A Stabilizer-Free Weak Galerkin Finite Element Method for the Stokes Equations

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Abstract. A stabilizer-free weak Galerkin finite element method is proposed for the Stokes equations in this paper. Here we omit the stabilizer term in the new method by increasing the degree of polynomial approximating spaces for the weak gradient operators. The new algorithm is simple in formulation and the computational complexity is also reduced. The corresponding approximating spaces consist of piecewise polynomials of degree $k \geq 1$ for the velocity and $k-1$ for the pressure, respectively. Optimal order error estimates have been derived for the velocity in both H^1 and L^2 norms and for the pressure in L^2 norm. Numerical examples are presented to illustrate the accuracy and convergency of the method.

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1 Introduction

In this paper, we propose a stabilizer-free WG finite element method for the Stokes equations. For simplicity, we consider the Stokes equations with homogeneous Dirichlet boundary condition which seeks unknown vector-valued function $\mathbf{u}$ and scalar function p satisfying

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega, \tag{1.1c}$$

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where Ω is a polygonal domain in $\mathbb{R}^d$ and $\mathbf{f} \in [L^2(\Omega)]^d$ is the unit external volumetric force acting on the fluid.

The variational formulation for the Stokes equations (1.1a)-(1.1c) is finding $\mathbf{u} \in [H^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ that satisfy $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ such that

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \tag{1.2a}$$

$$(\nabla\cdot\mathbf{u}, q) = 0, \tag{1.2b}$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$, where $H_0^1(\Omega)$ is defined in (2.1) and $L_0^2(\Omega)$ is defined as follows:

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega); \int_\Omega q\,dx = 0 \right\}. \tag{1.3}$$

Various numerical methods have been developed for solving the Stokes equations, such as the finite element methods (FEMs) [3,7,8], the finite volume methods (FVMs) [5,27,28], and the finite difference methods [4,16,17]. Taking the classical conforming FEMs as an example, they are based on the variational form (1.2a)-(1.2b) and finite dimensional approximating subspaces of $[H_0^1(\Omega)]^d \times L_0^2(\Omega)$ consisting of piecewise polynomials. In those methods, the $\inf-\sup$ condition [1,2] has to be satisfied, which causes some limitations in constructing elements and generating meshes.

As a generalization of the classical conforming FEMs, the weak Galerkin (WG) finite element method has been gradually studied by researchers. This method was first introduced by Wang and Ye [12,18,19] for second order elliptic problems in 2013. Then it was extended to other partial differential equations (PDEs), such as Stokes problems [20,21], Brinkman problems [10,24,29], linear elasticity problems [22,23], biharmonic problems [11,13,14], and parabolic problems [6,31]. The WG method for the Stokes problems would adopt the following form: find $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$ and $p_h \in W_h$ satisfying $\mathbf{u}_h = Q_b\mathbf{g}$ on $\partial\Omega$ and

$$(\nabla_d\mathbf{u}_h, \nabla_d\mathbf{v}_h) - (\nabla_d\cdot\mathbf{v}_h, p_h) + s(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_0), \tag{1.4a}$$

$$(\nabla_d\cdot\mathbf{u}_h, q_h) = 0, \tag{1.4b}$$

for all $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$ and $q_h \in W_h$. Here V_h and W_h are the properly defined WG finite element spaces for the scalar variables and vector-valued variables, respectively. ∇_d is a weak gradient operator and $\nabla_d\cdot$ is a weak divergence operator to be detailed in Section 2. As a parameter free stabilizer, the bilinear form $s(\cdot,\cdot)$ enforces a certain weak continuity for the approximating solutions across element boundaries. In 2016, a new WG finite element method has been developed for solving the Stokes equations based on two gradient operators in [21]. This method employs two parameter independent stabilizers $s(\cdot,\cdot)$ and $c(\cdot,\cdot)$ for velocity functions and pressure functions, respectively. The corresponding numerical scheme is: find $\mathbf{u}_h \in \tilde{V}_h$ and $p_h \in \tilde{W}_h$ satisfying $\mathbf{u}_h = Q_b\mathbf{g}$ on $\partial\Omega$ such that

$$(\nabla_d\mathbf{u}_h, \nabla_d\mathbf{v}) + (\mathbf{v}_0, \widetilde{\nabla}_d p_h) + s(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}_0),$$

$$(\mathbf{u}_0, \widetilde{\nabla}_d q) - c(p_h, q) = 0,$$

for all test functions

$$\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{V}_h^0 \quad \text{and} \quad q \in \tilde{W}_h^0,$$

where $\tilde{V}_h$ and $\tilde{W}_h$ are properly-defined finite element spaces.

The WG method has two key features: (1) the derivatives are taken as distributions or approximations of distributions, and (2) the approximating functions are discontinuous. But what comes with it is more degrees of freedom than the classical FEMs. In recent years, to circumvent this limitation, several improved numerical methods based on WG method have been developed to solve the Stokes equations. The modified weak Galerkin (MWG) finite element method [15] uses the average of the inside function $\mathbf{u}_0$ to replace $\mathbf{u}_b$ on the boundary of the element to eliminate the unknowns associated with element boundaries. The hybridized weak Galerkin (HWG) finite element method [30] uses a Lagrange multiplier defined on the element boundaries to relax certain restrictions such as continuity requirement. A Schur complement formulation of the HWG method eliminates all the interior unknowns and the Lagrange multipliers. Then in 2019, a simplified weak Galerkin (SWG) finite element method was proposed by Liu and Wang [9]. In this method, the degrees of freedom involves only those on the element boundary; i.e., the unknowns associated with the interior of each element in the original weak Galerkin are not used in SWG.

In this paper, we are concerning about a new numerical method that has a simple formulation without any stabilizer to solve the Stokes equations. To this end, we employ higher degree polynomials to compute the weak gradient ∇_d. It has a simple form close to its original PDE weak form (1.2a)-(1.2b) for discontinuous polynomials. This method has been applied to solve the second order elliptic problem [26] by Ye and Zhang in 2019. Raising the degree of polynomials in the computation of weak gradients will not increase the size of global stiffness matrix, but rather will reduce the computational complexity of programming since it removes the stabilizer term. In addition, it does not reduce the optimal-order of convergence, for both the second order elliptic equations and the Stokes equations.

This paper is organized as follows. In Section 2, we do some preliminaries for the stabilizer-free WG finite element method. The standard definition of Sobolev spaces and its associated inner products, norms, and seminorms are given. In particular, we introduce the weak function spaces, the weak gradient operator, and the weak divergence operator. In Section 3, we define three local L^2 projection operators. Then, we derive the stabilizer-free WG finite element scheme for the Stokes equations. In Section 4, we prove the existence and uniqueness of the solutions by establishing the usual inf$-$sup condition. Section 5 is devoted to derive the error equations for the numerical approximations. Optimal-order error estimates for the stabilizer-free WG finite element approximations are given in Section 6. Finally in Section 7, we present some numerical results to illustrate the accuracy and convergence derived in earlier sections.

2   Preliminaries and notations

In this section, we present some preparations and notations for Sobolev spaces and partition.

2.1   Sobolev spaces and partition

Throughout this paper, we adopt the standard definition of Sobolev spaces $H^s(\Omega)$. For any given open bounded domain $K \subseteq \Omega$, $(\cdot,\cdot)_{s,K}$, $\|\cdot\|_{s,K}$ and $|\cdot|_{s,K}$ are used to denote the inner product, norm and semi-norm, respectively. The norm $\|\cdot\|_{s,K}$ and semi-norm $|\cdot|_{s,K}$ are defined as follows

$$\|\varphi\|_{s,K} = \left(\sum_{j=0}^{s} |\varphi|_{j,K}^2 \right)^{\frac{1}{2}}, \quad |\varphi|_{s,K} = \left(\sum_{|\alpha|=s} \int_K |\partial^\alpha \varphi|^2 dK \right)^{\frac{1}{2}},$$

with the usual notation

$$\alpha = (\alpha_1, l \cdots, \alpha_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \partial^\alpha = \prod_{j=1}^{d} \partial_{x_j}^{\alpha_j}.$$

The space $H^0(K)$ coincides with $L^2(K)$, and the subscripts K in the inner product, norm, and semi-norm is dropped in the case of $K = \Omega$.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in 2D or polyhedra in 3D, and T be each element with ∂T as its boundary. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges in $\mathcal{T}_h$. For each $T \in \mathcal{T}_h$, denote by h_T and h_e the diameter of T and e, respectively. $h = \max_{T \in \mathcal{T}_h} h_T$ is the meshsize of $\mathcal{T}_h$.

In particular, the function space $H_0^1(\Omega)$ is defined as

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}, \tag{2.1}$$

and the space $H(div;\Omega)$ is defined as the set of vector-valued functions, which together with their divergence are square integrable, i.e.,

$$H(div;\Omega) = \left\{ \mathbf{v} \in [L^2(\Omega)]^d : \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.$$

The norm defined on $H(div;\Omega)$ is given by

$$\|\mathbf{v}\|_{H(div;\Omega)} = \left(\|\mathbf{v}\|_0^2 + \|\nabla \cdot \mathbf{v}\|_0^2 \right)^{\frac{1}{2}}.$$

2.2   Weak gradient operator and weak divergence operator

Since the central idea of the WG finite element method is to use the discrete differential operators to approximate classical differential operators, we define the discrete weak gradient operator and discrete weak divergence operator for the vector-valued function. To

this end, we first introduce the corresponding weak Galerkin finite element space for the vector-valued function **v** on the partition $\mathcal{T}_h$.

For a given integer $k \geq 1$, denote by V_h the WG finite element space for the vector-valued functions defined as follows

$$V_h = \left\{ \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [P_k(T)]^d, \ \mathbf{v}_b \in [P_k(e)]^d, \ e \subset \partial T, \ T \in \mathcal{T}_h \right\}. \tag{2.2}$$

We specify that $\mathbf{v}_b$ has only a single value on each edge $e \in \mathcal{E}_h$. The subspace of V_h is defined as

$$V_h^0 = \left\{ \mathbf{v} : \mathbf{v} \in V_h, \ \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega \right\}. \tag{2.3}$$

For the pressure variable, we define the weak Galerkin finite element space as follows

$$W_h = \left\{ q \in L_0^2(\Omega) : q|_T \in P_{k-1}(T) \right\}. \tag{2.4}$$

Then, we can derive the definition of the weak gradient operator and the weak divergence operator.

Definition 2.1. *For any finite element $T \in \mathcal{T}_h$ and a vector-valued function $v = \{v_0, v_b\} \in V_h + [H^1(\Omega)]^d$, denote by $\nabla_d v$ the weak gradient of v and it is a unique polynomial function in $[P_j(T)]^{d \times d}$ $(j > k)$ satisfying*

$$(\nabla_d v, \tau)_T = -(v_0, \nabla \cdot \tau)_T + \langle v_b, \tau \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \tau \in [P_j(T)]^{d \times d}, \tag{2.5}$$

where $\mathbf{n}$ is the outward normal direction to ∂T,

$$\langle v_b, \tau \cdot \mathbf{n} \rangle_{\partial T} = \int_{\partial T} v_b \tau \cdot \mathbf{n} \, ds$$

is the inner product of v_b and $\tau \cdot \mathbf{n}$ in $[P_j(T)]^{d \times d}$.

Similarly, we can define the weak divergence operator as follows

Definition 2.2. *For any $T \in \mathcal{T}_h$ and a vector-valued function $v \in V_h + [H^1(\Omega)]^d$, denote by $\nabla_d \cdot v$ the weak divergence of v defined as the unique polynomial function in $P_{k-1}(T)$ satisfying*

$$(\nabla_d \cdot v, q)_T = -(v_0, \nabla q)_T + \langle v_b \cdot \mathbf{n}, q \rangle_{\partial T}, \quad \forall q \in P_{k-1}(T), \tag{2.6}$$

where $\mathbf{n}$ is the outward normal direction to ∂T.

3 A stabilizer-free WG finite element method

In this section, we introduce the stabilizer-free WG finite element scheme and define three L^2 projection operators which will be used later in this paper. For simplicity of notation,

we define

$$(v,w) = (v,w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (v,w)_T = \sum_{T \in \mathcal{T}_h} \int_T vw dT,$$

$$\langle v,w \rangle = \langle v,w \rangle_{\partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \langle v,w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vw ds.$$

Then, we introduce two bilinear forms

$$a(\mathbf{v},\mathbf{w}) = (\nabla_d \mathbf{v}, \nabla_d \mathbf{w}) = \sum_{T \in \mathcal{T}_h} (\nabla_d \mathbf{v}, \nabla_d \mathbf{w})_T,$$

$$b(\mathbf{v},q) = (\nabla_d \cdot \mathbf{v}, q) = \sum_{T \in \mathcal{T}_h} (\nabla_d \cdot \mathbf{v}, q)_T.$$

The stabilizer-free weak Galerkin finite element scheme for the Stokes equations (1.1a)-(1.1c) is as follows

Algorithm 3.1. Find $\mathbf{u}_h \in V_h$ and $p_h \in W_h$ satisfying $\mathbf{u}_b = Q_h \mathbf{g}$ on $\partial \Omega$ and

$$a(\mathbf{u}_h,\mathbf{v}_h) - b(\mathbf{v}_h,p_h) = (\mathbf{f},\mathbf{v}_0), \qquad \forall \mathbf{v}_h \in V_h^0 + [H_0^1(\Omega)]^d, \tag{3.1a}$$

$$b(\mathbf{u}_h,q_h) = 0, \qquad \forall q_h \in W_h. \tag{3.1b}$$

For each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection operator from $[L^2(T)]^d$ onto $[P_k(T)]^d$. Similarly, for each edge $e \in \mathcal{E}_h$, denote by Q_b the L^2 projection operator from $[L^2(e)]^d$ onto $[P_k(e)]^d$. We shall combine Q_0 and Q_b by writing $Q_h = \{Q_0, Q_b\}$. Next, denote $\mathbf{Q}_h$ and $\mathbb{Q}_h$ the local L^2 projections from $[L^2(T)]^{d \times d}$ and $L^2(T)$ onto $[P_j(T)]^{d \times d}$ and $P_{k-1}(T)$, respectively.

4   Existence and uniqueness

In this section, we derive the existence and uniqueness of the solutions of Eqs. (3.1a)-(3.1b). First, we give the definition of a semi-norm $|||\mathbf{v}_h|||$ in V_h. Since the algorithm (3.1a)-(3.1b) is a typical saddle-point problem, we should introduce the inf$-$sup condition for the bilinear form $b(\cdot,\cdot)$ later.

First, for any $\mathbf{v}_h \in V_h + [H^1(\Omega)]^d$, we define the following tri-bar norm

$$|||\mathbf{v}_h|||^2 = \sum_{T \in \mathcal{T}_h} (\nabla_d \mathbf{v}_h, \nabla_d \mathbf{v}_h)_T, \tag{4.1}$$

Lemma 4.1. *For any $v = \{v_0, v_b\} \in V_h$ and $T \in \mathcal{T}_h$, the following inequality holds true*

$$\|v_b - v_0\|_{\partial T}^2 \leq C h_T \|\nabla_d v\|_T^2, \tag{4.2}$$

where C is a positive constant.

Proof. For any $\mathbf{v}_h \in V_h$ and $\tau \in [P_j(T)]^{d \times d}$, it follows from the definition of weak gradient ∇_d and integration by parts that

$$
\begin{aligned}
(\nabla_d \mathbf{v}_h, \tau)_T &= -(\mathbf{v}_0, \nabla \cdot \tau)_T + \langle \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla \mathbf{v}_0, \tau)_T - \langle \mathbf{v}_0 - \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T}.
\end{aligned}
\tag{4.3}
$$

Let n be the number of the edges/faces on a polygon/polyhadron. We can find the complete proof in [25] that there exists $\tau_0 \in [P_j(T)]^{d \times d}$, $j = n + k - 1$, such that

$$
\begin{aligned}
&(\nabla \mathbf{v}_0, \tau_0)_T = 0, \\
&\langle \mathbf{v}_0 - \mathbf{v}_b, \tau_0 \cdot \mathbf{n} \rangle_{\partial T \setminus e} = 0, \\
&\langle \mathbf{v}_b - \mathbf{v}_0, \tau_0 \cdot \mathbf{n} \rangle_e = \|\mathbf{v}_b - \mathbf{v}_0\|_e^2,
\end{aligned}
$$

and

$$
\|\tau_0\|_T \leq C h_T^{1/2} \|\mathbf{v}_b - \mathbf{v}_0\|_e.
\tag{4.4}
$$

Let $\tau = \tau_0$ in (4.3), we have

$$
(\nabla_d \mathbf{v}_h, \tau_0)_T = \|\mathbf{v}_b - \mathbf{v}_0\|_e^2.
\tag{4.5}
$$

Using the Cauchy-Schwarz inequality and substituting (4.4) into it, we get

$$
\|\mathbf{v}_b - \mathbf{v}_0\|_e^2 \leq C \|\nabla_d \mathbf{v}_h\|_T \|\tau_0\|_T \leq C h_T^{1/2} \|\nabla_d \mathbf{v}_h\|_T \|\mathbf{v}_b - \mathbf{v}_0\|_T,
$$

that is

$$
\|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \leq C h_T^{1/2} \|\nabla_d \mathbf{v}_h\|_T.
$$

This completes the proof of the lemma. □

Summing over all $T \in \mathcal{T}_h$, it gives the following inequality

$$
\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_b - \mathbf{v}_0\|_{\partial T}^2 \leq C \||\mathbf{v}_h\||^2.
\tag{4.6}
$$

Lemma 4.2. $\||v_h\||$ *defined in* (4.1) *denotes a norm in* V_h^0.

Proof. For $\mathbf{v}_h \in V_h^0$ and $\||\mathbf{v}_h\|| = 0$, according to the definition of $\||\cdot\||$ and ∇_d, we have

$$
\||\mathbf{v}_h\||^2 = \sum_{T \in \mathcal{T}_h} (\nabla_d \mathbf{v}_h, \nabla_d \mathbf{v}_h)_T = 0,
$$

i.e.,

$$
\|\nabla_d \mathbf{v}_h\|_T^2 = (\nabla_d \mathbf{v}_h, \nabla_d \mathbf{v}_h)_T = 0, \quad \forall T \in \mathcal{T}_h.
\tag{4.7}
$$

It follows from (4.2) and (4.7) that

$$\|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} = 0. \tag{4.8}$$

For any $T \in \mathcal{T}_h$ and $\tau \in [P_j(T)]^{d \times d}$, it follows from the definition of ∇_d and integration by parts that

$$\begin{aligned}
(\nabla_d \mathbf{v}_h, \tau)_T &= -(\mathbf{v}_0, \nabla \cdot \tau)_T + \langle \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla \mathbf{v}_0, \tau)_T - \langle \mathbf{v}_0 - \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T},
\end{aligned}$$

i.e.,

$$(\nabla \mathbf{v}_0, \tau)_T = (\nabla_d \mathbf{v}_h, \tau)_T + \langle \mathbf{v}_0 - \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T}.$$

Using the triangle inequality, the trace inequality, the inverse inequality, and (4.2), we obtain

$$\begin{aligned}
(\nabla \mathbf{v}_0, \tau)_T &\le \|\nabla_d \mathbf{v}_h\|_T \|\tau\|_T + \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \|\tau\|_{\partial T} \\
&\le \|\nabla_d \mathbf{v}_h\|_T \|\tau\|_T + Ch^{-1/2} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \|\tau\|_T \\
&\le C\|\nabla_d \mathbf{v}_h\|_T \|\tau\|_T.
\end{aligned}$$

Let $\tau = \nabla \mathbf{v}_0$, we have

$$\|\nabla \mathbf{v}_0\|_T \le C\|\nabla_d \mathbf{v}_h\|_T = 0,$$

which yields that $\mathbf{v}_0 = const$ on each $T \in \mathcal{T}_h$. Together with (4.8) and the fact $\mathbf{v}_b = 0$ on $\partial \Omega$, we have $\mathbf{v}_0 = 0$ and $\mathbf{v}_b = 0$. We have completed the proof. $\quad\square$

The following lemma gives the boundedness and coercivity for the bilinear form $a(\cdot, \cdot)$.

Lemma 4.3. *For any v, $w \in V_h^0$, we have*

$$|a(v, w)| \le |||v||| \, |||w|||, \tag{4.9a}$$

$$a(v, v) = |||v|||^2. \tag{4.9b}$$

As to the bilinear form $b(\cdot, \cdot)$, the following inf$-$sup condition holds true.

Lemma 4.4 ([20]). *There exist a positive constant β independent of h such that*

$$\sup_{v \in V_h^0} \frac{b(v, q)}{|||v|||} \ge \beta \|q\| \tag{4.10}$$

for all $q \in W_h$.

The above two lemmas imply the well-posedness of the scheme (3.1a)-(3.1b). Thus, we can prove the existence and uniqueness of the stabilizer-free WG method solution in the Theorem 4.1.

Theorem 4.1. *The stabilizer-free WG scheme (3.1a)-(3.1b) for Stokes equations has and only has one solution.*

5   Error equations

In this section, we do some preparatory work for the error estimates in Section 6. To this end, we derive the error equations for both the velocity variable and the pressure variable. Denote by **u**, p the exact solution of the Stokes equations (1.1a)-(1.1c), and $\mathbf{u}_h$, p_h the weak Galerkin finite element approximations obtained with equations (3.1a)-(3.1b), respectively. Let $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$ and $\varepsilon_h = Q_h p - p_h$, we will derive the error equations for $\mathbf{e}_h$ and ε_h in the following.

First, we introduce some properties of the projection operators defined in Section 3.

Lemma 5.1. *For the projection operators Q_h, $\mathbf{Q}_h$, and $\mathbb{Q}_h$, the following communicative property holds true*

$$\nabla_d \cdot (Q_h v) = \mathbb{Q}_h(\nabla \cdot v), \quad \forall v \in [H(div;\Omega)]^d. \tag{5.1}$$

Moreover, on any $T \in \mathcal{T}_h$, we have

$$\nabla_d v = \mathbf{Q}_h(\nabla v), \quad \forall v \in [H^1(\Omega)]^d. \tag{5.2}$$

Proof. From the definition of the weak divergence $\nabla_d \cdot$, Q_h and $\mathbb{Q}_h$, we have

$$\begin{aligned}
(\nabla_d \cdot (Q_h \mathbf{v}), q)_T &= -(Q_0 \mathbf{v}, \nabla q)_T + \langle Q_b \mathbf{v}, q\mathbf{n} \rangle_{\partial T} \\
&= -(\mathbf{v}, \nabla q)_T + \langle \mathbf{v}, q\mathbf{n} \rangle_{\partial T} \\
&= (\nabla \cdot \mathbf{v}, q)_T \\
&= (\mathbb{Q}_h(\nabla \cdot \mathbf{v}), q)_T, \quad \forall q \in P_{k-1}(T),
\end{aligned}$$

which completes the proof of (5.1).

As to (5.2), we also use the definition of the weak gradient ∇_d to get

$$\begin{aligned}
(\nabla_d \mathbf{v}, \tau)_T &= -(\mathbf{v}, \nabla \cdot \tau)_T + \langle \mathbf{v}, \tau \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla \mathbf{v}, \tau)_T \\
&= (\mathbf{Q}_h \nabla \mathbf{v}, \tau)_T, \quad \forall \tau \in [P_j(T)]^{d \times d}.
\end{aligned}$$

Thus, we complete the proof.   □

Lemma 5.2. *Denote e_h and ε_h the errors of the stabilizer-free weak Galerkin finite element approximation arising from (3.1a)-(3.1b). For any $v_h = \{v_0, v_b\} \in V_h^0$ and $q_h \in W_h$, we have*

$$a(e_h, v_h) - b(v_h, \varepsilon_h) = l(u, v_h) + \theta(p, v_h), \tag{5.3a}$$
$$b(e_h, q_h) = 0, \tag{5.3b}$$

where

$$l(u, v_h) = \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (\nabla u - \mathbf{Q}_h \nabla u) \cdot n \rangle_{\partial T},$$
$$\theta(p, v_h) = \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (p - Q_h p)n \rangle_{\partial T}.$$

Proof. Testing Eq. (1.1a) by $\mathbf{v}_0$, using integration by parts and the definition of projection operators $\mathbf{Q}_h$ and $\mathbb{Q}_h$, we get

$$
\begin{aligned}
-(\Delta\mathbf{u},\mathbf{v}_0) &= \sum_{T\in\mathcal{T}_h}(\nabla\mathbf{u},\nabla\mathbf{v}_0)_T - \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0,\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h}(\nabla\mathbf{v}_0,\mathbf{Q}_h\nabla\mathbf{u})_T - \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T} \\
&= -\sum_{T\in\mathcal{T}_h}(\mathbf{v}_0,\nabla\cdot(\mathbf{Q}_h\nabla\mathbf{u}))_T + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,\mathbf{Q}_h\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T} \\
&\quad + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_b,\mathbf{Q}_h\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}, \\
(\nabla p,\mathbf{v}_0) &= -\sum_{T\in\mathcal{T}_h}(p,\nabla\cdot\mathbf{v}_0)_T + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0,p\mathbf{n}\rangle_{\partial T} \\
&= -\sum_{T\in\mathcal{T}_h}(\mathbb{Q}_hp,\nabla\cdot\mathbf{v}_0)_T + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,p\mathbf{n}\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h}(\nabla(\mathbb{Q}_hp),\mathbf{v}_0)_T - \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbb{Q}_hp)\mathbf{n}\rangle_{\partial T} \\
&\quad - \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_b,(\mathbb{Q}_hp)\mathbf{n}\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,p\mathbf{n}\rangle_{\partial T},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_b,\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T} &= 0, \\
\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_b,p\mathbf{n}\rangle_{\partial T} &= 0.
\end{aligned}
$$

Using the definition of weak gradient ∇_d and weak divergence $\nabla_d\cdot$, we can easily arrive at

$$
-(\Delta\mathbf{u},\mathbf{v}_0) = \sum_{T\in\mathcal{T}_h}(\nabla_d\mathbf{v}_h,\mathbf{Q}_h\nabla\mathbf{u})_T + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbf{Q}_h\nabla\mathbf{u}-\nabla\mathbf{u})\cdot\mathbf{n}\rangle_{\partial T}, \tag{5.4a}
$$

$$
(\nabla p,\mathbf{v}_0) = -\sum_{T\in\mathcal{T}_h}(\mathbb{Q}_hp,\nabla_d\cdot\mathbf{v}_h)_T + \sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbb{Q}_hp-p)\mathbf{n}\rangle_{\partial T}. \tag{5.4b}
$$

Combining (5.4a) and (5.4b), we obtain

$$
(\mathbf{f},\mathbf{v}_0) = \sum_{T\in\mathcal{T}_h}(\nabla_d\mathbf{v}_h,\mathbf{Q}_h\nabla\mathbf{u})_T - \sum_{T\in\mathcal{T}_h}(\mathbb{Q}_hp,\nabla_d\cdot\mathbf{v}_h)_T - l(\mathbf{u},\mathbf{v}_h) - \theta(p,\mathbf{v}_h). \tag{5.5}
$$

Since the numerical solution $(\mathbf{u}_h,p_h)\in V_h\times W_h$ satisfies Eq. (3.1a) for any $\mathbf{v}_h\in V_h^0$, applying it to (5.5), we have

$$
a(\mathbf{u},\mathbf{v}_h) - b(\mathbf{v}_h,\mathbb{Q}_hp) - l(\mathbf{u},\mathbf{v}_h) - \theta(p,\mathbf{v}_h) = a(\mathbf{u}_h,\mathbf{v}_h) - b(\mathbf{v}_h,p_h),
$$

which completes the proof of (5.3a).

As to Eq. (5.3b), testing (1.1b) by $q_h \in W_h$, we arrive at

$$0 = (\nabla \cdot \mathbf{u}, q_h) = (\nabla_d \cdot \mathbf{u}, q_h).$$

With (3.1b), we can finally prove Eq. (5.3b).   □

6 Error estimate

The goal of this section is to derive the detailed analysis of error estimates in H^1 and L^2 norms for the velocity variable $\mathbf{u}_h$ and in L^2 norm for the pressure variable p_h.

Before deriving the optimal-order error estimates, we present some preparation work and inequalities.

Lemma 6.1. *For any $\boldsymbol{w} \in [H^{k+1}(\Omega)]^d$, $q \in H^k(\Omega)$, $\boldsymbol{v}_h = \{v_0, v_b\} \in V_h^0$ and $q_h \in W_h$, we have*

$$|l(\boldsymbol{w}, \boldsymbol{v}_h)| \leq Ch^k \|\boldsymbol{w}\|_{k+1} |||\boldsymbol{v}_h|||, \tag{6.1a}$$

$$|\theta(q, \boldsymbol{v}_h)| \leq Ch^k \|q\|_k |||\boldsymbol{v}_h|||, \tag{6.1b}$$

where C is a positive constant independent with h.

Proof. It follows from the Cauchy-Schwarz inequality, the trace inequality, the projection inequality, and (4.6) that

$$|l(\mathbf{w}, \mathbf{v}_h)| = \left| \sum_{T \in \mathcal{T}_h} \langle (\nabla \mathbf{w} - \mathbf{Q}_h \nabla \mathbf{w}) \cdot \mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial T} \right|$$

$$\leq C \left(\sum_{T \in \mathcal{T}_h} h_T \|\nabla \mathbf{w} - \mathbf{Q}_h \nabla \mathbf{w}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\leq Ch^k \|\mathbf{w}\|_{k+1} |||\mathbf{v}_h|||,$$

which completes the proof of (6.1a).

Similarly, we can get the inequality related to $\theta(q, \mathbf{v}_h)$, that is

$$|\theta(q, \mathbf{v}_h)| = \left| \sum_{T \in \mathcal{T}_h} \langle (q - \mathbb{Q}_h q) \mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial T} \right|$$

$$\leq C \left(\sum_{T \in \mathcal{T}_h} h_T \|q - \mathbb{Q}_h q\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\leq Ch^k \|q\|_k |||\mathbf{v}_h|||.$$

Then, we complete the proof.   □

Lemma 6.2. *For any $w \in [H^{k+1}(\Omega)]^d$, the following inequality holds true*

$$|||w - Q_h w||| \leq Ch^k ||w||_{k+1}. \tag{6.2}$$

Proof. For any $\tau \in [P_j(T)]^{d \times d}$, from the definition of weak gradient operator ∇_d, integration by parts, the trace inequality, the projection inequality, and the definition of Q_b, we get

$$
\begin{aligned}
(\nabla_d(\mathbf{w} - Q_h \mathbf{w}), \tau)_T &= -(\mathbf{w} - Q_0 \mathbf{w}, \nabla \cdot \tau)_T + \langle \mathbf{w} - Q_b \mathbf{w}, \tau \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla(\mathbf{w} - Q_0 \mathbf{w}), \tau)_T - \langle Q_b \mathbf{w} - Q_0 \mathbf{w}, \tau \cdot \mathbf{n} \rangle_{\partial T} \\
&\leq \|\nabla(\mathbf{w} - Q_0 \mathbf{w})\|_T \|\tau\|_T + \|Q_b \mathbf{w} - Q_0 \mathbf{w}\|_{\partial T} \|\tau\|_{\partial T} \\
&\leq \|\nabla(\mathbf{w} - Q_0 \mathbf{w})\|_T \|\tau\|_T + Ch^{-1/2} \|\mathbf{w} - Q_0 \mathbf{w}\|_{\partial T} \|\tau\|_T \\
&\leq Ch^k \|\mathbf{w}\|_{k+1} \|\tau\|_T.
\end{aligned}
$$

Setting $\tau = \nabla_d(\mathbf{w} - Q_h \mathbf{w})$ and summing over all $T \in \mathcal{T}_h$, we have

$$|||\mathbf{w} - Q_h \mathbf{w}||| \leq Ch^k \|\mathbf{w}\|_{k+1},$$

which gives a complete proof of the lemma.   □

Theorem 6.1. *Assume $(\mathbf{u}; p) \in [H_0^1(\Omega) \cap H^{k+1}(\Omega)]^d \times (L_0^2(\Omega) \cap H^k(\Omega))$ with $k \geq 1$ is the exact solution of the Stokes equations (1.1a)-(1.1c) and $(\mathbf{u}_h; p_h) \in V_h \times W_h$ is the numerical solution of (3.1a)-(3.1b), respectively. Let $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$ and $\varepsilon_h = Q_h p - p_h$ be the corresponding errors. Then there exist a constant C independent of h such that*

$$|||\mathbf{e}_h||| + \|\varepsilon_h\| \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{6.3}$$

Proof. Letting $\rho_h = Q_h \mathbf{u} - \mathbf{u}_h$ and $v_h = \rho_h$ in (5.3a), we have

$$
\begin{aligned}
|||\mathbf{e}_h|||^2 &= (\nabla_d \mathbf{e}_h, \nabla_d \mathbf{e}_h)_{\mathcal{T}_h} \\
&= (\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h} + (\nabla_d \mathbf{e}_h, \nabla_d \rho_h)_{\mathcal{T}_h} \\
&= (\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h} + a(\mathbf{e}_h, \rho_h) \\
&= l(\mathbf{u}, \rho_h) + \theta(p, \rho_h) + b(\rho_h, \varepsilon_h) + (\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h}. \tag{6.4}
\end{aligned}
$$

Since we have the fact that

$$
\begin{aligned}
b(\mathbf{u}_h, \varepsilon_h) &= (\nabla_d \cdot \mathbf{u}_h)_{\mathcal{T}_h} = 0, \\
b(Q_h \mathbf{u}, \varepsilon_h) &= (\nabla_d \cdot Q_h \mathbf{u}, \varepsilon_h)_{\mathcal{T}_h} = (Q_h(\nabla \cdot \mathbf{u}), \varepsilon_h)_{\mathcal{T}_h} = (\nabla \cdot \mathbf{u}, \varepsilon_h)_{\mathcal{T}_h} = 0,
\end{aligned}
$$

which implies $b(\rho_h, \varepsilon_h) = 0$. We can get

$$|||\mathbf{e}_h|||^2 = l(\mathbf{u}, \rho_h) + \theta(p, \rho_h) + (\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h},$$

where we have used the definition of $||| \cdot |||$, $l(\cdot, \cdot)$, and $\theta(\cdot, \cdot)$.

We give the estimates of the last three terms as follows. For any $\tau \in [P_j(T)]^{d \times d}$ and $\mathbf{w} \in V_h$, from the Young's inequality and (6.2), we have

$$
\begin{aligned}
(\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h} &\leq C |||\mathbf{e}_h||| \, |||\mathbf{u} - Q_h \mathbf{u}||| \\
&\leq \frac{1}{4} |||\mathbf{e}_h|||^2 + C |||\mathbf{u} - Q_h \mathbf{u}|||^2 \leq \frac{1}{4} |||\mathbf{e}_h|||^2 + C h^{2k} \|\mathbf{u}\|_{k+1}^2.
\end{aligned} \tag{6.5}
$$

As to $l(\mathbf{u}, \rho_h)$ and $\theta(p, \rho_h)$, we can derive the estimates from Lemma 6.1 and (6.2), that is

$$
\begin{aligned}
|l(\mathbf{u}, \rho_h) + \theta(p, \rho_h)| &\leq |l(\mathbf{u}, \rho_h)| + |\theta(p, \rho_h)| \\
&\leq C h^k \|\mathbf{u}\|_{k+1} |||\rho_h||| + C h^k \|p\|_k |||\rho_h||| \\
&\leq C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k) |||\rho_h||| \\
&\leq C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k)(|||\mathbf{u} - Q_h \mathbf{u}||| + |||\mathbf{e}_h|||) \\
&\leq C h^{2k} (\|\mathbf{u}\|_{k+1} + \|p\|_k)^2 + C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k) |||\mathbf{e}_h||| \\
&\leq C h^{2k} (\|\mathbf{u}\|_{k+1} + \|p\|_k)^2 + \frac{1}{4} |||\mathbf{e}_h|||^2,
\end{aligned}
$$

where we have used the Young's inequality. Then, we have

$$
\begin{aligned}
|||\mathbf{e}_h|||^2 &= l(\mathbf{u}, \rho_h) + \theta(p, \rho_h) + (\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h} \\
&\leq |l(\mathbf{u}, \rho_h) + \theta(p, \rho_h)| + |(\nabla_d \mathbf{e}_h, \nabla_d(\mathbf{u} - Q_h \mathbf{u}))_{\mathcal{T}_h}| \\
&\leq C h^{2k} (\|\mathbf{u}\|_{k+1} + \|p\|_k)^2 + \frac{1}{2} |||\mathbf{e}_h|||^2.
\end{aligned}
$$

i.e.,

$$
|||\mathbf{e}_h||| \leq C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{6.6}
$$

Then we consider $\|\varepsilon_h\|$. It is easy to get from (5.3a) that

$$
\begin{aligned}
|b(\mathbf{v}_h, \varepsilon_h)| &= |a(\mathbf{e}_h, \mathbf{v}_h) - l(\mathbf{u}, \mathbf{v}_h) + \theta(p, \mathbf{v}_h)| \\
&\leq |a(\mathbf{e}_h, \mathbf{v}_h)| + |l(\mathbf{u}, \mathbf{v}_h)| + |\theta(p, \mathbf{v}_h)| \\
&\leq C |||\mathbf{e}_h||| \, |||\mathbf{v}_h||| + C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k) |||\mathbf{v}_h||| \\
&\leq C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k) |||\mathbf{v}_h|||.
\end{aligned}
$$

From the inf−sup condition, we have

$$
\|\varepsilon_h\| \leq C \frac{|b(\mathbf{v}_h, \varepsilon_h)|}{|||\mathbf{v}_h|||} \leq C h^k (\|\mathbf{u}\|_{k+1} + \|p\|_k), \tag{6.7}
$$

which completes the proof. □

Then we derive the optimal order error estimate in L^2 norm for the velocity function. To this end, we adopt the standard duality argument. Denote by $\rho_h = Q_h\mathbf{u} - \mathbf{u}_h$ another error related to $\mathbf{u}$. Consider the dual problem that seeks $\psi \in [H^2(\Omega)]^d$ and $\phi \in H^1(\Omega)$ satisfying

$$-\Delta\psi + \nabla\phi = \rho_0 \qquad \text{in } \Omega, \tag{6.8a}$$

$$\nabla\cdot\psi = 0 \qquad \text{in } \Omega, \tag{6.8b}$$

$$\psi = 0 \qquad \text{on } \partial\Omega. \tag{6.8c}$$

Assume that the above dual problem satisfies the usual $[H^2(\Omega)]^d \times H^1(\Omega)$-regularity, which means the following priori estimate holds true

$$\|\psi\|_2 + \|\phi\|_1 \leq C\|\rho_0\|. \tag{6.9}$$

Theorem 6.2. *Assume $(\mathbf{u};p) \in [H_0^1(\Omega)\cap H^{k+1}(\Omega)]^d \times (L_0^2(\Omega)\cap H^k(\Omega))$ with $k\geq 1$ is the exact solution of the Stokes equations (1.1a)-(1.1c), and $(\mathbf{u}_h;p_h) \in V_h \times W_h$ is the numerical solution of (3.1a)-(3.1b), respectively. Moreover, assume the priori estimate (6.9) holds. Then, there exist a constant C and the following estimate holds true*

$$\|e_0\| \leq Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{6.10}$$

Proof. Testing (6.8a) by ρ_0, we get

$$\|\rho_0\|^2 = (\rho_0,\rho_0) = -(\Delta\psi,\rho_0) + (\nabla\phi,\rho_0).$$

From (5.4a) and (5.4b), we have

$$\begin{aligned}
\|\rho_0\|^2 &= (\nabla_d\rho_h,\nabla_d\psi)_{\mathcal{T}_h} - (\nabla_d\rho_h,Q_h\phi)_{\mathcal{T}_h} + l(\psi,\rho_h) - \theta(\phi,\rho_h) \\
&= a(\rho_h,\psi) - b(\rho_h,Q_h\phi) + l(\psi,\rho_h) - \theta(\phi,\rho_h).
\end{aligned}$$

Since $\nabla\cdot\psi = 0$ in Ω, it is easy to derive

$$b(Q_h\psi,\varepsilon_h) = (\nabla_d\cdot Q_h\psi, Q_hp - p_h)_{\mathcal{T}_h} = (Q_h(\nabla\cdot\psi), Q_hp - p_h)_{\mathcal{T}_h} = 0. \tag{6.11}$$

It follows from (5.3b), (6.11), and (5.3a) that

$$\begin{aligned}
\|\rho_0\|^2 &= a(\mathbf{e}_h,\psi) - b(Q_h\psi,\varepsilon_h) + l(\psi,\rho_h) - \theta(\phi,\rho_h) + a(Q_h\mathbf{u} - \mathbf{u},\psi) \\
&= a(\mathbf{e}_h,Q_h\psi) - b(Q_h\psi,\varepsilon_h) + l(\psi,\rho_h) - \theta(\phi,\rho_h) + a(Q_h\mathbf{u} - \mathbf{u},\psi) + a(\mathbf{e}_h,\psi - Q_h\psi) \\
&= l(\mathbf{u},Q_h\psi) - \theta(p,Q_h\psi) + l(\psi,\rho_h) - \theta(\phi,\rho_h) + a(Q_h\mathbf{u} - \mathbf{u},\psi) + a(\mathbf{e}_h,\psi - Q_h\psi) \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

With the definition of Q_b and the fact that $\psi = 0$ on $\partial\Omega$, it is obvious that

$$\sum_{T\in\mathcal{T}_h}\langle\psi - Q_b\psi,(p - Q_hp)\mathbf{n}\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\langle\psi - Q_b\psi,p\mathbf{n}\rangle_{\partial T} = 0. \tag{6.12}$$

Now, we start to estimate all the 6 terms on the right-hand side. From the Cauchy-Schwarz inequality, the trace inequality, and the projection inequality, we have

$$
\begin{aligned}
|I_1| = |l(\mathbf{u}, Q_h \psi)| &= \left| \sum_{T \in \mathcal{T}_h} \langle Q_0 \psi - Q_b \psi, (\nabla \mathbf{u} - \mathbf{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \| Q_0 \psi - Q_b \psi \|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T \| \nabla \mathbf{u} - \mathbf{Q}_h \nabla \mathbf{u} \|_{\partial T}^2 \right)^{1/2} \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \| Q_0 \psi - \psi \|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T \| \nabla \mathbf{u} - \mathbf{Q}_h \nabla \mathbf{u} \|_{\partial T}^2 \right)^{1/2} \\
&\leq C h^{k+1} \| \psi \|_2 \| \mathbf{u} \|_{k+1}.
\end{aligned}
$$

Similarly, we can derive the estimate of I_2 that

$$
\begin{aligned}
|I_2| = |\theta(p, Q_h \psi)| &= \left| \sum_{T \in \mathcal{T}_h} \langle Q_0 \psi - Q_b \psi, (p - Q_h p) \mathbf{n} \rangle_{\partial T} \right| \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \| Q_0 \psi - \psi \|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T \| p - Q_h p \|_{\partial T}^2 \right)^{1/2} \\
&\leq C h^{k+1} \| \psi \|_2 \| p \|_k.
\end{aligned}
$$

As to I_3 and I_4, it follows from (6.1a), (6.1b), (6.2), and (6.6) that

$$
\begin{aligned}
|I_3| = |l(\psi, \rho_h)| &\leq C h \| \psi \|_2 \|| \rho_h \|| \\
&\leq C h \| \psi \|_2 (\|| \mathbf{e}_h \|| + \|| Q_h \mathbf{u} - \mathbf{u} \||) \\
&\leq C h^{k+1} \| \psi \|_2 \| \mathbf{u} \|_{k+1}, \\
|I_4| = |\theta(\phi, \rho_h)| &\leq C h \| \phi \|_1 \|| \rho_h \|| \\
&\leq C h^{k+1} \| \phi \|_1 \| \mathbf{u} \|_{k+1}.
\end{aligned}
$$

Before we estimate I_5, a new projection operator should be defined. Denote by $\tilde{\mathbf{Q}}_h$ the L^2 projection from $[L^2(T)]^{d \times d}$ onto $[P_1(T)]^{d \times d}$. For any $T \in \mathcal{T}_h$, the following equation holds true

$$
(\nabla_d (Q_h \mathbf{u} - \mathbf{u}), \tilde{\mathbf{Q}}_h \nabla_d \psi)_T = -(Q_0 \mathbf{u} - \mathbf{u}, \nabla \cdot \tilde{\mathbf{Q}}_h \nabla_d \psi) + \langle Q_b \mathbf{u} - \mathbf{u}, \tilde{\mathbf{Q}}_h \nabla_d \psi \cdot \mathbf{n} \rangle_{\partial T} = 0.
$$

It follows from the equation above, the Cauchy-Schwarz inequality, the projection inequality, and (6.2) that

$$
\begin{aligned}
|I_5| = |a(Q_h \mathbf{u} - \mathbf{u}, \psi)| &= \left| \sum_{T \in \mathcal{T}_h} (\nabla_d (Q_h \mathbf{u} - \mathbf{u}), \nabla_d \psi)_T \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} (\nabla_d (Q_h \mathbf{u} - \mathbf{u}), \nabla_d \psi - \tilde{\mathbf{Q}}_h \nabla_d \psi)_T \right|
\end{aligned}
$$

$$= \left| \sum_{T \in \mathcal{T}_h} (\nabla_d (Q_h \mathbf{u} - \mathbf{u}), \nabla \psi - \tilde{\mathbf{Q}}_h \nabla \psi)_T \right|$$

$$\leq \left(\sum_{T \in \mathcal{T}_h} \| \nabla_d (Q_h \mathbf{u} - \mathbf{u}) \|_T^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} \| \nabla \psi - \tilde{\mathbf{Q}}_h \nabla \psi \|_T^2 \right)^{1/2}$$

$$\leq C h^{k+1} \| \mathbf{u} \|_{k+1} \| \psi \|_2.$$

Using (6.6) and (6.2), we have

$$|I_6| = |a(\mathbf{e}_h, \psi - Q_h \psi)| = \left| \sum_{T \in \mathcal{T}_h} (\nabla_d \mathbf{e}_h, \nabla_d (\psi - Q_h \psi))_T \right|$$

$$\leq C |||\mathbf{e}_h||| \, |||\psi - Q_h \psi||| \leq C h^{k+1} \| \mathbf{u} \|_{k+1} \| \psi \|_2.$$

By combining the estimates of all the terms, we obtain

$$\| \rho_0 \|^2 \leq C h^{k+1} (\| \mathbf{u} \|_{k+1} + \| p \|_k)(\| \psi \|_2 + \| \phi \|_1).$$

The regularity assumption implies

$$\| \rho_0 \| \leq C h^{k+1} (\| \mathbf{u} \|_{k+1} + \| p \|_k). \tag{6.13}$$

With (6.13), the triangle inequality, and the projection inequality, we get

$$\| \mathbf{e}_0 \| \leq \| \rho_0 \| + \| \mathbf{u} - Q_0 \mathbf{u} \| \leq C h^{k+1} (\| \mathbf{u} \|_{k+1} + \| p \|_k),$$

which completes the proof.   □

7 Numerical results

In this section, we will test the SF WG method for solving the Stokes equations (1.1a)-(1.1c). The computation domain is the unit square $\Omega = (0,1) \times (0,1)$, and the exact solution is chosen as follows:

$$\mathbf{u} = \begin{pmatrix} \sin^2(\pi x) \sin(2\pi y) \\ -\sin(2\pi x) \sin^2(\pi y) \end{pmatrix}, \quad p = (x - y)^3. \tag{7.1}$$

We first use triangular grids as shown in Fig. 1 for the computation. The resulting linear systems are solved by the Matlab backslash sparse solver. In this computation on triangular grids, the weak gradient is computed in the space of $P_{k+1}^{2 \times 2}$ polynomials, and the weak divergence is defined in the P_k space. The errors and the convergence rates are listed in Table 1. Optimal order of convergence is achieved in all cases.

To see why there is no superconvergence and possible superconvergence, we plot the P_2 solution $(\mathbf{u}_h)_1$ and its error on the 4th level grid in Fig. 2. We know most P_2 finite
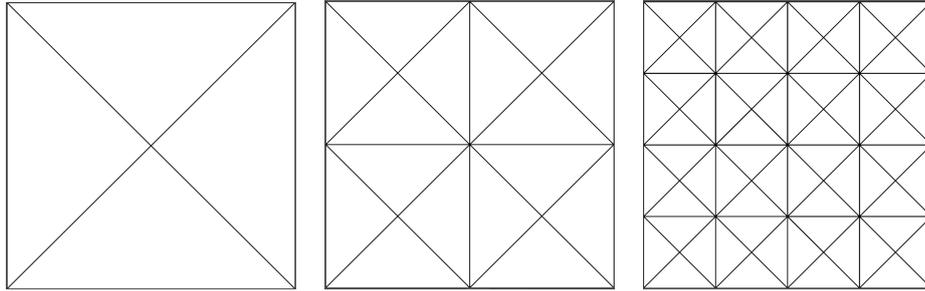
Figure 1: The level 1, 2, and 3 triangular grids for the computation of Table 1.

elements are superconvergent in L^2 norm, for example, the P_2 Taylor-Hood element. The error of the P_2 weak Galerkin velocity does cross the zero-plane on each element. This indicates there are superconvergent points for the weak Galerkin elements. Thus the weak Galerkin solution is superconvergent to some locally interpolated solution, which needs further studies to be found.

   Next, we use polygonal (pentagon and octagon) grids as shown in Fig. 3 to solve the test case (7.1). The weak gradient polynomial degree j in (2.5) is chosen to be $k+2$, much less than the theoretic proved lower bound $k+8-1$, for the computation on polygonal

Table 1: Error profile for $e_{\mathbf{u}} = Q_h\mathbf{u} - \mathbf{u}_h$ and $e_p = Q_h p - p_h$, for the SF WG on triangular grids (Fig. 1).

| level	$\|e_{\mathbf{u}}\|_0$	h^r	$\|\|e_{\mathbf{u}}\|\|$	h^r	$\|e_p\|_0$	h^r
	The P_1^2-P_0 WG element on triangular mesh					
4	0.4602E-01	2.0	0.2592E+01	0.9	0.1037E+01	1.5
5	0.1116E-01	2.0	0.1324E+01	1.0	0.3363E+00	1.6
6	0.2724E-02	2.0	0.6665E+00	1.0	0.1164E+00	1.5
7	0.6714E-03	2.0	0.3340E+00	1.0	0.4553E-01	1.4
	The P_2^2-P_1 element on triangular mesh					
3	0.1893E-01	1.8	0.9486E+00	0.6	0.3487E+00	0.9
4	0.2543E-02	2.9	0.2486E+00	1.9	0.7217E-01	2.3
5	0.3219E-03	3.0	0.6266E-01	2.0	0.1396E-01	2.4
6	0.4031E-04	3.0	0.1567E-01	2.0	0.2756E-02	2.3
	The P_3^2-P_2 element on triangular mesh					
3	0.1647E-02	4.4	0.1275E+00	3.3	0.2225E+00	3.6
4	0.1028E-03	4.0	0.1632E-01	3.0	0.1476E-01	3.9
5	0.6489E-05	4.0	0.2055E-02	3.0	0.9460E-03	4.0
6	0.4078E-06	4.0	0.2574E-03	3.0	0.6225E-04	3.9
	The P_4^2-P_3 element on triangular mesh					
2	0.1310E-02	5.8	0.6342E-01	5.1	0.1284E-01	5.2
3	0.1476E-03	3.1	0.1357E-01	2.2	0.3084E-02	2.1
4	0.4868E-05	4.9	0.8664E-03	4.0	0.1632E-03	4.2
5	0.1526E-06	5.0	0.5428E-04	4.0	0.8082E-05	4.3
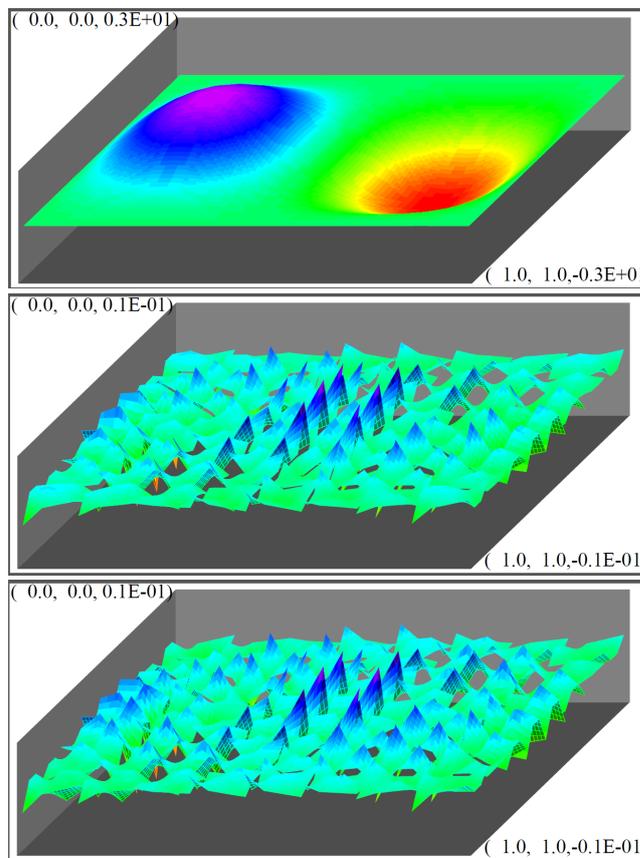
Figure 2: The P_2 WG solution of $(\mathbf{u})_1$ on the level 4 triangular grid (top), the error $(\mathbf{u})_1-(\mathbf{u}_h)_1$ on the same grid (middle), and the error $(\mathbf{u})_1-(\mathbf{u}_h)_1$ on the 4-th level polygonal grid shown as in Fig. 3 (bottom).
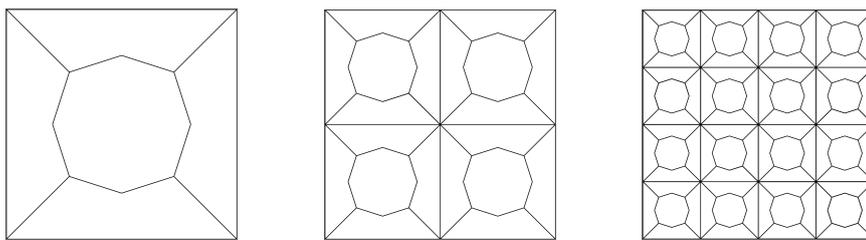

Figure 3: The level 1, 2, and 3 (pentagon and octagon) polygonal grids for the computation of Table 2.

(including octagon) grids. The weak divergence is still computed by P_k polynomials. It is an advantage of the WG finite element method over the traditional finite element method, in using arbitrary polygonal meshes. The numerical results are listed in Table 2. Optimal orders of convergence are achieved in all cases of computation on polygonal grids (cf.

Table 2: Error profile for $e_{\mathbf{u}} = Q_h \mathbf{u} - \mathbf{u}_h$ and $e_p = \mathbb{Q}_h p - p_h$, for WG finite elements on pentagon/octagon grids (Fig. 3).

level	$\|e_{\mathbf{u}}\|_0$	h^r	$\|\|\|e_{\mathbf{u}}\|\|\|$	h^r	$\|e_p\|_0$	h^r
	The P_1^2-P_0 WG element on pentagon/octagon grids					
3	0.1544E+00	2.4	0.9334E+01	1.2	0.1580E+01	1.5
4	0.3526E-01	2.1	0.4853E+01	0.9	0.5593E+00	1.5
5	0.8665E-02	2.0	0.2456E+01	1.0	0.1782E+00	1.7
6	0.2159E-02	2.0	0.1233E+01	1.0	0.5998E-01	1.6
	The P_2^2-P_1 WG element on pentagon/octagon grids					
3	0.2747E-01	1.7	0.2579E+01	0.9	0.2858E+00	0.9
4	0.3535E-02	3.0	0.6630E+00	2.0	0.6049E-01	2.2
5	0.4373E-03	3.0	0.1668E+00	2.0	0.1150E-01	2.4
6	0.5410E-04	3.0	0.4175E-01	2.0	0.2175E-02	2.4
	The P_3^2-P_2 WG element on pentagon/octagon grids					
3	0.4808E-02	4.3	0.6643E+00	3.3	0.2735E-01	4.1
4	0.3062E-03	4.0	0.8466E-01	3.0	0.2018E-02	3.8
5	0.1925E-04	4.0	0.1063E-01	3.0	0.1580E-03	3.7
6	0.1205E-05	4.0	0.1331E-02	3.0	0.1461E-04	3.4
	The P_4^2-P_3 WG element on pentagon/octagon grids					
2	0.9970E-02	5.8	0.9015E+00	5.0	0.4039E-01	5.2
3	0.7529E-03	3.7	0.1381E+00	2.7	0.3020E-02	3.7
4	0.2434E-04	5.0	0.8758E-02	4.0	0.1645E-03	4.2
5	0.7671E-06	5.0	0.5494E-03	4.0	0.1019E-04	4.0

Table 2), which are consistent with the theory developed previously. For a comparison, we plot an error on the 4th polygonal grid (cf. Fig. 3) in Fig. 2.

Acknowledgements

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