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Finite-Time Stability and Instability of Nonlinear Impulsive Systems

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Abstract. In this paper, the finite-time stability and instability are studied for nonlinear impulsive systems. There are mainly four concerns. 1) For the system with stabilizing impulses, a Lyapunov theorem on global finite-time stability is presented. 2) When the system without impulsive effects is globally finite-time stable (GFTS) and the settling time is continuous at the origin, it is proved that it is still GFTS over any class of impulse sequences, if the mixed impulsive jumps satisfy some mild conditions. 3) For systems with destabilizing impulses, it is shown that to be finite-time stable, the destabilizing impulses should not occur too frequently, otherwise, the origin of the impulsive system is finite-time instable, which are formulated by average dwell time (ADT) conditions respectively. 4) A theorem on finite-time instability is provided for system with stabilizing impulses. For each GFTS theorem of impulsive systems considered in this paper, the upper boundedness of settling time is given, which depends on the initial value and impulsive effects. Some numerical examples are given to illustrate the theoretical analysis.

AMS subject classifications: 34A37, 93D40

Key words: Impulsive systems, finite-time stability, finite-time instability, nonlinear systems.

1 Introduction

Impulsive systems combine continuous evolution (typically modelled by ordinary differential equations) with instantaneous state jumps or resets (also referred to as impulses) (see [6]). Due to the rich applications, impulsive systems have attracted lots of researchers' attention (e.g., [1, 3, 5, 9, 11–14, 16, 17, 23, 24, 26–28]).

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Recently, finite-time stability of impulsive systems has also attracted lots of attention (e.g., [3, 11, 17–22]), because comparing with infinite-time stable system, the finite-time stable system has faster convergence, better robustness and better disturbance rejection (see [2, 25]). Meanwhile, finite-time stability has rich applications in practical systems, such as spacecraft system [4], continuously stirred tank reactor system [7] and mechanical system [10]. In [3], by using the existing results about continuous systems, the finite-time stability of nonlinear impulsive systems is obtained; in [17], the finite-time stability for systems with stabilizing impulses and the finite-time stabilization of impulsive dynamical systems are studied; in [11, 19–22], the finite-time stability of system with stabilizing impulses and finite-time stability of system with destabilizing impulses are studied, and upper boundedness of settling time is estimated; in [18], for the coupled impulsive neural networks with time-varying delays and saturating actuators, the finite-time stabilization is achieved by using the finite-time stability theorem in [11, Theorem 1], which is a special case of Theorem 3.1 of this paper.

On the one hand, in [3, 11, 17], the original systems without impulsive effects is required to be finite-time stable. On the other hand, the upper right-hand Dini derivative of the Lyapunov function in the finite-time stability theorems depends on the initial value and the first impulsive time (e.g., [21]) or a finite-time stable function pair (e.g., [19, 20, 22]). Hence, this agrees us to study the finite-time stability for more general impulsive systems. In our finite-time stability theorems, the system without impulsive effects may not be finite-time stable, the upper right-hand Dini derivative of the Lyapunov function may not be related to the finite-time stable function pair and may be independent of the initial value and the first impulsive time. Because the impulsive events occur in a finite or infinite sequence of time (see [24]), we study the finite-time stability for the nonlinear impulsive systems, whose impulse sequence may be finite or infinite, instead of only finite impulse sequence (e.g., [19,21,22]) or only infinite impulse sequence (e.g., [20]). Besides, the results about finite-time stability of impulsive systems in [11,17] are involved in our results.

For the system with stabilizing impulses, we provide a Lyapunov theorem on the global finite-time stability. It is shown that the more frequently the impulses occur, the faster the system state reaches the origin, which is formulated by an ADT condition. For systems with mixed impulses (some impulses are stabilizing and some are destabilizing), we show that GFTS system with settling time being continuous at the origin is still GFTS under impulsive effects, if the jumps satisfy some mild conditions. Next, we give a rigorous proof on the global finite-time stability of system with destabilizing impulses. It is shown that the impulses should not occur too frequently, otherwise the origin of the impulsive system is finite-time instable, which are formulated by ADT conditions respectively. In addition, the finite-time instability of system with stabilizing impulses is also studied.

The main contributions of this paper include: 1) Our results can be applied to more general impulsive systems, since our results not only cover all results about the finite-time stability of impulsive systems in [11,17], but also cover some systems whose finite-time stability cannot be analyzed by the existing results about impulsive systems (e.g., [3, 11, 17–21]); 2) The global finite-time stability of systems with mixed impulses is studied; 3) For system with destabilizing impulses, we study the global finite-time stability of the origin, and show the relationship between the global finite-time stability and the destabilizing impulses.

The organization of this paper is as follows. In Section 2, we introduce some notations and preliminary results. In Section 3, three finite-time stability theorems on nonlinear impulsive systems are developed and two finite-time instability theorems on nonlinear impulsive systems are provided in Section 4. In Section 5, some numerical examples are given to illustrate the theoretical analysis and some concluding remarks are given in Section 6.

2 Notations and preliminary results

The following notations will be used throughout this paper:

- $\mathbb{Z}_+$ denotes the set of all positive integers;
- $\mathbb{R}_+$ denotes the set of all nonnegative real numbers;
- $\mathbb{R}^n$ denotes the real n-dimensional space;
- $\mathcal{K}$ denotes the set of all functions $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero;
- $\mathcal{C}^{1,1}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ denotes the family of all functions $V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, which have continuous partial derivatives with respect to t and x;
- $\mathcal{L}^1(\mathbb{R}_+; \mathbb{R})$ denotes the family of functions $l: \mathbb{R}_+ \to \mathbb{R}$ such that
$$\int_0^t l(s)ds < \infty \quad \text{for} \quad \forall t > 0.$$

Consider the following system

$$\begin{cases} \dot{x}(t) = f(t,x(t)), & t \neq t_k, \quad t \geq 0, \\ x(t) = h_k(x(t^-)), & t = t_k, \quad k \in \{1,2,\cdots\} =: \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where t_k is assumed to be strictly increasing with respect to $k \in \mathbb{N}$ on $(0,\infty)$, $f: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with $f(t,0) = 0$ for $t \in \mathbb{R}_+$, $x(t^-)$ denotes the left limit of $x(t)$, $h_k: \mathbb{R}^n \to \mathbb{R}^n$ satisfies $h_k(0) = 0$ for $\forall k \in \mathbb{N}$, and $x_0 \in \mathbb{R}^n$ is a constant. The impulse sequence $\{t_k\} = \{t_k : k \in \mathbb{N}\}$ is finite or infinite and unbounded. Let $N(t,s)$ denote the number of impulse times in interval $(s,t]$, $t_0 := 0$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, and $t_{k_0+1} := \infty$ for a finite impulse sequence $\{t_1, \cdots, t_{k_0}\}$ if there is no explicit illustration.

Since the global finite-time stability of system (2.1) is related to $\{t_k\}$, and it is interesting to characterize the global finite-time stability over the class of impulse sequences $\{t_k\}$, we give the following definition, which is motivated by [6,11].

Definition 2.1. *For a given impulse sequence $\{t_k\}$, the origin of system* (2.1) *is said to be globally finite-time stable (GFTS), if there exists a function $T(x_0,\{t_k\})$ taking values in $\mathbb{R}_+$ such that the following two statements hold:*

(i) Finite-time convergence: for $\forall x_0 \in \mathbb{R}^n \setminus \{0\}$, each solution $x(t)$ of system (2.1) *satisfies $\lim_{t\to T(x_0,\{t_k\})} x(t)=0$ and $x(t)=0$ for $\forall t \geq T(x_0,\{t_k\})$;*

(ii) Stability: for $\forall \epsilon > 0$, there exists a constant $\delta(\epsilon)>0$ such that $|x(t)|<\epsilon$ for $t\geq 0$ and $|x_0|<\delta$.

$T(x_0,\{t_k\})$ is called as the settling time of system (2.1) *with respect to the initial value $x(0)=x_0$ and impulse sequence $\{t_k\}$. The origin of system* (2.1) *is said to be GFTS over the class $\mathcal{S}$ of impulse sequences, if the origin of system* (2.1) *is GFTS for $\forall \{t_k\}\in \mathcal{S}$.*

The origin of system (2.1) is said to be finite-time instable for an impulse sequence, if the origin of system (2.1) is not finite-time convergent or stable.

To study the global finite-time stability and instability of system (2.1), we give the following definition.

Definition 2.2 ([1]). *We say that the function $V:\mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}_+$ belongs to the class v_0, if*

1) for $\forall k \in \mathbb{N}$, the function V is continuous in $[t_{k-1},t_k)\times\mathbb{R}^n$, and

$$\lim_{t\to t_k,\ t\in(t_{k-1},t_k)} V(t,x(t)) = V(t_k^-,x(t_k^-))$$

exists;

2) V is locally Lipschitzian in x.

For $V\in v_0$, the upper right-hand Dini derivative of V along the solution of (2.1) is

$$D^+V(t,x)=\limsup_{h\to 0^+}\frac{1}{h}\big(V(t+h,x+hf(t,x))-V(t,x)\big). \tag{2.2}$$

If $V\in \mathcal{C}^{1,1}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$, then

$$D^+V(t,x)=V'(t,x)=\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}f(t,x).$$

The average dwell time (ADT) condition will be used to study the finite-time stability and instability. Similar to [6], define

$$\mathcal{S}_{\rm r-avg}[\tau^*,N_0] := \Big\{t_k : N(t,s)\geq \frac{t-s}{\tau^*}-N_0\Big\}, \tag{2.3a}$$

$$\mathcal{S}_{\rm avg}[\tau^*,N_0] := \Big\{t_k : N(t,s)\leq \frac{t-s}{\tau^*}+N_0\Big\}, \tag{2.3b}$$

where $\tau^*>0$ denotes the average impulsive interval of the impulse sequence (see [16]) and $N_0>0$ is a constant.

The following lemma is very important to study the finite-time stability and instability of system (2.1).

Lemma 2.1. *If there exist a v_0 function V and positive constants M_k, $k\in\mathbb{N}$, such that*

$$V(t_k,x(t_k))\le M_kV(t_k^-,x(t_k^-)),\quad \forall k\in\mathbb{N}, \tag{2.4}$$

then

$$V(t,x(t))\le\left(\prod_{k=1}^{N(t,0)}M_k\right)V(0,x_0)+\int_0^t\left(\prod_{k=N(s,0)+1}^{N(t,0)}M_k\right)D^+V(s,x(s))ds, \tag{2.5}$$

where

$$\begin{aligned}
&\int_t^\tau D^+V(s,x(s))ds=\int_{t_k}^{t_{k+1}^-}D^+V(s,x(s))ds, &&N(t_{k+1}^-,0)=k,\\
&\prod_{j=k+1}^{k}M_j=1, &&M_0=1,
\end{aligned}$$

for $t=t_k$, $\tau=t_{k+1}$ and $k\in\mathbb{N}_0$. Especially, if $M_k=\bar{M}$ for $\forall k\in\mathbb{N}$, where $\bar{M}$ is a positive constant, then (2.5) *can be written as*

$$V(t,x(t))\le\bar{M}^{N(t,0)}V(0,x_0)+\int_0^t\bar{M}^{N(t,s)}D^+V(s,x(s))ds. \tag{2.6}$$

Proof. We will prove (2.5) by induction. It is obvious that for $t\in[0,t_1)$,

$$V(t,x(t))\le V(0,x_0)+\int_0^tD^+V(s,x(s))ds, \tag{2.7}$$

so (2.5) holds for $t\in[0,t_1)$.

We assume that (2.5) holds for $t\in[t_{k-1},t_k)$. Then it follows from (2.4) that

$$\begin{aligned}
V(t,x(t))\le&V(t_k,x(t_k))+\int_{t_k}^tD^+V(s,x(s))ds\\
\le&M_kV(t_k^-,x(t_k^-))+\int_{t_k}^tD^+V(s,x(s))ds\\
\le&M_k\left[\left(\prod_{j=1}^{N(t_k^-,0)}M_j\right)V(0,x_0)+\int_0^{t_k^-}\left(\prod_{j=N(s,0)+1}^{N(t_k^-,0)}M_j\right)D^+V(s,x(s))ds\right]\\
&+\int_{t_k}^t\left(\prod_{j=N(s,0)+1}^{N(t,0)}M_j\right)D^+V(s,x(s))ds,
\end{aligned} \tag{2.8}$$

for $t\in[t_k,t_{k+1})$. Note that for $s\in[0,t_k^-)$ and $t\in[t_k,t_{k+1})$,

$$M_k\left(\prod_{j=1}^{N(t_k^-,0)}M_j\right)=\prod_{j=1}^{N(t,0)}M_j,$$
$$M_k\left(\prod_{j=N(s,0)+1}^{N(t_k^-,0)}M_j\right)=\prod_{j=N(s,0)+1}^{N(t,0)}M_j.$$

Thus, it follows from (2.8) that (2.5) holds for $t\in[t_k,t_{k+1})$.

When $M_k=\bar{M}$ for $\forall k\in\mathbb{N}$. Let $0\le s\le t$, $t\in[t_k,t_{k+1})$ and $s\in[t_i,t_{i+1})$. If $i<k$,

$$\prod_{j=N(s,0)+1}^{N(t,0)}M_j=\bar{M}^{k-i}=\bar{M}^{N(t,s)},$$

and if $i=k$, it follows from $\prod_{j=k+1}^{k}M_j=1$, that

$$\prod_{j=N(s,0)+1}^{N(t,0)}M_j=\bar{M}^{N(t,s)},$$

which yields (2.6), and the proof is completed. □

Remark 2.1. The estimation of the solution of impulsive systems is also considered in [1, 24], whereas, the estimation in [1] concerns the inverse of some functions, and the estimation in [24] is about the linear impulsive system. Hence, we develop Lemma 2.1 to analyze the finite-time stability of the nonlinear impulsive systems.

3 Global finite-time stability

In this section, we will investigate the global finite-time stability for three classes of nonlinear impulsive systems.

3.1 The global finite-time stability of systems with stabilizing impulses

First, we study the global finite-time stability of system (2.1) with stabilizing impulses.

Theorem 3.1. *If there exist a v_0 function V, a continuous function $l:\mathbb{R}_+\to\mathbb{R}\backslash\mathbb{R}_+$, two $\mathcal{K}$ functions α_1 and α_2 and constants $\delta_0\in(0,1)$ and $\eta\in[0,1)$ such that*

$$\alpha_1(|x|)\le V(t,x(t))\le\alpha_2(|x|), \tag{3.1a}$$
$$V(t_k,x(t_k))\le\delta_0 V(t_k^-,x(t_k^-)),\qquad \forall k\in\mathbb{N}, \tag{3.1b}$$
$$D^+V(t,x(t))\le l(t)V^{\eta}(t,x(t)),\qquad t\ne t_k,\quad k\in\mathbb{N}. \tag{3.1c}$$

Then, the origin of system (2.1) *is GFTS with*

$$(1-\eta)\mu(T(x_0,\{t_k\}))\leq V^{1-\eta}(0,x_0), \tag{3.2}$$

over $\mathcal{S}_{l(t),\delta_0}$, *which denotes the class of impulse sequences* $\{t_k\}$ *satisfying*

$$\mu(t)=-\int_0^t l(s)\delta_0^{-(1-\eta)N(s,0)}ds\to\infty \quad \text{as} \quad t\to\infty. \tag{3.3}$$

Proof. Note that $\mu(t)$ is continuous and increasing with $\mu(0)=0$. Thus, it follows from (3.3) that there exists a positive constant $T_1(x_0,\{t_k\})$ depending on x_0 and $\{t_k\}$ such that for $x_0\neq 0$,

$$\mu(T_1)=\bar{V}(0,x_0), \tag{3.4}$$

where

$$\bar{V}(t,x(t)):=V^{1-\eta}(t,x(t))/(1-\eta).$$

By (3.1b) and (3.1c), we have

$$\begin{aligned}
&\bar{V}(t_k,x(t_k))\leq\delta_0^{1-\eta}\bar{V}(t_k^-,x(t_k^-)), && \forall k\in\mathbb{N}, && \text{(3.5a)}\\
&D^+\bar{V}(t,x(t))\leq l(t), && t\neq t_k \quad \text{and} \quad k\in\mathbb{N}. && \text{(3.5b)}
\end{aligned}$$

These together with Lemma 2.1, $l(s)\leq 0$ and $N(t,s)=N(t,0)-N(s,0)$ yield that for $\forall\{t_k\}\in\mathcal{S}_{l(t),\delta_0}$ and $t\geq 0$,

$$\begin{aligned}
0\leq\bar{V}(t,x(t))&\leq\delta_0^{(1-\eta)N(t,0)}\bar{V}(0,x_0)+\int_0^t\delta_0^{(1-\eta)N(t,s)}D^+\bar{V}(s,x(s))ds\\
&\leq\delta_0^{(1-\eta)N(t,0)}\big(\bar{V}(0,x_0)-\mu(t)\big),
\end{aligned} \tag{3.6}$$

which together with (3.4) and the definition of $\bar{V}(t,x)$ yield that

$$V(T_1,x(T_1))=0.$$

So $x(T_1)=0$ due to (3.1a) and $\alpha_1\in\mathcal{K}$.

Since T_1 is a nonnegative constant for the given initial value and impulse sequence, there exists a $k_0\in\mathbb{N}_0$ such that $T_1\in[t_{k_0},t_{k_0+1})$. Then, it follows from (3.1a), (3.1c) and $l(t)\leq 0$ that for $\forall t\in[T_1,t_{k_0+1})$,

$$\alpha_1(|x(t)|)\leq V(t,x(t))\leq V(T_1,x(T_1))\leq\alpha_2(|x(T_1)|). \tag{3.7}$$

Since α_1, $\alpha_2\in\mathcal{K}$ and $x(T_1)=0$, we have $x(t)=0$ for $\forall t\in[T_1,t_{k_0+1})$. Because of (3.1b), we can prove $x(t)=0$ for $\forall t\in[t_k,t_{k+1})$, $k>k_0$ in a similar way. Then we have

$$x(t)=0 \quad \text{for} \quad t\geq T_1. \tag{3.8}$$

Hence, the origin of system (2.1) is finite-time convergent and the settling time is $T(x_0,\{t_k\})\leq T_1$, and (3.2) follows from the monotonically increasing property of μ and (3.4).

For $\forall\epsilon>0$, let $\delta\in(0,\alpha_2^{-1}(\alpha_1(\epsilon)))$. Then it follows from (3.1a)-(3.1c) and Lemma 2.1 that for any $|x_0|<\delta$ and $t\geq 0$,

$$\alpha_1(|x(t)|)\leq V(t,x(t))\leq V(0,x_0)\leq\alpha_2(|x_0|)<\alpha_1(\epsilon). \tag{3.9}$$

Noting that $\alpha_1\in\mathcal{K}$, thus, we have

$$|x(t)|<\epsilon \quad \text{for } t\geq 0 \quad \text{and} \quad |x_0|<\delta.$$

The proof is completed. □

Remark 3.1. The main differences between Theorem 3.1 and the existing finite-time stability results of impulsive systems (e.g., [3,11,17,19–22]) are as follows:

1) the original system without impulsive effects is not necessary to be finite-time stable;

2) D^+V does not depend on the initial value or the first impulsive time;

3) $l(t)$ may be not a finite-time stable function, i.e., there are two constants $\lambda_1>0$ and $\lambda_2\geq 0$ such that
$$\int_s^t l(t)dt\leq-\lambda_1(t-s)+\lambda_2.$$

Corollary 3.1. *Assume that conditions of Theorem* 3.1 *hold with* $l(t)=-c_0$*, then:*

1) the origin of system (2.1) *is GFTS with*

$$T(x_0,\{t_k\})\leq-\frac{\tau^*}{(1-\eta)\ln\delta_0}\ln\left(1-\frac{V^{1-\eta}(0,x_0)\ln\delta_0}{c_0\delta_0^{N_0(1-\eta)}\tau^*}\right)=:T_1'' \tag{3.10}$$

over $\mathcal{S}_{\rm r-avg}[\tau^*,N_0]$ *for all* $N_0>0$ *and* $\tau^*>0$*;*

2) we can obtain the GFTS results of impulsive systems in [17, Theorem 3.1] and [11, Theorem 1], namely, the origin of system (2.1) *is GFTS over any class of impulse sequences with* $T(x_0,\{t_k\})\leq\Gamma_{x_0}$*, and over* $\mathcal{S}_N:=\{t_1,\cdots,t_N\}$ *with* $T(x_0,\{t_k\})\leq\gamma^N\Gamma_{x_0}$*, where* c_0 *is a positive constant,*

$$\Gamma_{x_0}:=\frac{\bar{V}(0,x_0)}{c_0},\quad t_N\leq\frac{\gamma^{N-1}\Gamma_{x_0}(\gamma-\beta)}{1-\beta},\quad \beta=\delta_0^{1-\eta}\quad \text{and}\quad \gamma\in(\beta,1).$$

Proof. 1) For $l(t)=-c_0$ and $\{t_k\}\in\mathcal{S}_{\text{r-avg}}[\tau^*,N_0]$, we obtain

$$\begin{aligned}\mu(t)\geq&\int_0^t c_0\delta_0^{(1-\eta)(-\frac{s}{\tau^*}+N_0)}ds\\=&\tau^*c_0\delta_0^{N_0(1-\eta)}(1-\delta_0^{-t(1-\eta)/\tau^*})/((1-\eta)\ln\delta_0)=:\mu_1(t)\end{aligned}\tag{3.11}$$

with $\mu_1(T_1'')=\bar{V}(0,x_0)$. Then it follows from (3.6) that

$$\begin{aligned}0\leq&\bar{V}(T_1'',x(T_1''))\leq\delta_0^{(1-\eta)N(T_1'',0)}(\bar{V}(0,x_0)-\mu(T_1''))\\\leq&\delta_0^{(1-\eta)N(T_1'',0)}(\bar{V}(0,x_0)-\mu_1(T_1''))=0,\end{aligned}\tag{3.12}$$

i.e., $\mu(T_1'')=\bar{V}(0,x_0)$. In addition, $\mu(t)\geq\mu_1(t)$ and $\mu_1(t)\to\infty$ as $t\to\infty$ yield (3.3). The desired result 1) follows from Theorem 3.1.

2) For any given class of impulse sequences $\{t_k\}$, we have $\mu(t)\geq c_0t$, which yields (3.3). In addition, (3.6) implies

$$0\leq\bar{V}(0,x_0)-\mu(\Gamma_{x_0})\leq\bar{V}(0,x_0)-c_0\Gamma_{x_0}=0,$$

so we have $\mu(\Gamma_{x_0})=\bar{V}(0,x_0)$. Hence, by Theorem 3.1, we obtain that the origin of system (2.1) is GFTS over any class of impulse sequences with $T(x_0,\{t_k\})\leq\Gamma_{x_0}$.

For the impulse sequence $\mathcal{S}_N$, let $t_{N+1}=\gamma^N\Gamma_{x_0}$. Since $t_j\leq t_N$ for $1\leq j\leq N$ and $\gamma\in(0,1)$, we have that for $\forall j\in\{1,2,\cdots,N\}$, $t_j\leq t_N\leq\gamma^{j-1}\Gamma_{x_0}(\gamma-\beta)/(1-\beta)$, i.e., $\beta\gamma^{j-1}+t_j(1-\beta)/\Gamma_{x_0}\leq\gamma^j$, which together with (3.6) and $c_0=\bar{V}(0,x_0)/\Gamma_{x_0}$ yields that

$$\begin{aligned}0\leq&\delta_0^{(1-\eta)N(t,0)}\big(\bar{V}(0,x_0)-\mu(t)\big)\\=&\beta^j\bar{V}(0,x_0)-c_0\sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}\beta^{j-k+1}ds-\int_{t_j}^{t}c_0ds\\=&\beta^j\bar{V}(0,x_0)+c_0\sum_{k=1}^{j}t_k\beta^{j-k}(1-\beta)-c_0t\\=&\bar{V}(0,x_0)\big(\beta^{j-1}(\beta+(1-\beta)t_1/\Gamma_{x_0})+\sum_{k=2}^{j}\beta^{j-k}t_k(1-\beta)/\Gamma_{x_0}\big)-c_0t\\\leq&\gamma^j\bar{V}(0,x_0)-c_0t,\end{aligned}$$

for $t\in[t_j,t_{j+1})$ and $0\leq j\leq N$. Besides, $\gamma^N\bar{V}(0,x_0)-c_0t=0$ at $t=t_{N+1}$. Thus, $\mu(t_{N+1})=\bar{V}(0,x_0)$. Therefore, it follows from Theorem 3.1 that the origin of (2.1) is GFTS over $\mathcal{S}_N$ with $T(x_0,\{t_k\})\leq t_{N+1}=\gamma^N\Gamma_{x_0}$. The proof is completed. □

Remark 3.2. Let $\varphi(t):=t\ln(1+c/t)$ with c being a positive constant. Then $\varphi''(t)<0$, $t>0$ and $\lim_{t\to\infty}\varphi'(t)=0$, which implies that $\varphi'(t)>0$ for $t>0$. Thus, T_1'' in (3.10) is an increasing function with respect to τ^*, which shows that the more frequently the stabilizing impulses happen, the faster the state reaches the origin.

3.2 The global finite-time stability of systems with mixed impulses

Impulses may be stabilizing and destabilizing. Now, we study the global finite-time stability of systems with stabilizing and destabilizing impulses, which are called as the systems with mixed impulses in this paper.

Theorem 3.2. *If there exist a v_0 function V, a function $l:\mathbb{R}_+\to\mathbb{R}\backslash\mathbb{R}_+$, two $\mathcal{K}$ functions α_1 and α_2, and constants $\theta_1,\theta_2>0$, $\eta\in[0,1)$ and positive constants M_k, $k\in\mathbb{N}$ such that*

$$\alpha_1(|x|)\le V(t,x)\le\alpha_2(|x|),\qquad \forall t\in\mathbb{R}_+,\quad \forall x\in\mathbb{R}^n, \tag{3.13a}$$

$$V(t_k,x(t_k))\le M_kV(t_k^-,x(t_k^-)),\qquad \forall k\in\mathbb{N}, \tag{3.13b}$$

$$D^+V(t,x(t))\le l(t)V^\eta(t,x(t)),\qquad t\neq t_k,\quad k\in\mathbb{N}, \tag{3.13c}$$

$$\theta_1\le\prod_{k=N(s,0)+1}^{N(t,0)}M_k\le\theta_2,\qquad t\ge s\ge 0, \tag{3.13d}$$

then the origin of system (2.1) *is GFTS with*

$$\nu(T(x_0,\{t_k\}))\le\frac{\theta_2^{1-\eta}}{\theta_1^{1-\eta}(1-\eta)}V^{1-\eta}(0,x_0), \tag{3.14}$$

over any impulse sequence, where the continuous function $\nu(t)=-\int_0^t l(s)ds$ satisfies

$$\nu(t)<\infty\quad\text{and}\quad \nu(t)\to\infty\quad\text{as}\ \ t\to\infty. \tag{3.15}$$

Proof. Since $\nu(t)$ is continuous and increasing and $\nu(0)=0$. It follows from (3.15) that there exists a positive constant $T_2(x_0,\{t_k\})$ depending on x_0 and $\{t_k\}$ such that for $x_0\neq 0$,

$$\theta_1^{1-\eta}\nu(T_2)=\theta_2^{1-\eta}\bar{V}^{1-\eta}(0,x_0), \tag{3.16}$$

where $\bar{V}(t,x)=V^{1-\eta}(t,x)/(1-\eta)$. It follows from (3.13b) and (3.13c) that

$$\bar{V}(t_k,x(t_k))\le M_k^{1-\eta}\bar{V}(t_k^-,x(t_k^-)),\qquad \forall k\in\mathbb{N}, \tag{3.17a}$$

$$D^+\bar{V}(t,x(t))\le l(t),\qquad t\neq t_k,\quad k\in\mathbb{N}. \tag{3.17b}$$

Note that (3.13d) implies

$$\theta_1^{1-\eta}\le\prod_{k=N(s,0)+1}^{N(t,0)}M_k^{1-\eta}\le\theta_2^{1-\eta},\quad t\ge s\ge 0. \tag{3.18}$$

Thus, by Lemma 2.1 and $l(t)\le 0$, we obtain that for $\forall\{t_k\}$ and $0\le t\le T_2$,

$$\begin{aligned}0\le\bar{V}(t,x(t))&\le\theta_2^{1-\eta}\bar{V}(0,x_0)+\theta_1^{1-\eta}\int_0^t l(s)ds\\&=\theta_2^{1-\eta}\bar{V}(0,x_0)-\theta_1^{1-\eta}\nu(t).\end{aligned} \tag{3.19}$$

By (3.16) and the definition of $\bar{V}(t,x)$, we obtain $V(T_2,x(T_2))=0$. Then it follows from (3.13a) and $\alpha_1\in\mathcal{K}$ that $x(T_2)=0$. In a similar way to prove (3.8), we have

$$x(t)=0 \quad \text{for} \;\; t\geq T_2. \tag{3.20}$$

Hence, the origin of system (2.1) is finite-time convergent for any given impulse sequence and the settling time is $T(x_0,\{t_k\})\leq T_2$. Since ν is increasing, (3.14) follows from (3.16).

For $\forall\epsilon>0$, let $\delta\in(0,\alpha_2^{-1}(\alpha_1(\epsilon)/\theta_2))$. Then it follows from (3.13a)-(3.13d), $l(t)\leq 0$ and Lemma 2.1 that for any $|x_0|<\delta$, $t\geq 0$,

$$\alpha_1(|x(t)|)\leq V(t,x(t))\leq\theta_2 V(0,x_0)\leq\theta_2\alpha_2(|x_0|)<\alpha_1(\epsilon), \tag{3.21}$$

which together with $\alpha_1\in\mathcal{K}$ yield that

$$|x(t)|<\epsilon$$

for $t\geq 0$ and $|x_0|<\delta$. The proof is completed. □

Remark 3.3. The finite-time stability with mixed impulses is also studied in [21], where $l(t)$ depends on the initial value, and θ_2 is less than 1, but in Theorem 3.2, θ_2 is an any positive constant.

Remark 3.4. By [8, Lemma 4.3], we know that (3.13a) always holds when $V(t,x)$ is a positive definite function for $\forall t\geq 0$. In addition, by [2, Theorem 4.3], we obtain that (3.13a) and (3.13c) hold if the continuous dynamics (the dynamics of the system without impulsive effects) are GFTS with the settling time being continuous at the origin. Thus, if the continuous dynamics are GFTS with the settling time being continuous at the origin and impulse jumps satisfy (3.13d), the impulsive system (2.1) with $f(t,x)=f(x)$ is GFTS over any class of impulse sequences. This case is covered by Theorem 3.2. However, the finite-time stability cannot be analyzed by the existing finite-time stable results, such as Theorem 3.1 in this paper and the results in [3,11,17,19–22].

3.3 The global finite-time stability of systems with destabilizing impulses

It is shown in [6] that the destabilizing impulses should not happen too frequently, because the destabilizing impulses destroy the input-to-state stability. In this subsection, we will extend the result to the global finite-time stability of system (2.1) with destabilizing impulses.

Theorem 3.3. *If there exist a v_0 function V, two $\mathcal{K}$ functions α_1 and α_2 and constants $\delta_1\geq 1$, $\eta\in[0,1)$ and $c_1>0$ such that*

$$\alpha_1(|x|)\leq V(t,x(t))\leq\alpha_2(|x|), \tag{3.22a}$$

$$V(t_k,x(t_k))\leq\delta_1 V(t_k^-,x(t_k^-)), \qquad k\in\mathbb{N}, \tag{3.22b}$$

$$D^+V(t,x(t))\leq -c_1V^{\eta}(t,x(t)), \qquad t\neq t_k, \quad k\in\mathbb{N}, \tag{3.22c}$$

then the origin of system (2.1) *is finite-time convergent over* $\mathcal{S}^1_{c_1,\delta_1}$. *Here* $\mathcal{S}^1_{c_1,\delta_1}$ *denotes a class of impulse sequences* $\{t_k\}$ *such that*

$$\vartheta(t)=0 \tag{3.23}$$

has a positive solution $T_3(x_0,\{t_k\})$ *depending on* x_0 *and* $\{t_k\}$ *for* $x_0\neq 0$, *where*

$$\vartheta(t):=\delta_1^{(1-\eta)N(t,0)}\bar{V}(0,x_0)-c_1\int_0^t \delta_1^{(1-\eta)N(t,s)}ds,$$
$$\bar{V}(t,x):=V^{1-\eta}(t,x)/(1-\eta).$$

Furthermore, let $\mathcal{S}^2_{c_1,\delta_1}:=\{t_k\in\mathcal{S}^1_{c_1,\delta_1}:$ *there exists a positive constant* σ *such that* $\sup\{N(T_3(x_0,\{t_k\}),0):|x_0|<\sigma\}<\infty\}$. *Then the origin of system* (2.1) *is GFTS with* $T(x_0,\{t_k\})\leq T_3$ *over* $\mathcal{S}^2_{c_1,\delta_1}$.

Proof. It follows from (3.22b) and (3.22c) that

$$\bar{V}(t_k,x(t_k))\leq\delta_1^{1-\eta}V(t_k^-,x(t_k^-)),\qquad \forall k\in\mathbb{N}, \tag{3.24a}$$
$$D^+\bar{V}(t,x(t))\leq -c_1,\qquad t\neq t_k\quad\text{and}\quad k\in\mathbb{N}. \tag{3.24b}$$

Then by Lemma 2.1, we obtain

$$0\leq\bar{V}(t,x(t))\leq\vartheta(t). \tag{3.25}$$

Note that $\vartheta(T_3)=0$ for $\{t_k\}\in\mathcal{S}^1_{c_1,\delta_1}$ and $x_0\neq 0$. Thus, $\bar{V}(T_3,x(T_3))=0$, which implies $V(T_3,x(T_3))=0$. Then it follows from (3.22a) and $\alpha_1\in\mathcal{K}$ that $x(T_3)=0$. Similarly to the proof of (3.8), we can prove that

$$x(t)=0\quad\text{for}\ \ t\geq T_3. \tag{3.26}$$

So the origin of system (2.1) is finite-time convergent over $\mathcal{S}^1_{c_1,\delta_1}$ and the settling time is $T(x_0,\{t_k\})\leq T_3$.

For any given $\epsilon>0$ and $\{t_k\}\in\mathcal{S}^2_{c_1,\delta_1}$, the construction of $\mathcal{S}^2_{c_1,\delta_1}$ implies that there exists a constant $\sigma>0$ such that $\rho:=\sup\{N(T_3(x_0,\{t_k\}),0):|x_0|<\sigma\}<\infty$. Let $M:=\delta_1^\rho$ and $\delta:=\min\{\sigma,\alpha_2^{-1}(\alpha_1(\epsilon)/M)\}$. Then it follows from (3.22a), (3.22b), (3.22c) and Lemma 2.1 that for $\forall|x_0|<\delta$, $\forall t\in[0,T_3]$,

$$\alpha_1(|x(t)|)\leq V(t,x(t))\leq\delta_1^{N(t,0)}V(0,x_0)\leq M\alpha_2(|x_0|)<\alpha_1(\epsilon). \tag{3.27}$$

Then we have that when $\{t_k\}\in\mathcal{S}^2_{c_1,\delta_1}$, $|x(t)|<\epsilon$ for $\forall|x_0|<\delta$ and $\forall t\in[0,T_3]$. This together with (3.26) yields the stability of the origin of system (2.1) for $\forall\{t_k\}\in\mathcal{S}^2_{c_1,\delta_1}$. The proof is completed. □

Now, we give two special cases of Theorem 3.3 in the following remark and corollary.

Remark 3.5. Let $\beta := \delta_1^{1-\eta}$, $\Gamma_\sigma := \alpha_2^{1-\eta}(\sigma)/(c_1(1-\eta))$ and $\mathcal{S}^1$ denote the class of impulse sequences $\{t_k\}$ satisfying

$$\min\Big\{j\in\mathbb{Z}_+ : \frac{t_j}{\beta^{j-1}} \geq \frac{\alpha_2^{1-\eta}(\sigma)}{c_1(1-\eta)}\Big\} := N_1 < \infty. \tag{3.28}$$

Assume that conditions of Theorem 3.3 hold. It follows from the construction of $\bar{V}$, (3.22a) and $\delta_1 \geq 1$ that for $\{t_k\}\in\mathcal{S}^1$ and $|x_0|\leq\sigma$,

$$\vartheta(t) = \beta^{N(t,0)}\bar{V}(0,x_0) - c_1\int_0^t \beta^{N(t,s)}ds \leq \beta^{N(t,0)}\alpha_2^{1-\eta}(\sigma)/(1-\eta) - c_1 t.$$

So

$$\vartheta(t) \leq \beta^{N_1-1}c_1\Gamma_\sigma - c_1 t \quad \text{for } t\in[t_{N_1-1}, t_{N_1}).$$

Note that $\{t_k\}\in\mathcal{S}^1$ implies $t_j < \beta^{j-1}\Gamma_\sigma$ for $j=1,\cdots,N_1-1$ and $t_{N_1}\geq\beta^{N_1-1}\Gamma_\sigma$, which together with $\beta\geq 1$ yield that $t_{N_1-1} < \beta^{N_1-1}\Gamma_\sigma \leq t_{N_1}$. Thus, if $\beta^{N_1-1}\Gamma_\sigma < t_{N_1}$, we have $\vartheta(\beta^{N_1-1}\Gamma_\sigma)\leq 0$, and if $\beta^{N_1-1}\Gamma_\sigma = t_{N_1}$, we obtain $\vartheta(t_{N_1}^-)\leq 0$, which together with $\vartheta(t_{N_1}) = \delta_1^{1-\eta}\vartheta(t_{N_1}^-)$ yields $\vartheta(\beta^{N_1-1}\Gamma_\sigma)\leq 0$. Then it follows from (3.25) that $\vartheta(\beta^{N_1-1}\Gamma_\sigma) = 0$. Hence, [11, Theorem 2] can be viewed as a special case of Theorem 3.3 with $\mathcal{S}^1_{c_1,\delta_1} = \mathcal{S}^1$ and $T(x_0,\{t_k\}) \leq \beta^{N_1-1}\Gamma_\sigma$. Besides, since $N(T_3,0)\leq N_1$ for $|x_0|<\sigma$ and $\forall\{t_k\}\in\mathcal{S}^1$, Theorem 3.3 implies that the origin of system (2.1) is finite-time stable over $\mathcal{S}^1$.

Corollary 3.2. *Assume that conditions of Theorem* 3.3 *hold with* $\delta_1 > 1$, *then the origin of system* (2.1) *is GFTS with* $T(x_0,\{t_k\}) \leq T^*$, *over* $\mathcal{S}_{\text{r}-\text{avg}}[\tau^*,N_0]\cap\mathcal{S}_{\text{avg}}[\tau^*,N_0]$ *for all* $N_0 > 0$ *and* τ^* *satisfying*

$$\tau^* > \ln(\delta_1)\delta_1^{2(1-\eta)N_0}V^{1-\eta}(0,x_0)/c_1, \tag{3.29}$$

where

$$T^* := \frac{\tau^*}{(1-\eta)\ln\delta_1}\ln\Big(\frac{c_1\tau^*}{c_1\tau^* - \delta_1^{2(1-\eta)N_0}V^{1-\eta}(0,x_0)\ln\delta_1}\Big).$$

Proof. For $\forall\{t_k\}\in\mathcal{S}_{\text{r}-\text{avg}}[\tau^*,N_0]\cap\mathcal{S}_{\text{avg}}[\tau^*,N_0]$, we have

$$\begin{aligned}\vartheta(t) &\leq \delta_1^{\frac{t(1-\eta)}{\tau^*}+N_0(1-\eta)}\bar{V}(0,x_0) - c_1\int_0^t \delta_1^{\frac{(t-s)(1-\eta)}{\tau^*}-N_0(1-\eta)}ds\\ &= \delta_1^{\frac{t(1-\eta)}{\tau^*}}\Big(\delta_1^{N_0(1-\eta)}\bar{V}(0,x_0) - \frac{c_1\tau^*\delta_1^{-N_0(1-\eta)}}{(1-\eta)\ln\delta_1}\Big) + \frac{c_1\tau^*\delta_1^{-N_0(1-\eta)}}{(1-\eta)\ln\delta_1}\\ &=: \sigma_1(t).\end{aligned} \tag{3.30}$$

Note that $\sigma_1(T^*)=0$ and (3.29) implies $T^*>0$. Thus, it follows from (3.25) that

$$0 \leq \bar{V}(T^*,x(T^*)) \leq \vartheta(T^*) \leq \sigma_1(T^*) = 0,$$

i.e., $\vartheta(T^*)=0$. Note that $N(T^*,0)<T^*/\tau^*+N_0$ for $\{t_k\}\in\mathcal{S}_{\rm avg}[\tau^*,N_0]$ and

$$T^*<\frac{\tau^*}{(1-\eta)\ln\delta_1}\ln\Big(\frac{c_1\tau^*}{c_1\tau^*-\delta_1^{2(1-\eta)N_0}\alpha_2^{1-\eta}(1)\ln\delta_1}\Big)$$

for $|x_0|<1$. Thus, $\sup\{N(T^*,0):|x_0|<1\}<\infty$ for $\forall\{t_k\}\in\mathcal{S}_{\rm avg}[\tau^*,N_0]$, and the desired result follows from Theorem 3.3. □

Remark 3.6. Note that the impulse sequence in Theorem 3.3 may be finite or infinite and unbounded. However, in [11,19,21,22], only finite impulse sequence is considered.

4 Finite-time instability

In this section, for the finite-time instability of system (2.1), we mainly investigate two cases: finite-time stable system with destabilizing impulses and finite-time instable system with stabilizing impulses.

4.1 The finite-time instability of systems with destabilizing impulses

The finite-time instability of systems with destabilizing impulses is studied in the following theorem.

Theorem 4.1. *If there exist a function $V\in\mathcal{C}^{1,1}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$, two $\mathcal{K}_\infty$ functions α_1 and α_2, and constants $\delta_1>1$, $\eta\in[0,1)$, and $c_1>0$ such that*

$$\alpha_1(|x|)\le V(t,x)\le\alpha_2(|x|),\qquad \forall t\ge 0,\quad \forall x\in\mathbb{R}^n, \tag{4.1a}$$
$$V'(t,x)=-c_1V^\eta(t,x),\qquad t\neq t_k,\quad k\in\mathbb{N}, \tag{4.1b}$$
$$V(t_k,x(t_k))=\delta_1V(t_k^-,x(t_k^-)),\qquad k\in\mathbb{N}, \tag{4.1c}$$

then the origin of system (2.1) *is finite-time instable for $\forall\{t_k\}\in\mathcal{S}_{\rm r-avg}[\tau^*,N_0]\cap\mathcal{S}_{\rm avg}[\tau^*,N_0]$, where $N_0>0$ and τ^* satisfy*

$$0<\tau^*\le\frac{V^{1-\eta}(0,x_0)\ln\delta_1}{c_1\delta_1^{2(1-\eta)N_0}}. \tag{4.2}$$

Proof. Let $\bar V(t,x):=V^{1-\eta}(t,x(t))/(1-\eta)$. Then by (4.1b) and (4.1c), we obtain that for $t\neq t_k$ and $k\in\mathbb{N}$,

$$\bar V'(t,x)=-c_1,\quad \bar V(t_k,x(t_k))=\delta_1^{1-\eta}\bar V(t_k^-,x(t_k^-)). \tag{4.3}$$

Following the proof of Lemma 2.1, we have

$$\bar V(t,x(t))=\delta_1^{(1-\eta)N(t,0)}\bar V(0,x_0)-c_1\int_0^t\delta_1^{(1-\eta)N(t,s)}ds. \tag{4.4}$$

Since $\{t_k\}\in\mathcal{S}_{\rm r-avg}[\tau^*,N_0]\cap\mathcal{S}_{\rm avg}[\tau^*,N_0]$, by (2.3a) and (2.3b), we obtain

$$\begin{aligned}\bar{V}(t,x(t))\geq&\delta_1^{(1-\eta)\left(\frac{t}{\tau^*}-N_0\right)}\bar{V}(0,x_0)-c_1\int_0^t\delta_1^{(1-\eta)\left(\frac{t-s}{\tau^*}+N_0\right)}ds\\ =&\delta_1^{(1-\eta)t/\tau^*}\left(\delta_1^{-(1-\eta)N_0}\bar{V}(0,x_0)-\frac{c_1\tau^*\delta_1^{(1-\eta)N_0}}{(1-\eta)\ln\delta_1}\right)+\frac{c_1\tau^*\delta_1^{(1-\eta)N_0}}{(1-\eta)\ln\delta_1},\end{aligned}\tag{4.5}$$

which together with (4.2) yield that $\bar{V}(t,x(t))>0$ for $t\geq0$ and $x_0\neq0$. Then it follows from the construction of $\bar{V}$ and (4.1a) that $x(t)\neq0$ for $t\geq0$ and $x_0\neq0$. The proof is completed. □

Remark 4.1. According to [15, Theorem 2.2], the origin of system (2.1) without impulsive effects is finite-time stable if the conditions of Theorem 4.1 hold. If the impulses are destabilizing, in order to guarantee the origin of the system to be finite-time stable, the impulses should not occur frequently (see Corollary 3.2); otherwise, the origin of the system may be finite-time instable (see Theorem 4.1). This result will be illustrated by the numerical simulations of system (5.4) in Section 5.

4.2 The finite-time instability of systems with stabilizing impulses

In the following, we present a class of systems, which cannot be stabilized by the stabilizing impulses.

Theorem 4.2. *If there exist a function $V\in\mathcal{C}^{1,1}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$, a function $l\in\mathcal{L}^1(\mathbb{R}_+;\mathbb{R})$, two $\mathcal{K}$ functions α_1 and α_2 and a positive constant $\delta_0\in(0,1)$ such that for $\forall x\in\mathbb{R}^n$,*

$$\alpha_1(|x|)\leq V(t,x)\leq\alpha_2(|x|),\tag{4.6a}$$

$$V'(t,x)\geq l(t)V(t,x),\qquad t\neq t_k,\quad k\in\mathbb{N},\tag{4.6b}$$

$$V(t_k,x(t_k))=\delta_0V(t_k^-,x(t_k^-)),\qquad k\in\mathbb{N},\tag{4.6c}$$

then the origin of system (2.1) *is finite-time instable for any given impulse sequence with $N(t,0)<\infty$ for $\forall t>0$.*

Proof. Let

$$\bar{V}(t,x):=e^{-\int_0^t l(s)ds}V(t,x).$$

Then it follows from (4.6b) and (4.6c) that $\bar{V}'(t,x(t))\geq0$ and $\bar{V}(t_k,x(t_k))=\delta_0\bar{V}(t_k^-,x(t_k^-))$ for $t\neq t_k$ and $k\in\mathbb{N}$, which together with the proof of Lemma 2.1 yield that

$$\bar{V}(t,x(t))\geq\delta_0^{N(t,0)}\bar{V}(0,x(0)),\quad t\geq0.\tag{4.7}$$

Combining this inequality with the definition of $\bar{V}(t,x(t))$ and (4.6a) yield that

$$\begin{aligned}\alpha_2(|x(t)|)\geq V(t,x(t))\geq&\delta_0^{N(t,0)}e^{\int_0^t l(s)ds}V(0,x_0)\\ \geq&\delta_0^{N(t,0)}e^{\int_0^t l(s)ds}\alpha_1(|x_0|).\end{aligned}\tag{4.8}$$

Then we have that there is no finite time $t^*>0$ such that $x(t^*)=0$ for $x_0\neq 0$ and any impulse sequence satisfying $N(t,0)<\infty$. The proof is completed. $\square$

Remark 4.2. By (4.8), it is shown that the origin of system (2.1) without impulsive effects is finite-time instable if the conditions of Theorem 4.2 hold. Thus, we obtain that some finite-time instable systems cannot be stabilized by the stabilizing impulses.

5 Numerical examples

In this section, we provide some numerical examples to illustrate the obtained results on finite-time stability.

Example 5.1. Consider the following system

$$\begin{cases} \dot{x}(t)=-2^{-\frac{t}{2}}x^{-\frac{1}{3}}(t), \quad t\neq k, \quad k\in\mathbb{Z}_+, \\ x(k)=2^{-\frac{3}{4}}x(k^-), \end{cases} \tag{5.1}$$

with the initial value $x(0)=3$.

It follows from Theorem 3.1 that the origin of system (5.1) is finite-time stable, since by letting

$$V(t,x(t))=\frac{3}{2}x^{\frac{4}{3}}(t),$$

we have $\alpha_1(s)=\alpha_2(s)=3s^{4/3}/2$, $\eta=0$, $l(t)=-2^{-t/2+1}$, $\delta_0=1/2$ and

$$\mu(t)\geq\int_0^t 2^{-\frac{s}{2}+1}2^{s-1}ds=\frac{2}{\ln 2}(2^{\frac{t}{2}}-1).$$

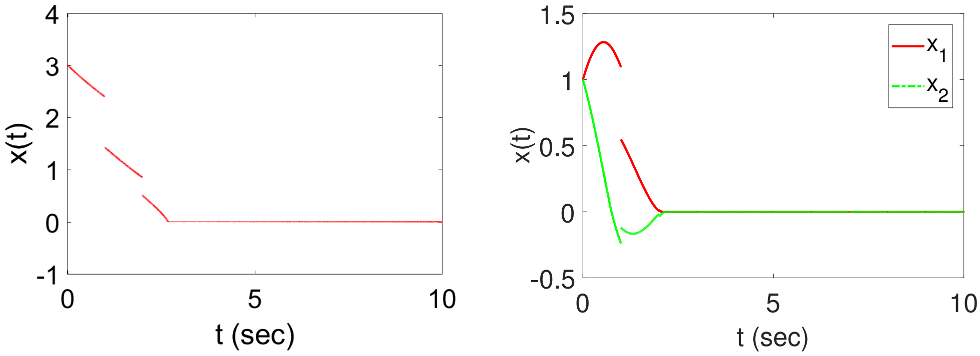
In addition, the solution of system (5.1) without impulsive jump is

$$x^{\frac{4}{3}}(t)=\frac{8}{3\ln 2}\left(2^{-t/2}-1+\frac{3\ln 2}{8}3^{\frac{4}{3}}\right), \tag{5.2}$$

which cannot reach the origin in finite time, and $l(t)$ has no relation with the initial value and the first impulsive time, and is not a finite-time stable function, thus, the existing finite-time stability results (e.g., [3,11,17,19–22]) cannot be used to analyze the the finite-time stability of (5.1). Fig. 1(a) shows that the state of system (5.1) converges to the origin in finite time.

Example 5.2. Consider the following system

$$\begin{cases} \dot{x}_1(t)=-\dfrac{t}{2}x_1^{\frac{1}{3}}+x_2, \quad t\neq k, \quad k\in\mathbb{Z}_+, \\ \dot{x}_2(t)=-\dfrac{t}{2}x_2^{\frac{1}{3}}-x_1, \quad t\neq k, \\ x(k)=2^{(-1)^k}x(k^-). \end{cases} \tag{5.3}$$



(a) State of system (5.1) (b) State of system (5.3)

Figure 1: Simulations of systems (5.1) and (5.3).

Let $V(t,x):=x_1^2+x_2^2$. Then the conditions of Theorem 3.2 hold with $\alpha_1(s)=\alpha_2(s)=s^2$, $l(t)=-t$, $\eta=2/3$, $\theta_1=1/2$ and $\theta_2=2$. Hence, the origin of system (5.3) is finite-time stable. We provide the simulations for the state of system (5.3) with $x(0)=(1,1)^T$ in Fig. 1(b), which show that the state of system (5.3) converges to the origin in finite time.

Example 5.3. Consider the following system

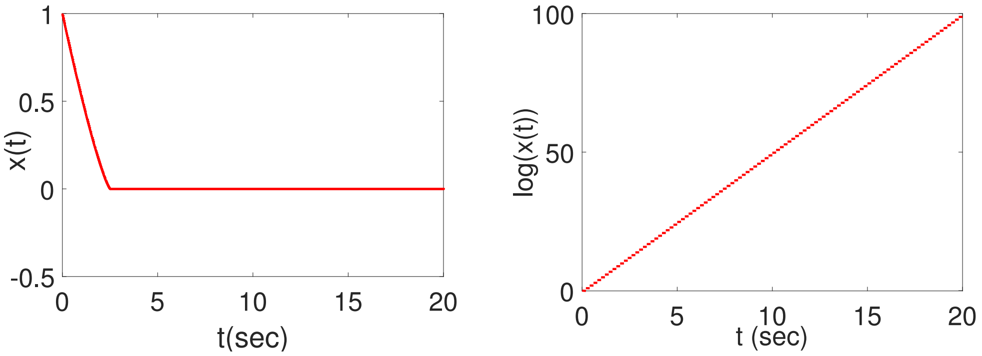
$$\begin{cases} \dot{x}(t)=-\dfrac{1}{2}x^{\frac{1}{5}}(t), & t\neq t_k, \\ x(t_k)=ex(t_k^-), \end{cases} \tag{5.4}$$

with $x(0)=1$.

Let $V(t,x):=x^2$. Then

$$V'(t,x(t))=-V^{\frac{3}{5}}(t,x(t)) \quad \text{for} \quad t\neq t_k \quad \text{and} \quad V(t_k,x(t_k))=e^2V(t_k^-,x(t_k^-)).$$

For the case of $\{t_k:t_k=4k,\ k\in\mathbb{Z}_+\}$, we have $\tau^*=4$ and $N_0=1$. Then system (5.4) with $\{t_k:t_k=4k,\ k\in\mathbb{Z}_+\}$ satisfies the conditions of Corollary 3.2. Namely, the origin of system (5.4) with $\{t_k:t_k=4k,\ k\in\mathbb{Z}_+\}$ is finite-time stable. We provide the simulations of system (5.4) with $\{t_k:t_k=4k,\ k\in\mathbb{Z}_+\}$ and $\{t_k:t_k=\frac{k}{5},\ k\in\mathbb{Z}_+\}$ in Fig. 2(a) and Fig. 2(b) respectively. Fig. 2(a) shows that state of system (5.4) with $\{t_k:t_k=4k,\ k\in\mathbb{Z}_+\}$ converges to the origin in finite time. However, when the impulse sequence is $\{t_k:t_k=\frac{k}{5},\ k\in\mathbb{Z}_+\}$, the state of system (5.4) cannot converge to the origin in Fig. 2(b). Fig. 2 shows that for the system with destabilizing impulses, the impulses should not occur frequently; otherwise, the origin of the system is finite-time instable.



(a) $\{t_k : t_k = 4k,\ k \in \mathbb{Z}_+\}$ (b) $\{t_k : t_k = \frac{k}{5},\ k \in \mathbb{Z}_+\}$

Figure 2: Simulations of system (5.4).

6 Concluding remarks

The global finite-time stability and instability of nonlinear impulsive systems are studied in this paper. For system with stabilizing impulses, we provide a Lyapunov theorem on global finite-time stability and show that the more frequently the stabilizing impulses happen, the faster the state converges to the origin. For system with mixed impulses, the global finite-time stability is studied, and it is shown that if the continuous system is GFTS with settling time being continuous at the origin, then the system is still GFTS under any impulsive effects if the jumps satisfy some mild conditions. For system with destabilizing impulses, the global finite-time stability and finite-time instability are studied and it is shown that to be finite-time stable, the destabilizing impulses should not occur too frequently, otherwise, the origin of the system is finite-time instable, which are formulated by ADT conditions respectively. A theorem of finite-time instability for system with stabilizing impulses is developed, and it is shown that a class of finite-time instable systems cannot be stabilized by the stabilizing impulses.

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