_x^{n+\frac{1}{2}}\right\|^2_{\delta_x J_x} + \left\|\delta_x J_y^{n+\frac{1}{2}}\right\|^2_{\delta_x J_y} + \left\|\delta_y J_x^{n+\frac{1}{2}}\right\|^2_{\delta_y J_x} + \left\|\delta_y J_y^{n+\frac{1}{2}}\right\|^2_{\delta_y J_y} \\ & + \left\|\delta_x H_z^{n+\frac{1}{2}}\right\|^2_{\delta_x H_z} + \left\|\delta_y H_z^{n+\frac{1}{2}}\right\|^2_{\delta_y H_z} + \left\|\delta_x K_z^{n+1}\right\|^2_{\delta_x K_z} + \left\|\delta_y K_z^{n+1}\right\|^2_{\delta_y K_z}, \end{aligned}$$

$$E^n_{L22D}=\|E^n_x\|^2_{E_x}+\|E^n_y\|^2_{E_y}+\|J^{n+\frac{1}{2}}_x\|^2_{J_x}+\|J^{n+\frac{1}{2}}_y\|^2_{J_y}+\|H^{n+\frac{1}{2}}_z\|^2_{H_z}+\|K^{n+1}_z\|^2_{K_z}.$$

Let ErrH1s2D and ErrL22D be the absolute errors in the H^1 semi-norm and in the L^2 norm, respectively.

$$\begin{aligned}\text{ErrH1s2D}=\Big(&\|\delta_x\mathcal{E}^n_x\|^2_{\delta_xE_x}+\left\|\delta_x\mathcal{E}^n_y\right\|^2_{\delta_xE_y}+\|\delta_y\mathcal{E}^n_x\|^2_{\delta_yE_x}+\left\|\delta_y\mathcal{E}^n_y\right\|^2_{\delta_yE_y}\\&+\left\|\delta_x\mathcal{J}^{n+\frac{1}{2}}_x\right\|^2_{\delta_xJ_x}+\left\|\delta_x\mathcal{J}^{n+\frac{1}{2}}_y\right\|^2_{\delta_xJ_y}+\left\|\delta_y\mathcal{J}^{n+\frac{1}{2}}_x\right\|^2_{\delta_yJ_x}+\left\|\delta_y\mathcal{J}^{n+\frac{1}{2}}_y\right\|^2_{\delta_yJ_y}\\&+\left\|\delta_x\mathcal{H}^{n+\frac{1}{2}}_z\right\|^2_{\delta_xH_z}+\left\|\delta_y\mathcal{H}^{n+\frac{1}{2}}_z\right\|^2_{\delta_yH_z}+\left\|\delta_x\mathcal{K}^{n+1}_z\right\|^2_{\delta_xK_z}+\left\|\delta_y\mathcal{K}^{n+1}_z\right\|^2_{\delta_yK_z}\Big)^{\frac{1}{2}},\\\text{ErrL22D}=\Big(&\|\mathcal{E}^n_x\|^2_{E_x}+\|\mathcal{E}^n_y\|^2_{E_y}+\|\mathcal{J}^{n+\frac{1}{2}}_x\|^2_{J_x}+\|\mathcal{J}^{n+\frac{1}{2}}_y\|^2_{J_y}+\|\mathcal{H}^{n+\frac{1}{2}}_z\|^2_{H_z}+\|\mathcal{K}^{n+1}_z\|^2_{K_z}\Big)^{\frac{1}{2}}.\end{aligned}$$

Example 4.1. We choose a rectangular physical domain $\Omega=(0,1)\times(0,1)$ and $\epsilon_0=\mu_0=1$, $\Gamma_m=\Gamma_e=\omega_{pm}=\omega_{pe}=\pi$. The spatial partition is $[0:h:0.5-h,0.5:h/2:1]\times[0:k:0.5-k,0.5:k/2:1]$, which is the same as in [14]. The exact solution of this 2D problem is given by:

$$\begin{cases}\mathbf{E}=\mathbf{W}e^{-\pi t}, \quad H_z=\cos(\pi x)\cos(\pi y)e^{-\pi t},\\ \mathbf{J}=\mathbf{W}e^{-\pi t}\pi^2 t, \quad K_z=\cos(\pi x)\cos(\pi y)e^{-\pi t}\pi^2 t,\\ \mathbf{g}=\mathbf{W}e^{-\pi t}\pi^2 t,\\ f=\cos(\pi x)\cos(\pi y)e^{-\pi t}(-3\pi+\pi^2 t),\end{cases} \tag{4.1}$$

where $\mathbf{g}=(g_x,g_y)^T,f$ are the right-hand side source term of Eqs. (2.1)-(2.3), respectively, $\mathbf{W}=(\cos(\pi x)\sin(\pi y),-\sin(\pi x)\cos(\pi y))^T$.

In order to verify the Theorems 3.1 and 3.2, we consider (4.1) without source term. We take the mesh size $h=k=1/64$, then $h_{\min}=k_{\min}=1/128$, the time step size constraint as following

$$\Delta t < \min\left\{\frac{1}{2}\sqrt{\frac{h^2_{\min}k^2_{\min}}{h^2_{\min}+k^2_{\min}}},\frac{\sqrt{2}}{\sqrt{3}\pi},\frac{1}{\pi}\right\}=2.8\times10^{-3}. \tag{4.2}$$

Thus, we can choose $\Delta t=10^{-3}$, $C=3$ in Theorems 3.1 and 3.2, then run total 10000 time steps. The energy stability results are presented in Fig. 1. From Fig. 1, we can clearly the energy in H^1 sei-norm and L^2 norm at any time are bounded by three times the initial energy.

Inspired by the result of [14], the covergence result in the L^2 norm of Yee scheme is superconvergence on non-uniform rectangular meshes. Here, we make a thorough inquiry about he convergence result in the H^1 semi-norm of Yee sheme. We take the mesh step sizes $h=k$ varying from $1/4$ to $1/256$, then, choose the fixed time step size $\Delta t=$

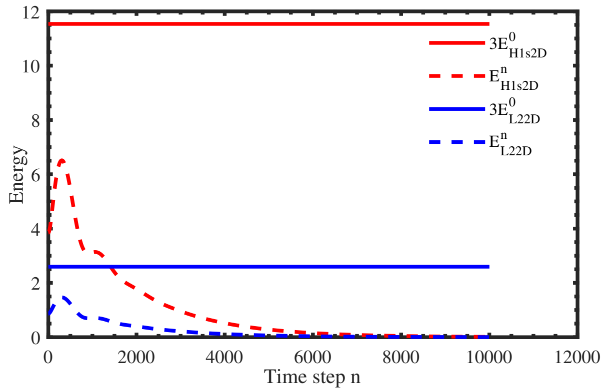

Figure 1: Time evolution of energy in H^1 semi-norm and L^2 norm in Example 4.1.

Table 1: Errors and convergence rates of the Yee scheme on non-uniform rectangular grids.

h	ErrH1s2D	Rate	ErrL22D	Rate
1/4	4.1889E-02	-	7.8334E-03	-
1/8	1.1201E-02	1.9030	1.9518E-03	2.0048
1/16	2.8803E-03	1.9593	4.8781E-04	2.0004
1/32	7.2943E-04	1.9814	1.2195E-04	2.0001
1/64	1.8348E-04	1.9911	3.0485E-05	2.0001
1/128	4.6006E-05	1.9958	7.6204E-06	2.0002
1/256	1.1515E-05	1.9983	1.9042E-06	2.0007

10^{-4}, run total 10000 time steps to calculate the absolute errors, ErrH1s2D and ErrL22D, respectively. The numerical results are presented in Table 1. From Table 1, we can clearly see the Yee scheme on non-uniform rectangular grids is stable and the convergence rate in space is approximate second order. This shows that the Yee scheme on non-uniform rectangular grids is superconvergence in the discrete H^1 norm.

Similar to the definition of the energy and the absolute errors in 2D case, we can easily define E^n_{H1s3D}, E^n_{L23D}, ErrH1s3D and ErrL23D, in the 3D case

$$
\begin{aligned}
E^n_{H1s3D} &= \|\delta E^n\|^2_{\delta E} + \left\|\delta H^{n+\frac{1}{2}}\right\|^2_{\delta H} + \left\|\delta J^{n+\frac{1}{2}}\right\|^2_{\delta J} + \left\|\delta K^{n+1}\right\|^2_{\delta K}, \\
E^n_{L23D} &= \|E^n_x\|^2_{E_x} + \|E^n_y\|^2_{E_y} + + \|E^n_z\|^2_{E_z} + \|J^{n+\frac{1}{2}}_x\|^2_{J_x} + \|J^{n+\frac{1}{2}}_y\|^2_{J_y} + + \|J^{n+\frac{1}{2}}_z\|^2_{J_z} \\
&\quad + \|H^{n+\frac{1}{2}}_x\|^2_{H_x} + \|H^{n+\frac{1}{2}}_y\|^2_{H_y} + \|H^{n+\frac{1}{2}}_z\|^2_{H_z} + \|K^{n+1}_x\|^2_{K_x} + \|K^{n+1}_y\|^2_{K_y} + \|K^{n+1}_z\|^2_{K_z}, \\
\text{ErrH1s3D} &= \left(\|\delta \mathcal{E}^n\|^2_{\delta E} + \left\|\delta \mathcal{H}^{n+\frac{1}{2}}\right\|^2_{\delta H} + \left\|\delta \mathcal{J}^{n+\frac{1}{2}}\right\|^2_{\delta J} + \left\|\delta \mathcal{K}^{n+1}\right\|^2_{\delta K} \right)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\text{ErrL23D}=\Big(&\|\mathcal{E}_x^n\|_{E_x}^2+\|\mathcal{E}_y^n\|_{E_y}^2++\|\mathcal{E}_z^n\|_{E_z}^2+\|\mathcal{J}_x^{n+\frac{1}{2}}\|_{J_x}^2+\|\mathcal{J}_y^{n+\frac{1}{2}}\|_{J_y}^2++\|\mathcal{J}_z^{n+\frac{1}{2}}\|_{J_z}^2\\
&+\|\mathcal{H}_x^{n+\frac{1}{2}}\|_{H_x}^2+\|\mathcal{H}_y^{n+\frac{1}{2}}\|_{H_y}^2+\|\mathcal{H}_z^{n+\frac{1}{2}}\|_{H_z}^2+\|\mathcal{K}_x^{n+1}\|_{K_x}^2+\|\mathcal{K}_y^{n+1}\|_{K_y}^2+\|\mathcal{K}_z^{n+1}\|_{K_z}^2\Big)^{\frac{1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
\|\delta\mathcal{V}^n\|_{\delta\mathcal{V}}^2=&\|\delta_x\mathcal{V}_x^n\|_{\delta_xV_x}^2+\left\|\delta_x\mathcal{V}_y^n\right\|_{\delta_xV_y}^2+\|\delta_x\mathcal{V}_z^n\|_{\delta_xV_z}^2+\|\delta_y\mathcal{V}_x^n\|_{\delta_yV_x}^2\\
&+\left\|\delta_y\mathcal{V}_y^n\right\|_{\delta_yV_y}^2+\|\delta_y\mathcal{V}_z^n\|_{\delta_yV_z}^2+\|\delta_z\mathcal{V}_x^n\|_{\delta_zV_x}^2+\left\|\delta_z\mathcal{V}_y^n\right\|_{\delta_zV_y}^2+\|\delta_z\mathcal{V}_z^n\|_{\delta_zV_z}^2.
\end{aligned}
$$

Example 4.2. In this example, we choose the physical domain is the unit cube $\Omega=[0,1]^3$ and all physical parameters being one (i.e., $\epsilon_0=\mu_0=\Gamma_m=\Gamma_e=\omega_{pm}=\omega_{pe}=1$). The spatial partition is $[0{:}h{:}0.5-h,0.5{:}h/2{:}1]\times[0{:}k{:}0.5-k,0.5{:}k/2{:}1]\times[0{:}l{:}0.5-l,0.5{:}l/2{:}1]$, which is the similar with Example 4.1. The initial condition and the right side of the equation are computed according to the analytic solution [6] given as below.

$$\mathbf{E}=\mathbf{U}e^{-t}\cos t,\quad \mathbf{J}=\mathbf{U}e^{-t}\sin t, \tag{4.3a}$$

$$\mathbf{H}=\mathbf{V}e^{-t}\cos t,\quad \mathbf{K}=\mathbf{V}e^{-t}\sin t, \tag{4.3b}$$

where

$$
\mathbf{U}=\begin{pmatrix} A\cos\pi x\sin\pi y\sin\pi z\\ B\sin\pi x\cos\pi y\sin\pi z\\ C\sin\pi x\sin\pi y\cos\pi z\end{pmatrix},\qquad \mathbf{V}=\begin{pmatrix} \pi(C-B)\sin\pi x\cos\pi y\cos\pi z\\ \pi(A-C)\cos\pi x\sin\pi y\cos\pi z\\ \pi(B-A)\cos\pi x\cos\pi y\sin\pi z\end{pmatrix}.
$$

The source term at the right hand side of the equation can be derived as follow

$$
\mathbf{f}=\epsilon_0\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{H}+\mathbf{J}=\begin{pmatrix}(-A-3A\pi^2)\cos\pi x\sin\pi y\sin\pi z\\ (-B-3B\pi^2)\sin\pi x\cos\pi y\sin\pi z\\ (-C-3C\pi^2)\sin\pi x\sin\pi y\cos\pi z\end{pmatrix}e^{-t}\cos t,
$$

where the constants $A=1$, $B=1/3$ and $C=-4/3$.

In this example, we explore the energy stability and convergence results of Yee scheme on nonuniform rectangular meshes in three dimensional. For the energy stability test, we consider (4.3) without source term. We take the mesh size $h=k=l=1/32$, then, we can choose proper parameters $\Delta t=10^{-3}$, $C=1$, then run total 10000 time steps. The energy stability results are presented in Fig. 2. In Fig. 2, the energy curves show that energy in H^1 sei-norm and L^2 norm for all time steps are bounded by the initial energy.

In this example, we take a fixed time step size $\Delta t=10^{-4}$, the mesh step sizes $h=k=l$ varying from $1/4$ to $1/64$ and then runs total 10000 time steps to calculate the absolute errors, ErrH1s3D and ErrL23D, in the H^1 semi-norm and L^2 norm, respectively. The numerical results are presented in Table 2. From Table 2, we can clearly see the Yee scheme on non-uniform cubic grids is superconvergence in the discrete H^1 norm.

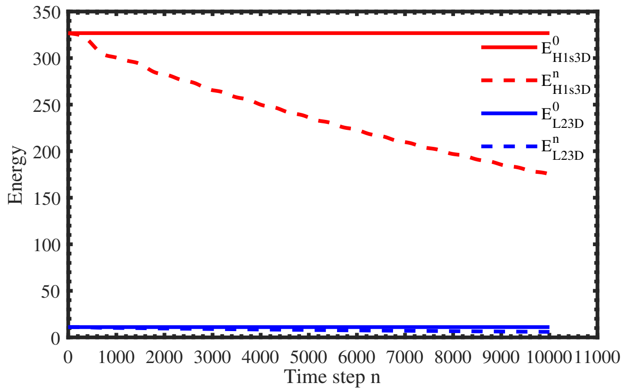

Figure 2: Time evolution of energy in H^1 semi-norm and L^2 norm in Example 4.2.

Table 2: Errors and convergence rates of the Yee scheme on non-uniform cubic grids.

h	ErrH1s3D	Rate	ErrL23D	Rate
1/4	2.2247E-01	-	3.9970E-02	-
1/8	5.7572E-02	1.9502	1.0052E-02	1.9914
1/16	1.4616E-02	1.9778	2.5170E-03	1.9978
1/32	3.6810E-03	1.9894	6.2948E-04	1.9995
1/64	9.2385E-04	1.9944	1.5738E-04	1.9999
1/128	2.3135E-04	1.9976	3.9343E-05	2.0001

Example 4.3. In this example, we use the Yee scheme to simulate the classic example of electromagnetic wave propagations on a non-uniform rectangular grids in metamaterials originally introduced by Ziolkowski [33]. The metamaterial slab of size $[0.024,0.054]\text{m}\times[0.002,0.062]\text{m}$ occupies in subdomain $[0,0.07]\text{m}\times[0,0.064]\text{m}$ and the other domain is filled with vacuum with ϵ_0 and μ_0. In addition, the perfectly matched layers (PML) around the physical domain. The velocity in vacuum is $c_0=1/\sqrt{\epsilon_0\mu_0}=3.0\times10^8\text{m/s}$, and the frequency is chosen as $f_0=3\times10^{10}\text{Hz}$. The H_z filed are excited with a line source located at $x=0.004\text{m}$ and $y\in[0.025,0.035]\text{m}$. The input signal in space as $e^{-(x-0.03)^2/(0.01)^2}$ and in time:

$$f(t)=\begin{cases} g_{\text{on}}\sin(\omega_0 t) & \text{for} \quad 0\le t<mT_p, \\ \sin(\omega_0 t) & \text{for} \quad mT_p\le t<(m+n)T_p, \\ g_{\text{off}}\sin(\omega_0 t) & \text{for} \quad (m+n)T_p\le t<(2m+n)T_p, \\ 0 & \text{for} \quad (2m+n)\le t, \end{cases}$$

where $T_p=1/f_0$, and $g_{\text{on}}(t)=10x_{\text{on}}^3-15x_{\text{on}}^4+6x_{\text{on}}^5$, $g_{\text{off}}(t)=1-[10x_{\text{off}}^3-15x_{\text{off}}^4+6x_{\text{off}}^5]$ with $x_{\text{on}}=t/mT_p$ and $x_{\text{off}}=(t-(m+n)T_p)/mT_p$.

In this simulation, we take $m=2$, $n=12$, $\omega_0=2\pi f_0$, $h=10^{-4}\text{m}$, the metamaterial

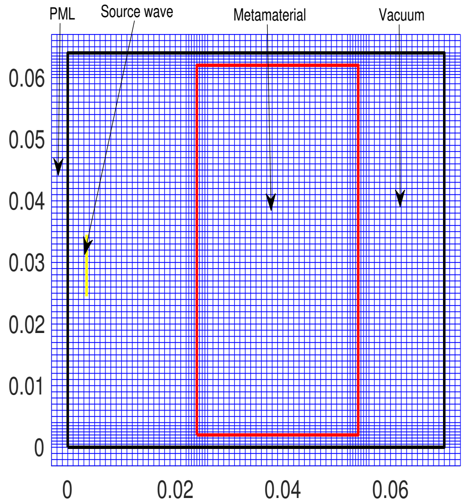

Figure 3: The non-uniform mesh sketch.

parameters as following:

$$\Gamma_e = \Gamma_m = 10^8 \text{s}^{-1}, \qquad \omega_{pe} = \omega_{pm} = 2\pi\sqrt{2}\cdot 3\times 10^9 \text{s}^{-1},$$
$$\epsilon_0 = 8.85\times 10^{-12}\text{C}^2\text{N}^{-1}\text{m}^{-2}, \qquad \mu_0 = 4\pi\times 10^{-7}\text{H},$$

the time step size $\Delta t = 10^{-13}\text{s} = 0.1\text{ps}$ (picosecond), run 5000 time steps.

The initial spatial partition in Fig. 3, then the non-uniform mesh uniformly refined by dividing each edge into two equal parts. The calculated $|H_z|$ fields and **E** fields at various times are presented in Fig. 4. The simulation clearly shows the special phenomena (backward wave propagation) in metamaterials.

5 Conclusions

In this paper, several new energy identities of Maxwell's equations in metamaterials with the PEC boundary condition have been derived. These identities give us new energy methods in studying Yee scheme on non-uniform meshes. It was proved that the Yee scheme of metamaterial Maxwell's equations with the PEC boundary condition on non-uniform meshes is conditionally stable in the discrete L^2 and H^1 norms. Numerical experiments confirm the analysis on stability of Yee scheme. Moreover, we find that the Yee scheme has superconvergence in discrete H^1 and L^2 norms on non-uniform rectangular and cubic meshes. In the future, we will give the convergence analysis of the superconvergence in discrete H^1 norm.

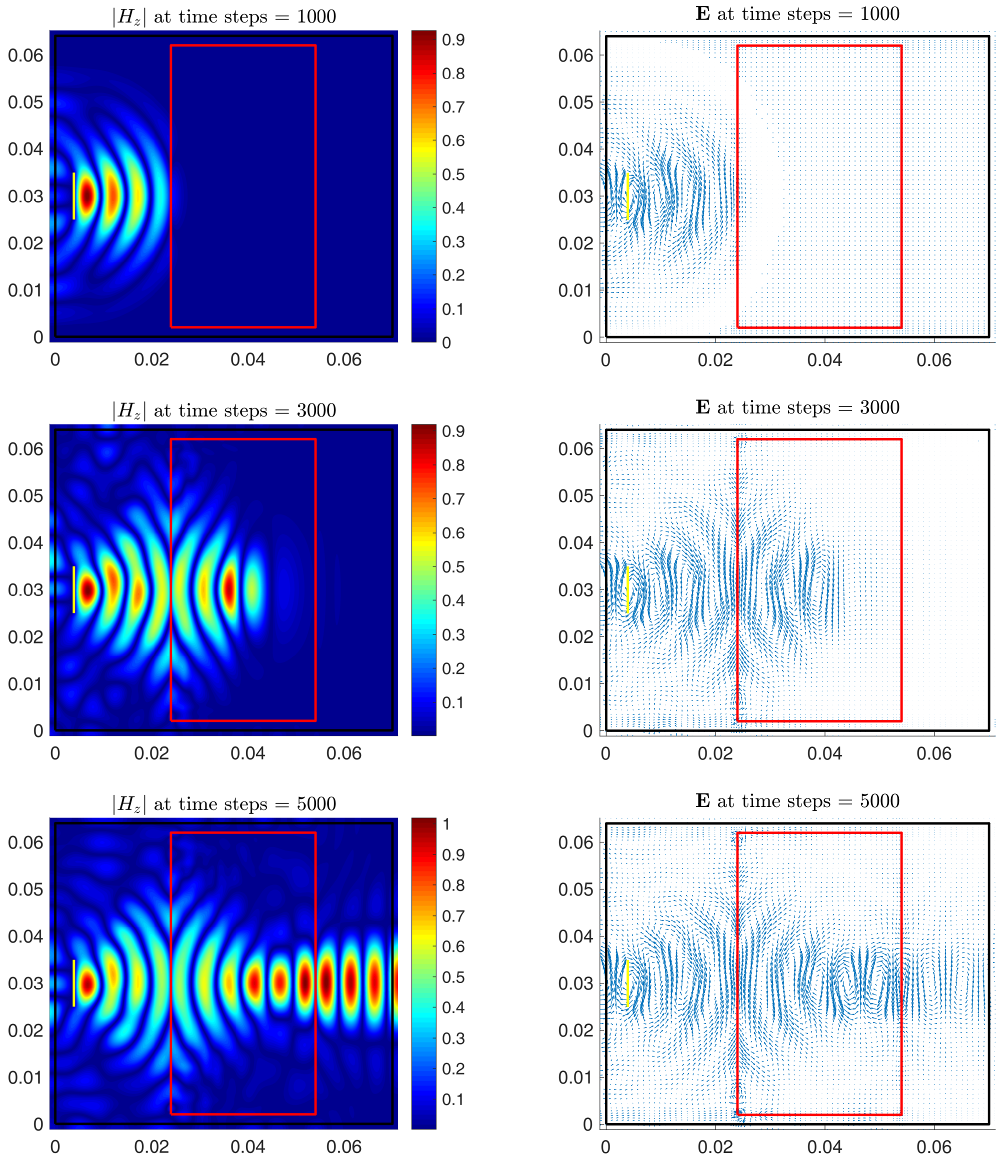

Figure 4: $|H_z|$ fields (left) and $\mathbf{E}$ fields (right) at various time steps.

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