

Avoiding Small Denominator Problems by Means of the Homotopy Analysis Method

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Abstract. The so-called “small denominator problem” was a fundamental problem of dynamics, as pointed out by Poincaré. Small denominators appear most commonly in perturbative theory. The Duffing equation is the simplest example of a non-integrable system exhibiting all problems due to small denominators. In this paper, using the forced Duffing equation as an example, we illustrate that the famous “small denominator problems” never appear if a non-perturbative approach based on the homotopy analysis method (HAM), namely “the method of directly defining inverse mapping” (MDDiM), is used. The HAM-based MDDiM provides us great freedom to directly define the inverse operator of an undetermined linear operator so that all small denominators can be completely avoided and besides the convergent series of multiple limit-cycles of the forced Duffing equation with high nonlinearity are successfully obtained. So, from the viewpoint of the HAM, the famous “small denominator problems” are only artifacts of perturbation methods. Therefore, completely abandoning perturbation methods but using the HAM-based MDDiM, one would be never troubled by “small denominators”. The HAM-based MDDiM has general meanings in mathematics and thus can be used to attack many open problems related to the so-called “small denominators”.

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1 Origin of “small denominator problem”

Poincaré [1] pointed out that the so-called “small denominator problem” was “the fundamental problem of dynamics”. The small denominator was first mentioned by Delaunay [2] in his 900 pages book about celestial motions using perturbation method.

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Poincaré [1] first recognized that, when small denominator appears, the coefficients of perturbation series may grow too large too often, threatening the convergence of the series. As pointed out by Pérez [3], “small denominators are found most commonly in the perturbative theory”. It often appears when perturbation methods are used to solve problems in classical and celestial mechanics [4], fluid mechanics [5, 6], and so on [7, 8].

What is the origin of the so-called “small denominator problem”? As pointed out by Giorgilli [9], the Duffing equation [10] “is perhaps the simplest example of a non-integrable system exhibiting all problems due to the small denominators”. So, without loss of generality, let us focus on the forced Duffing equation

$$\mathcal{N}[u(t)] = u''(t) + 2\zeta u'(t) + u(t) + \beta u^3(t) - \alpha \cos(\Omega t) = 0, \quad (1.1)$$

where \mathcal{N} is a nonlinear operator, the prime denotes the differentiation with respect to the time t , α and Ω is the amplitude and frequency of the external force $F = \alpha \cos(\Omega t)$, $\zeta > 0$ is the resistance coefficient, and $\beta > 0$ is a physical parameter related to nonlinearity, respectively.

As pointed out by Kartashova [11], “physical classification of PDEs is based not on the form of equations, but on the form of solutions”. So, let us consider here the stationary periodic limit-cycle of $u(t)$ as $t \rightarrow +\infty$ of the forced Duffing equation (1.1), which can be expressed in the form:

$$u(t) = \sum_{n=1}^{+\infty} \left\{ a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \right\}, \quad (1.2)$$

where a_n, b_n are constants and

$$\omega_n = (2n-1)\Omega, \quad n \geq 1. \quad (1.3)$$

This is mainly because the common solution

$$A \exp(-\zeta t) \cos(t) + B \exp(-\zeta t) \sin(t)$$

of the linear equation

$$u''(t) + 2\zeta u'(t) + u(t) = 0$$

tends to zero as $t \rightarrow +\infty$ for arbitrary constants A and B , and thus disappear in the so-called “solution-expression” (1.2) of the limit-cycle.

Let us first show how perturbation technique [12, 13] can bring the so-called small denominators into the above-mentioned problem. Let β be a small parameter and assume that $u(t)$ can be expanded in such a series

$$u(t) = u_0(t) + \sum_{n=1}^{+\infty} u_n(t) \beta^n. \quad (1.4)$$

Substituting it into (1.1) and equating the like power of β , we have the perturbation equations at different orders of β :

$$\beta^0: u_0''(t) + 2\zeta u_0'(t) + u_0(t) = \alpha \cos(\Omega t), \quad (1.5a)$$

$$\beta^1: u_1''(t) + 2\zeta u_1'(t) + u_1(t) = -u_0^3(t), \quad (1.5b)$$

$$\beta^2: u_2''(t) + 2\zeta u_2'(t) + u_2(t) = -3u_0^2(t)u_1(t), \quad (1.5c)$$

$$\beta^3: u_3''(t) + 2\zeta u_3'(t) + u_3(t) = -3u_0^2(t)u_2(t) - 3u_0(t)u_1^2(t), \dots \quad (1.5d)$$

The above perturbation equations have the *unique* linear operator

$$\mathcal{L}_0[u(t)] = u''(t) + 2\zeta u'(t) + u(t), \quad (1.6)$$

whose inverse operator \mathcal{L}_0^{-1} reads

$$\mathcal{L}_0^{-1}[\cos(\omega t)] = \frac{(1-\omega^2)\cos(\omega t) + 2\zeta\omega\sin(\omega t)}{(1-\omega^2)^2 + 4\zeta^2\omega^2}, \quad (1.7a)$$

$$\mathcal{L}_0^{-1}[\sin(\omega t)] = \frac{(1-\omega^2)\sin(\omega t) - 2\zeta\omega\cos(\omega t)}{(1-\omega^2)^2 + 4\zeta^2\omega^2}, \quad (1.7b)$$

where ω is a frequency. Note that the denominator

$$(1-\omega^2)^2 + 4\zeta^2\omega^2$$

becomes rather small when $\omega \rightarrow 1$ and $\zeta \rightarrow 0$ so that small denominators appear. This is the origin of the so-called “small denominator problem”. Thus, for Eq. (1.1), the “small denominator problem” occurs when ζ is small and $\Omega = 1/(2n-1)$, i.e., $\omega_n = (2n-1)\Omega = 1$, for any a positive integer $n \geq 1$. Without loss of generality, let us focus on here the fixed values $\alpha = 1$ and $\Omega = 1/3$, say, $\omega_2 = 3\Omega = 1$, but investigate different values of ζ and β . In this case, the so-called “resonance” occurs when $\zeta = 0$ and the “near resonance” occurs for a small ζ , corresponding to the “small denominator problem”.

In the frame of the perturbation approach, the *unique* initial guess is given according to (1.5a) and (1.7a), say,

$$u_0(t) = \frac{(1-\Omega^2)\cos(\Omega t) + 2\zeta\Omega\sin(\Omega t)}{(1-\Omega^2)^2 + 4\zeta^2\Omega^2}. \quad (1.8)$$

Note that small denominator appears when $\Omega \rightarrow 1$ and $\zeta \rightarrow 0$ for this perturbation initial approximation! This is the reason why we choose $\Omega = 1/3 = \omega_1$ in this paper, otherwise the perturbation method fails at the very beginning.

Let us first consider the perturbation method in case of $\Omega = 1/3$ and $\alpha = 1$. It is found that, when $\zeta = 1/100$, say, there exists the so-called “small denominator problem” for the terms $\cos(\omega_2 t)$ and $\sin(\omega_2 t)$, where $\omega_2 = 3\Omega = 1$, the perturbation series is divergent even for a small value $\beta = 0.012$, corresponding to a rather weak nonlinearity. Thus,

when the so-called “small denominator problem” appears, the perturbation method is indeed invalid in practice. Besides, it is found that, when $\zeta = 1$, the perturbation series is also divergent for $\beta \geq 0.55$, indicating that the perturbation series is divergent for high nonlinearity even if the “small denominator problem” does not occur! Therefore, the perturbation approach indeed does not work for high nonlinearity and/or when the so-called “small denominator problem” occurs. As mentioned by Arnol’d [4], there often exist two difficulties with perturbation method in many classical and celestial problems: (I) the appearance of small denominator, and (II) the divergence of solution series.

The above-mentioned perturbation approach has the following disadvantages:

- (1). Small/large physical parameters should exist;
- (2). There is no freedom to choose its linear operator;
- (3). There is no freedom to choose its initial approximation;
- (4). It is convergent only for weak nonlinearity.

These limitations of perturbation methods are well-known. They are the origin of the so-called “small denominator problem” and the divergence of solution series.

Note that, due to some historic reasons, the so-called “small denominator problem” has very close relationships with perturbation methods. Indeed, “small denominators are found most commonly in the perturbative theory”, as pointed out by Pérez [3]. Is perturbation method the only way to solve these problems? What happens if we completely abandon perturbation methods?

In this paper, we use the forced Duffing equation (1.1) to illustrate that the so-called “small denominator problem” can never appear if we completely abandon perturbation techniques but use a non-perturbative technique, namely the homotopy analysis method (HAM) [14–26], which is based on the basic concept “homotopy” in topology and can overcome all restrictions of perturbation methods. A new HAM-based approach is proposed, which provides us great freedom to directly define the inverse operator of an auxiliary linear operator so that all small denominators can be completely avoided. Convergent series of multiple limit-cycles of the Duffing equation are successfully obtained, although the directly defined inverse operators might be beyond the traditional mathematical theories. Thus, from the viewpoint of the HAM, the famous “small denominator problems” are only artifacts of perturbation methods. Therefore, completely abandoning perturbation methods, one would be never troubled by small denominators, as illustrated below in this paper.

2 Basic ideas of the HAM

Can we avoid the famous “small denominator problem” in a systematic way? The answer is yes. Here, we give an approach based on the homotopy analysis method (HAM) [14–16], which can completely avoid the “small denominator problem”.

“Small denominators are found most commonly in the perturbative theory”, as pointed out by Pérez [3]. So, in order to avoid “small denominator problem”, we *must* abandon perturbation methods *completely*. The homotopy analysis method (HAM) was proposed by Liao in 1992 in his dissertation [14]. Based on the basic concept homotopy in topology [27], i.e., a continuous deformation, the HAM [15–26] has the following advantages:

- (a). Unlike perturbation techniques, the HAM works even if there exist no small/large physical parameters;
- (b). The HAM provides great freedom to choose an auxiliary linear operator;
- (c). The HAM provides great freedom to choose an initial guess;
- (d). Different from other approximation methods, the HAM can guarantee the convergence of solution series even for highly nonlinear problem.

The HAM has been broadly used and its above-mentioned advantages have been verified and confirmed in thousands of articles by scientists and engineers all over the world [28–42]. In this paper, we use the forced Duffing equation (1.1) as an example to illustrate how to completely avoid the “small denominator problem” by means of a HAM-based approach.

First, let us briefly describe the basic ideas of the HAM using (1.1) as an example. Let

$$\mathcal{S} = \sum_{n=1}^{+\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \quad (2.1)$$

denote a vector space, where ω_n is defined by (1.3) and A_n, B_n are arbitrary constants. Let $u(t) \in \mathcal{S}$, \mathcal{L} denote an auxiliary linear operator with the property $\mathcal{L}[0] = 0$, which we have great freedom to choose, $q \in [0, 1]$ be a homotopy parameter, $u_0(t) \in \mathcal{S}$ be an initial guess of $u(t)$, c_0 be a constant having no physical meanings, namely “the convergence-control parameter”, whose value will be determined later, respectively. Then, we construct a continuous deformation $\phi(t, q) \in \mathcal{S}$ from the initial guess $u_0 \in \mathcal{S}$ to the solution $u(t) \in \mathcal{S}$ of the forced Duffing equation (1.1), governed by the so-called zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t, q) - u_0(t)] = c_0 q \mathcal{N}[\phi(t, q)], \quad q \in [0, 1], \quad (2.2)$$

where the nonlinear operator $\mathcal{N}[u]$ is defined by Eq. (1.1). When $q=0$, due to the property $\mathcal{L}[0] = 0$ of the auxiliary linear operator, we have the solution

$$\phi(t, 0) = u_0(t) \quad (2.3)$$

of Eq. (2.2). When $q=1$, Eq. (2.2) is exactly the same as the original equation (1.1), thus we have the solution

$$\phi(t, 1) = u(t), \quad (2.4)$$

where $u(t)$ is the solution (limiting cycle) of the original Duffing equation (1.1). So, as q increases from 0 to 1, $\phi(t,q)$ deforms *continuously* from the initial guess $u_0(t)$ to the solution $u(t)$ of the original nonlinear equation (1.1), since both of $u_0(t) \in \mathcal{S}$ and $u(t) \in \mathcal{S}$ can be expressed by the so-called “solution expression” (1.2). Then, expanding $\phi(t,q)$ in a power series of q , we have according to (2.3) the homotopy-series

$$\phi(t,q) = u_0(t) + \sum_{k=1}^{+\infty} u_k(t)q^k. \tag{2.5}$$

Note that $\phi(t,q) \in \mathcal{S}$ is dependent upon the so-called convergence-control parameter c_0 , which has no physical meanings. Therefore, $u_k(t) \in \mathcal{S}$ ($k \geq 1$) in (2.5) is also dependent upon c_0 so that the convergence radius of the series (2.5) is determined by c_0 . Here, it should be emphasized that we have great *freedom* to choose the auxiliary linear operator \mathcal{L} , the initial guess u_0 and the convergence-control parameter c_0 . This is the key point of the HAM. Assuming that the auxiliary linear operator \mathcal{L} , the initial guess u_0 and the convergence-control parameter c_0 are so properly chosen that the Maclaurin series (2.5) is convergent at $q=1$, we have due to (2.4) the homotopy-series solution

$$u(t) = u_0(t) + \sum_{k=1}^{+\infty} u_k(t). \tag{2.6}$$

So, even for given auxiliary linear operator \mathcal{L} and initial guess u_0 , the convergence-control parameter c_0 provides us an additional way to guarantee the convergence of the solution series, which overcomes the limitations of perturbation methods mentioned above, as illustrated below in this paper and other publications [28–42].

Substituting the power series (2.5) into the zeroth-order deformation equation (2.2) and equating the like-power of q , we have the high-order deformation equation

$$\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = c_0 R_{m-1}(t), \quad m \geq 1, \tag{2.7}$$

where

$$R_k(t) = \frac{1}{k!} \left. \frac{d^k \mathcal{N}[\phi(t,q)]}{dq^k} \right|_{q=0} \tag{2.8}$$

in general and

$$R_k(t) = \begin{cases} u_0'' + 2\xi u_0' + u_0 + \beta u_0^3 - \alpha \cos(\Omega t), & \text{when } k=0, \\ u_k'' + 2\xi u_k' + u_k + \beta \sum_{i=0}^k \sum_{j=0}^i u_{k-i} u_{i-j} u_j, & \text{when } k \geq 1, \end{cases} \tag{2.9}$$

for the forced Duffing equation (1.1) considered here, and besides

$$\chi_m = \begin{cases} 0, & \text{when } m=1, \\ 1, & \text{when } m > 1. \end{cases} \tag{2.10}$$

The general solution of the linear m th-order deformation equation (2.7) reads

$$u_m(t) = \chi_m u_{m-1}(t) + c_0 \mathcal{L}^{-1} [R_{m-1}(t)] + \sum_{n=1}^{\mu} A_{m,n} \psi_n(t), \quad m \geq 1, \quad (2.11)$$

where μ is a positive integer, \mathcal{L}^{-1} is the inverse operator of \mathcal{L} , $A_{m,n}$ is an arbitrary constant, and $\psi_n(t) \in \mathcal{S}$ is a base function satisfying

$$\mathcal{L} \left[\sum_{n=1}^{\mu} A_{m,n} \psi_n(t) \right] = 0, \quad 1 \leq n \leq \mu. \quad (2.12)$$

In other words, we have

$$\ker[\mathcal{L}] = \sum_{n=1}^{\mu} A_{m,n} \psi_n(t), \quad (2.13)$$

say, the kernel of the auxiliary linear operator \mathcal{L} is a vector space in dimension μ . Note that the linear part (1.6) of the original Duffing equation (1.1) is a second-order differential equation, whose kernel is a vector space in two dimension. However, we have great freedom to choose the auxiliary linear operator \mathcal{L} and its kernel $\ker[\mathcal{L}]$, as mentioned below, which might be a breakthrough in nonlinear differential equations.

The M th-order HAM approximation is given by

$$u^* \approx u_0(t) + \sum_{k=1}^M u_k(t). \quad (2.14)$$

Since the HAM provides us great freedom to choose the initial guess u_0 , we can further use the above M th-order approximation as a new initial guess, say, $u_0 = u^*$, to gain another M th-order approximation, and so on. This provides us the M th-order iteration approach of the HAM. Note that, for the HAM iteration approach, in order to avoid the exponential increment in growth of the terms in the solution expression (1.2), we eliminate the terms of $\cos(\omega_n t)$, $\sin(\omega_n t)$ whose coefficients are less than a small value, such as 10^{-20} for the forced Duffing equation (1.1) considered in this paper.

It should be emphasized once again that, different from all other approximation methods (including perturbation techniques), the homotopy analysis method (HAM) can guarantee the convergence of solution series by means of choosing a proper value of the so-called "convergence-control parameter" c_0 . This is the fundamental difference of the HAM from all other approaches! The optimal value of the "convergence-control parameter" c_0 is determined by the minimum of the residual error square

$$\mathcal{E} = \int_0^T (\mathcal{N}[u(t)])^2 dt \approx \frac{1}{K+1} \sum_{j=0}^K (\mathcal{N}[u(j\Delta t)])^2, \quad (2.15)$$

where $\Delta t = T/(K+1)$ is a time-step for numerical simulation, T is the period of the limiting cycle for the considered problem, $K > 0$ is a large enough integer, $u(t)$ is an approximation of limiting cycle of the original equation (1.1), \mathcal{N} is the nonlinear operator defined by (1.1), respectively.

What is the relationship between perturbation method and the HAM? Generally speaking, perturbation approach is often a special case of the HAM, if we choose the perturbation initial approximation as u_0 , the original linear operator as the auxiliary linear operator, i.e., $\mathcal{L} = \mathcal{L}_0$, and besides $c_0 = -1$. For example, when we choose (1.6) as the auxiliary linear operator \mathcal{L} , (1.8) as the initial approximation u_0 , and besides set $c_0 = -1$, the k th-order deformation equation (2.7) is exactly the same as the k th-order perturbation equation mentioned in Section 1. Therefore, the perturbation approach can be indeed regarded as a special case of the homotopy analysis method! However, the perturbation approach corresponds to only one choice, but there exist *many* other much *better* choices in the frame of the HAM, which can avoid the “small denominator problems” completely, as illustrated below.

In summary, the above-mentioned HAM has the following characteristics:

- (A). The homotopy-series (2.5) is expanded in the homotopy parameter $q \in [0,1]$ that has no physical meanings at all. So, the HAM has nothing to do with any small/large physical parameters: it works no matter whether small/large physical parameters exist or not;
- (B). The HAM provides us great freedom to choose its auxiliary linear operator;
- (C). The HAM provides us great freedom to choose its initial approximation;
- (D). The so-called convergence-control parameter c_0 has no physical meanings but can guarantee the convergence of the solution series even for high nonlinearity, as illustrated below and verified in many related publications [28–42].

Thus, the HAM can indeed overcome all limitations and restrictions of perturbation methods.

3 How to avoid “small denominator problem”

As mentioned above, the HAM provides us great freedom to choose the auxiliary linear operator \mathcal{L} and the initial guess u_0 : it is such kind of freedom that provides us possibility to avoid the so-called “small denominator problem”, as described below.

3.1 Choice of the auxiliary linear operator

Obviously, for the forced Duffing equation (1.1), the origin of the “small denominator problem” is mainly due to the original linear operator (1.6), which is unique from the viewpoint of perturbation theory. So, in order to avoid “small denominator problem”, we must abandon (1.6) thoroughly. Different from other approximation techniques, the HAM provides us great freedom to choose an auxiliary linear operator \mathcal{L} , as illustrated by Liao & Tan [21] and Liao & Zhao [43]. In most applications of the HAM, one often chooses

a proper auxiliary linear operator \mathcal{L} to gain the solution of the high-order deformation equation (2.7). However, the freedom is so large that we can here directly **define** its inverse operator \mathcal{L}^{-1} and the kernel of \mathcal{L} in (2.11). In fact, based on the HAM, Liao and Zhao [43] proposed the so-called “method of directly defining inverse mapping”, i.e., the MDDiM, which has been successfully applied to solve many types of nonlinear equations [44–56]. According to the solution expression (1.2) and the definition (2.9) of $R_m(t)$, the right-hand side of the high-order deformation equation (2.7) contains terms $\cos(\omega_n t)$ and $\sin(\omega_n t)$, where $\omega_n = (2n-1)\Omega$. Thus, we directly **define** here its inverse operator

$$\mathcal{L}^{-1} \left[A \cos(\omega_n t) + B \sin(\omega_n t) \right] = \frac{A \cos(\omega_n t) + B \sin(\omega_n t)}{\lambda^2 - \omega_n^2}, \quad |\lambda^2 - \omega_n^2| > \delta, \quad (3.1)$$

and its kernel

$$\mathcal{L} \left[A' \cos(\omega_n t) + B' \sin(\omega_n t) \right] = 0, \quad |\lambda^2 - \omega_n^2| \leq \delta, \quad (3.2)$$

for arbitrary constants A, B, A', B' , where we have great freedom to choose the two parameters $\lambda > 0$ and $\delta \geq 0$. Note that both of \mathcal{L} and \mathcal{L}^{-1} are linear, say,

$$\mathcal{L} \left[A \cos(\omega_n t) + B \sin(\omega_n t) \right] = A \mathcal{L} \left[\cos(\omega_n t) \right] + B \mathcal{L} \left[\sin(\omega_n t) \right], \quad (3.3a)$$

$$\mathcal{L}^{-1} \left[A' \cos(\omega_n t) + B' \sin(\omega_n t) \right] = A' \mathcal{L}^{-1} \left[\cos(\omega_n t) \right] + B' \mathcal{L}^{-1} \left[\sin(\omega_n t) \right], \quad (3.3b)$$

for arbitrary constants A, B, A' and B' . The above definitions are complete, according to the so-called “solution-expression” (1.2).

Let

$$W_{\lambda, \delta} = \left\{ \omega_n : |\lambda^2 - \omega_n^2| \leq \delta \right\} \quad (3.4)$$

denote a set containing all frequencies ω_n that satisfies $|\lambda^2 - \omega_n^2| \leq \delta$ for a given pair of λ and δ , where $\omega_n = (2n-1)\Omega$ is the frequency defined by (1.3). Then, its inverse operator (3.1) and the kernel (3.2) of the auxiliary linear operator \mathcal{L} can be rewritten by

$$\mathcal{L}^{-1} \left[A \cos(\omega_n t) + B \sin(\omega_n t) \right] = \frac{A \cos(\omega_n t) + B \sin(\omega_n t)}{\lambda^2 - \omega_n^2}, \quad \omega_n \notin W_{\lambda, \delta}, \quad (3.5)$$

and

$$\mathcal{L} \left[A' \cos(\omega_n t) + B' \sin(\omega_n t) \right] = 0, \quad \omega_n \in W_{\lambda, \delta}, \quad (3.6)$$

respectively, where A, B, A' and B' are arbitrary constants. Assume that $W_{\lambda, \delta}$ contains κ frequencies. Then, the kernel of the auxiliary linear operator \mathcal{L} defined by (3.1) and (3.2) is a vector space in dimension $\mu = 2\kappa$, say,

$$\ker[\mathcal{L}] = \sum_{\omega_n \in W_{\lambda, \delta}} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \quad (3.7)$$

for arbitrary constants A_n and B_n . Note that, in the frame of the HAM, we have great freedom to choose the dimension μ that is determined by λ and δ , as mentioned below. This is completely different from the traditional mathematical theory for a second-order differential equation. Thus, the general solution of the m th-order deformation equation (2.7) reads

$$u_m(t) = \chi_m u_{m-1}(t) + c_0 \mathcal{L}^{-1} \left[R_{m-1}(t) \right] + \sum_{\omega_n \in W_{\lambda, \delta}} \left[A_{m,n} \cos(\omega_n t) + B_{m,n} \sin(\omega_n t) \right], \quad (3.8)$$

where $A_{m,n}$ and $B_{m,n}$ are unknown constants. Note that $\mathcal{L}^{-1} [R_{m-1}(t)]$ can be directly obtained using the inverse operator **defined** by (3.1) or (3.5), and the term $u_{m-1}(t)$ is known. The unknown constants $A_{m,n}$ and $B_{m,n}$ are determined via $R_m(t)$ in the way described below.

According to the definition (3.1) or (3.5) of the inverse operator \mathcal{L}^{-1} , $R_m(t)$ can not contain the terms $\cos(\omega_n t)$ and $\sin(\omega_n t)$, where $\omega_n \in W_{\lambda, \delta}$, since there are **no** definitions on them. Substituting u_m defined by (3.8) into $R_m(t)$ defined by (2.9), we have

$$R_m(t) = \sum \left[Q_{m,n} \cos(\omega_n t) + S_{m,n} \sin(\omega_n t) \right]. \quad (3.9)$$

To avoid the appearance of $\cos(\omega_n t)$ and $\sin(\omega_n t)$ terms in the above expression, where $\omega_n \in W_{\lambda, \delta}$, we had to enforce the following coefficients to be zero:

$$Q_{m,n} = 0, \quad S_{m,n} = 0, \quad \omega_n \in W_{\lambda, \delta}, \quad (3.10)$$

which give us $\mu = 2\kappa$ linear algebraic equations that determine the $\mu = 2\kappa$ unknown coefficients $A_{m,n}$ and $B_{m,n}$ of u_m defined by (3.8). In this way, we successively gain the solution $u_m(t)$ of the m th-order deformation equation (2.7), where $m = 1, 2, 3$ and so on, *without* any small denominators.

According to the definition (3.1) or (3.5) of the linear inverse operator \mathcal{L}^{-1} , we can always choose a proper pair of λ and δ so as to avoid the so-called “small denominator problem”, as illustrated below in Section 4. Note that it is the HAM [21, 43] that provides us such kind of great freedom.

3.2 Choice of the initial guess

The perturbation method provides us the *unique* initial guess (1.8), which unfortunately contains the small denominator $(1 - \omega^2)^2 + 4\xi^2 \omega^2$ when the frequency ω is close to 1 and ξ is small. It is well-known that perturbation approaches become invalid when small denominators appear. In addition, in the frame of perturbation techniques, there is no freedom to choose initial guess. So, we had to abandon the initial guess (1.8) of the perturbation method.

In the frame of the HAM, for the limit cycle of the Duffing equation (1.1), all approximations should be in the form of (1.2), called the “solution expression”. So, the initial guess should agree with the “solution expression” (1.2). Besides, it should contain at least one or two lowest frequencies, such as $\cos(\omega_1 t)$, $\sin(\omega_1 t)$, $\cos(\omega_2 t)$ and $\sin(\omega_2 t)$ for the forced Duffing equation (1.1). In addition, it should contain the kernel (3.7) of the auxiliary linear operator \mathcal{L} , too. For the forced Duffing equation (1.1), considering the solution-expression (1.2) and the kernel (3.7) of the auxiliary linear operator \mathcal{L} , we choose the initial guess in the form

$$u_0(t) = \sum_{j=1}^{\gamma} \left[a_{0,j} \cos(\omega_j t) + b_{0,j} \sin(\omega_j t) \right] + \sum_{\omega_n \in W_{\lambda,\delta}} \left[A_{0,n} \cos(\omega_n t) + B_{0,n} \sin(\omega_n t) \right], \quad (3.11)$$

where $\gamma \geq 1$ is an integer, and $a_{0,j}$, $b_{0,j}$, $A_{0,n}$, $B_{0,n}$ are unknown constants. Note that the same terms in the above expression should be combined. All of these unknown constants in (3.11) are determined in the way mentioned below.

According to (2.9), $R_0(t) = \mathcal{N}[u_0(t)]$ denotes the residual error of the forced Duffing equation (1.1) for the initial guess u_0 . So, using the initial guess (3.11), we have

$$R_0 = \sum \left[Q_{0,n} \cos(\omega_n t) + S_{0,n} \sin(\omega_n t) \right]. \quad (3.12)$$

To avoid the appearance of the terms $\cos(\omega_n t)$, $\sin(\omega_n t)$ in the above expression, where $\omega_n \in W_{\lambda,\delta}$ as defined by (3.4), we had to enforce the following coefficients to be zero:

$$Q_{0,n} = 0, \quad S_{0,n} = 0, \quad \omega_n \in W_{\lambda,\delta}, \quad (3.13)$$

which provides us $\mu = 2\kappa$ algebraic equations. Besides, if necessary, we had better enforce the disappearance of the base functions with the lowest frequencies, such as $\cos(\omega_1 t)$, $\sin(\omega_1 t)$, $\cos(\omega_2 t)$ and $\sin(\omega_2 t)$, in the above expression of $R_0(t)$. In this way, all unknown constants in the initial guess (3.11) could be gained. Note that it is a set of nonlinear algebraic equations with a few unknowns, which can be solved by means of some well-known symbolic computation software, such as the commands **FindRoot** and **NSolve** of *mathematica*.

Assume that, in the iteration approach of the HAM mentioned in Section 2, we have a known approximation $u^*(t)$. Then, we choose the initial guess

$$u_0(t) = u^*(t) + \sum_{\omega_n \in W_{\lambda,\delta}} \left[A_{0,n} \cos(\omega_n t) + B_{0,n} \sin(\omega_n t) \right], \quad (3.14)$$

where the unknown constants $A_{0,n}$, $B_{0,n}$ are determined by enforcing the disappearance of the kernel terms of \mathcal{L} in $R_0(t)$ in the similar way as mentioned above.

In this way, we can avoid the “small denominator problem” in the initial guess $u_0(t)$, which occurs for the perturbation initial guess (1.8) when $\Omega \rightarrow 1$ and $\xi \rightarrow 0$. More importantly, a set of nonlinear algebraic equations often has multiple solutions, which might lead to multiple solutions of the limit-cycle for the forced Duffing equation (1.1), as described below.

Finally, we should emphasize that it is the HAM that provides us great freedom to choose the initial guess u_0 .

4 Some examples

In this section, let us use the forced Duffing equation (1.1) to illustrate the validity and novelty of the HAM approach mentioned in Sections 2 and 3. Without loss of generality, let us consider the case of $\alpha = 1$, $\Omega = 1/3$ but various values of β and ξ . Note that β is a measurement of the nonlinearity of the forced Duffing equation (1.1): the larger the value of β , the higher the nonlinearity of the forced Duffing equation (1.1).

Since $\Omega = 1/3$ is fixed, we always have $\omega_2 = 3\Omega = 1$. Thus, from the viewpoint of perturbation techniques, the so-called “small denominator problem” happens when ξ is small, such as $10^{-4} \leq \xi \leq 10^{-2}$, so that the perturbation method fails, as mentioned in Section 1. However, we illustrate here that such kind of “small denominator problem” **never** appears in the frame of the HAM approach, as long as we properly choose a pair of λ and δ .

Since $\Omega = 1/3$ and $\omega_n = (2n-1)\Omega$, we have

$$\omega_1 = \frac{1}{3}, \quad \omega_2 = 1, \quad \omega_3 = \frac{5}{3}, \quad \omega_4 = \frac{7}{3}, \quad \omega_5 = 3, \quad \omega_6 = \frac{11}{3}, \dots, \quad (4.1)$$

in this paper. In theory, there are an infinite number of ways to choose the values of λ and δ . In this section, we just consider the following two cases:

- (a). $\lambda = \sqrt{2}$ and $\delta = 0$;
- (b). $\lambda = \omega_1$ and $\delta = \left| \omega_1^2 - \omega_\kappa^2 \right|$ with $\kappa \geq 1$.

All of them can completely avoid the so-called “small denominator problem”.

4.1 In case of $\lambda = \sqrt{2}$ and $\delta = 0$

In case of $\lambda = \sqrt{2}$, according to the definitions (3.1) and (1.3), we have the denominator

$$\lambda^2 - \omega_n^2 = 2 - \frac{(2n-1)^2}{9} = \frac{17+4n-4n^2}{9}. \quad (4.2)$$

Setting $|\lambda^2 - \omega_n^2| = \delta = 0$, we have the two solutions $n \approx 2.62132$ and $n \approx -1.62132$, which however are *not* positive integers. Thus, nothing belongs to the set $W_{\lambda,\delta}$ when we choose

$\lambda = \sqrt{2}$ and $\delta = 0$, say, its kernel of the corresponding auxiliary linear operator \mathcal{L} has nothing, say, $\ker[\mathcal{L}] = \emptyset$, corresponding to $W_{\lambda, \delta} = \emptyset$. Besides, the corresponding denominators $(\lambda^2 - \omega_n^2)$ read

$$\frac{17}{9}, 1, -\frac{7}{9}, -\frac{31}{9}, -7, -\frac{103}{9}, \dots,$$

which are far away from zero, and therefore *all* denominators are **not** small so that the “small denominator problem” does **not** appear at all for *arbitrary* values of α , β and ξ .

In fact, in the case of $\lambda = \sqrt{2}$ and $\delta = 0$, the definitions (3.1) and (3.2) are equivalent to such an auxiliary linear operator

$$\mathcal{L}[u] = u'' + 2u, \quad (4.3)$$

with the property

$$\mathcal{L}\left[A \cos(\sqrt{2}t) + B \sin(\sqrt{2}t)\right] = 0, \quad (4.4)$$

whose inverse operator reads

$$\mathcal{L}^{-1}\left[A' \cos(\omega_n t) + B' \sin(\omega_n t)\right] = \frac{A' \cos(\omega_n t) + B' \sin(\omega_n t)}{2 - \omega_n^2}, \quad n \geq 1, \quad (4.5)$$

where A , B , A' , B' are arbitrary constants. Note that (4.3) has no relationship with the original linear operator \mathcal{L}_0 defined by (1.6).

In general, letting $n \geq 1$ be an integer, one can choose $\lambda = 2n\Omega$, which is far away from *all* $\omega_n = (2n-1)\Omega$, since $\lambda - \omega_n = \omega_{n+1} - \lambda = \Omega$. So, in theory there are an infinite number of ways to choose a proper λ (and $\delta = 0$) so that the so-called “small denominator problem” *never* appears for the forced Duffing equation (1.1)!

Note that the solution of the m th-order deformation equation (2.7) does not contain the terms $\cos(\sqrt{2}t)$ and $\sin(\sqrt{2}t)$, since they do not agree with the solution expression (1.2) and thus must be disappeared. This agrees with the conclusion that the kernel of the corresponding auxiliary linear operator \mathcal{L} is an empty set, say, $\ker[\mathcal{L}] = \emptyset$. Thus, the solution of the m th-order deformation equation (2.7) reads

$$u_m(t) = \chi_m u_{m-1}(t) + c_0 \mathcal{L}^{-1}[R_{m-1}(t)], \quad m \geq 1. \quad (4.6)$$

According to (3.11), since $\ker[\mathcal{L}] = \emptyset$, we choose the initial guess in the form

$$u_0(t) = \sum_{n=1}^2 [a_{0,n} \cos(\omega_n t) + b_{0,n} \sin(\omega_n t)], \quad (4.7)$$

where the four unknown constants $a_{0,n}$, $b_{0,n}$, ($n = 1, 2$) are determined by enforcing the disappearance of the terms $\cos(\omega_1 t)$, $\sin(\omega_1 t)$, $\cos(\omega_2 t)$ and $\sin(\omega_2 t)$ in $R_0(t)$. For details, please refer to Subsection 3.2.

Without loss of generality, let us first consider here the case of $\alpha = 1$, $\beta = 1$, $\Omega = 1/3$ and $\xi = 10^{-4}$, corresponding to the “small denominator problem” from the viewpoint

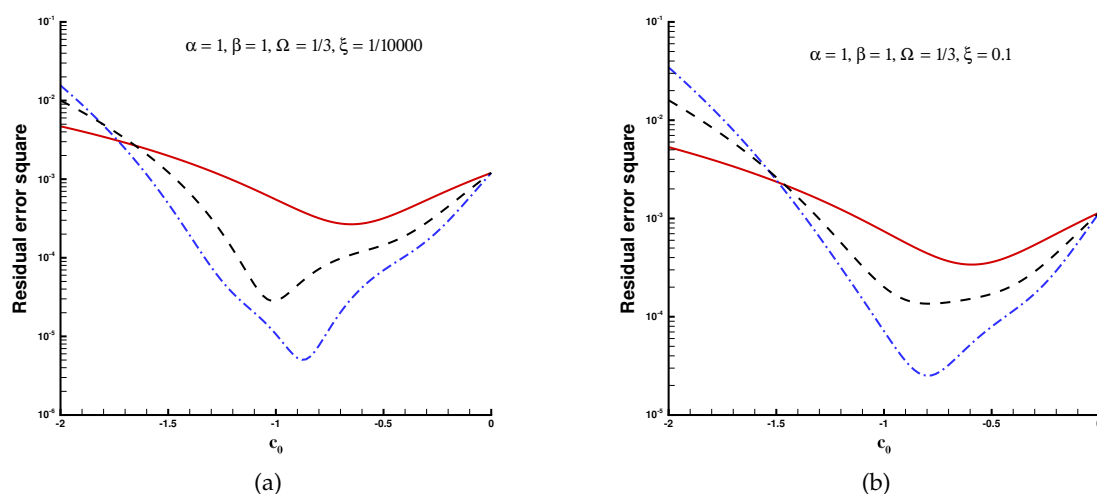


Figure 1: Residual error square of the forced Duffing equation (1.1) for $u(t)$ at different orders of approximation versus the convergence-control parameter c_0 in case of $\alpha=1$, $\beta=1$, $\Omega=1/3$ with different values of ζ , given by the HAM approach in the case of $\lambda=\sqrt{2}$ and $\delta=0$ described in Subsection 4.1. Solid line: 1st-order approximation; Dashed-line: 2nd-order approximation; Dash-dotted line: 3rd-order approximation. (a) $\zeta=10^{-4}$ using the initial guess (4.8); (b) $\zeta=0.1$.

of perturbation method. Following the method described in Subsection 3.2, we have its corresponding initial guess

$$u_0(t) = 0.775251 \cos(\omega_1 t) - 0.127485 \cos(\omega_2 t) + 4.98191 \times 10^{-5} \sin(\omega_1 t) - 5.24821 \times 10^{-5} \sin(\omega_2 t), \quad (4.8)$$

which is the *unique* real solution of the related set of nonlinear algebraic equations.

Although all physical parameters are given, we always have one unknown parameter, i.e., the convergence-control parameter c_0 , which has no physical means but can guarantee the convergence of solution series given by the HAM. To choose an optimal value of c_0 , we check the residual error squares of the first several orders of approximation, defined by (2.15), as shown in Fig. 1, which give us the optimal value $c_0 \approx -0.9$ for the considered case. It is found that, using $c_0 = -0.9$, the corresponding solution series indeed converge very quickly: the residual error square decreases about 20 orders of magnitude at the 30th-order of approximation, say, from 1.2×10^{-3} at the very beginning to 4.1×10^{-23} , as shown in Table 1. In fact, the 10th-order approximation already agrees quite well with the 30th-order approximation, as shown in Fig. 2. Similarly, we also check the validity of our HAM approach in the case of $\zeta = 0, 0.01$ and 0.1 , respectively, with $\alpha = 1$, $\Omega = 1/3$ and $\beta = 1$. In all of these cases, the solution series converge rather quickly, as shown in Table 1. It is found that, given fixed values of α , β and Ω , the series solutions for small values of $0 \leq \zeta \leq 0.01$ are almost the same, and the solution se-

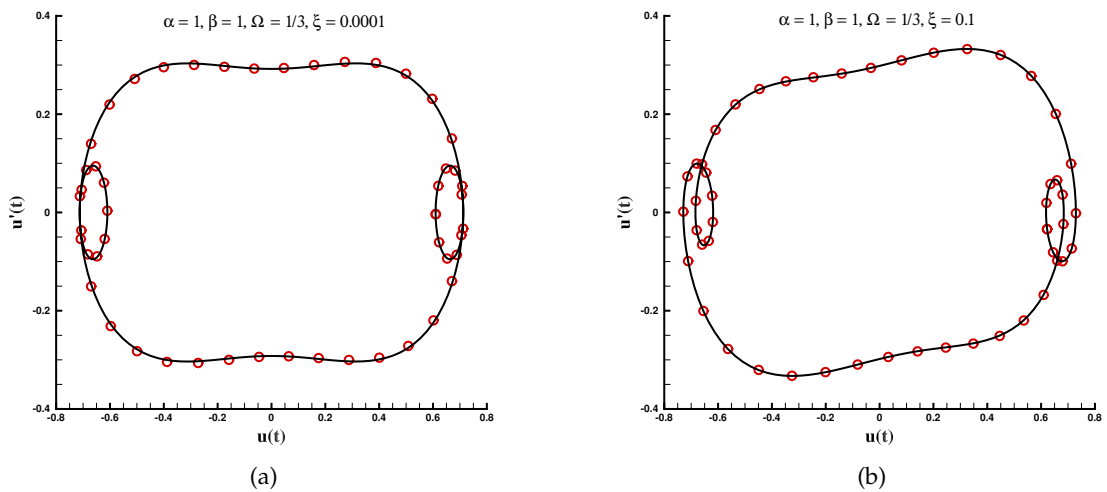


Figure 2: Convergent series solutions of limit-cycle of the forced Duffing equation (1.1) in cases of $\alpha = 1, \beta = 1, \Omega = 1/3$ and different values of ξ . Symbols: 10th-order approximation; Solid line: 30th-order approximation. (a) $\xi = 10^{-4}$ using $c_0 = -0.9$ and the initial guess (4.8); (b) $\xi = 1/10$ using $c_0 = -0.8$.

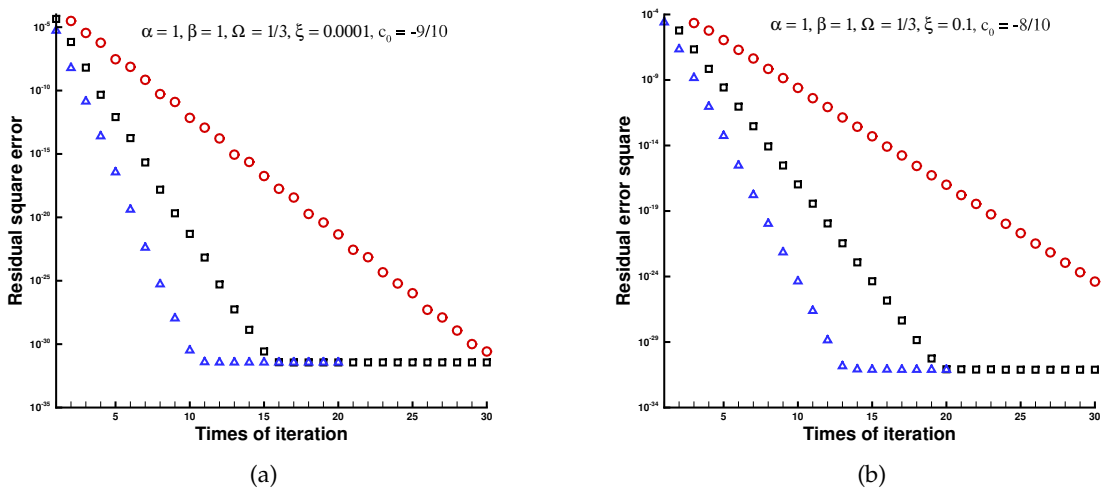


Figure 3: Residual error square of the forced Duffing equation (1.1) versus the times of iteration in case of $\alpha = 1, \beta = 1$ and $\Omega = 1/3$, given by the HAM iteration approach in the case of $\lambda = \sqrt{2}$ and $\delta = 0$ described in Subsection 4.1. Cycle: the 1st-order HAM iteration; Square: the 2nd-order HAM iteration; Delta: the 3rd-order HAM iteration. (a) $\xi = 0.0001$ using $c_0 = -0.9$ and the initial guess (4.8); (b) $\xi = 0.1$ using $c_0 = -0.8$.

ries converges almost at the same rate, as shown in Table 1. This is reasonable in physics, because the small resistance coefficient ξ has a very small influence on the limit-cycle. All

Table 1: Residual error square of the forced Duffing equation (1.1) for $u(t)$ at different order of approximations in case of $\alpha=1$, $\beta=1$, $\Omega=1/3$ and different values of ζ , given by the HAM approach in the case of $\lambda=\sqrt{2}$ and $\delta=0$ described in Subsection 4.1.

Order of approximation	$\zeta=0$ $c_0=-9/10$	$\zeta=10^{-4}$ $c_0=-9/10$	$\zeta=0.01$ $c_0=-9/10$	$\zeta=0.1$ $c_0=-8/10$
0	1.2E-3	1.2E-3	1.2E-3	4.9E-4
1	4.1E-4	4.1E-4	4.1E-4	4.4E-4
3	5.3E-6	5.3E-6	5.2E-6	2.5E-5
5	2.4E-7	2.4E-7	2.5E-7	2.1E-6
10	9.5E-11	9.5E-11	1.0E-10	4.6E-9
15	4.0E-14	4.0E-14	4.7E-14	1.4E-11
20	4.2E-17	4.2E-17	4.8E-17	5.1E-14
25	5.7E-20	5.7E-20	6.7E-20	2.1E-16
30	4.1E-23	4.1E-23	5.3E-23	9.1E-19

of these illustrate the validity of the HAM approach mentioned above.

As mentioned in Subsection 3.2, one M th-order HAM approximation can be used as a new initial guess to gain a better approximation, and so on. As shown in Fig. 3, in case of $\alpha=1$, $\beta=1$, $\Omega=1/3$ and $\zeta=0.0001$, using the initial guess (4.8), we gain the convergent series solution by means of the first, second and third-order HAM iteration approach, and the corresponding residual error square decreases rather quickly: from 1.2×10^{-3} at the beginning to 10^{-23} at the 16 iterations for the 2nd-order formula, or at the 11 iterations for the 3rd-order formula, respectively. It is found that the higher the order of the HAM iteration approach, the faster the solution series converges. Similarly, in case of $\zeta=0.1$, the iteration also converges rather quickly, as shown in Fig. 3. So, choosing an optimal convergence-control parameter c_0 and using HAM iteration approach, we can quickly gain the convergent series solution of the limiting cycle of the forced Duffing equation (1.1) by means of the HAM approach described in Sections 2 and 3. This illustrates the validity of our HAM iteration approach.

As shown in Fig. 3, the residual error square stops decreasing at the order of magnitude 10^{-23} . This is mainly because, in the solution expression (1.2) of the limiting cycle $u(t)$, we delete all terms $a_n \cos(\omega_n t)$ and $b_n \sin(\omega_n t)$, when $|a_n| < \epsilon$ and $|b_n| < \epsilon$, where we choose $\epsilon = 10^{-20}$ in this paper. This is specially necessary for the HAM iteration approach, otherwise the number of the base-functions increases exponentially so that the HAM iteration approach can not work. It is found that, when a smaller value of ϵ such as $\epsilon = 10^{-30}$ is used, the residual error square stops decreasing at a much smaller level. However, for the problem considered in this paper, $\epsilon = 10^{-20}$ is small enough for the cases under consideration.

Note that, from the viewpoint of perturbation method, the “small denominator problem” occurs when $\alpha=1$, $\beta=1$, $\Omega=1/3$ and $0.0001 \leq \zeta \leq 0.01$. However, as shown in Table 1, the series solutions given by our HAM approach converge almost in the same rate in case of $0 \leq \zeta \leq 0.01$. Therefore, our HAM approach indeed can **avoid** the small denominators.

In other words, from the viewpoint of the HAM approach (in the case of $\lambda=2$ and $\delta=0$), the so-called “small denominator problem” does **not** really exist at all: they are just the artifacts of perturbation methods.

4.2 In case of $\lambda = \omega_1$ and $\delta = |\omega_1^2 - \omega_\kappa^2|$ with $\kappa \geq 1$

Let us further consider the case $\lambda = \omega_1$ and $\delta = |\omega_1^2 - \omega_\kappa^2| = \omega_\kappa^2 - \omega_1^2$, where $\kappa \geq 1$ is an integer. According to (3.2), $|\lambda^2 - \omega_n^2| \leq \delta$ leads to the following equation

$$\omega_n^2 - \omega_1^2 \leq \omega_\kappa^2 - \omega_1^2, \quad n \geq 1, \quad (4.9)$$

which holds for $1 \leq n \leq \kappa$. Thus, the corresponding set $W_{\lambda,\delta}$ has κ members, say,

$$W_{\lambda,\delta} = \{\omega_1, \omega_2, \dots, \omega_\kappa\}. \quad (4.10)$$

In this case, (3.1) and (3.2) are equivalent to the following definitions:

$$\mathcal{L}^{-1} [A \cos(\omega_n t) + B \sin(\omega_n t)] = \frac{A \cos(\omega_n t) + B \sin(\omega_n t)}{\omega_1^2 - \omega_n^2}, \quad n > \kappa, \quad (4.11)$$

and

$$\mathcal{L} \left\{ \sum_{n=1}^{\kappa} [A'_{m,n} \cos(\omega_n t) + B'_{m,n} \sin(\omega_n t)] \right\} = 0, \quad (4.12)$$

for arbitrary constants $A, B, A'_{m,n}, B'_{m,n}$, where we have great freedom to choose the value of κ . According to (4.12), the kernel of the corresponding linear operator is a vector space with $\mu = 2\kappa$ dimension, say,

$$\ker[\mathcal{L}] = \sum_{n=1}^{\kappa} [A'_{m,n} \cos(\omega_n t) + B'_{m,n} \sin(\omega_n t)]. \quad (4.13)$$

Thus, in this case, the solution of the m th-order deformation equation (2.7) reads

$$u_m(t) = \chi_m u_{m-1}(t) + c_0 \mathcal{L}^{-1} [R_{m-1}(t)] + \sum_{n=1}^{\kappa} [A'_{m,n} \cos(\omega_n t) + B'_{m,n} \sin(\omega_n t)], \quad (4.14)$$

where $A'_{m,n}$ and $B'_{m,n}$ are 2κ unknown constants, which are determined by enforcing the coefficients of $\cos(\omega_n t)$ and $\sin(\omega_n t)$ in $R_m(t)$ being zero, where $1 \leq n \leq \kappa$.

In this case, we choose the initial guess in the form

$$u_0(t) = \sum_{n=1}^{\max\{2,\kappa\}} [a_{0,n} \cos(\omega_n t) + b_{0,n} \sin(\omega_n t)], \quad (4.15)$$

where the unknown coefficients $a_{0,n}$ and $b_{0,n}$ are determined by enforcing the coefficients of $\cos(\omega_n t)$ and $\sin(\omega_n t)$ in $R_0(t)$ being zero, where $1 \leq n \leq \max\{2,\kappa\}$.

It should be emphasized here that κ can be greater than 1, since we have great *freedom* to choose its value! For example, when $\kappa=2$, $\ker[\mathcal{L}]$, i.e., the kernel of the corresponding auxiliary linear operator \mathcal{L} , is a vector space of 4 dimension. When $\kappa=3$, $\ker[\mathcal{L}]$ is a vector space of 6 dimension! Note that the forced Duffing equation (1.1) is just a 2nd-order nonlinear differential equation. In the frame of the perturbation method, the forced Duffing equation (1.1) is transferred into an infinite number of 2nd-order linear differential equations, as shown in Section 1. Note also that, according to the traditional mathematical theory, the kernel of a second-order linear differential operator is a vector space of 2 dimension only. Thus, in case of $\kappa > 1$, our HAM approach is beyond the traditional mathematical theory about differential equations. This also indicates the novelty of our HAM approach in mathematics.

4.2.1 Results when $\kappa=2$

In this case, we have $\delta=\omega_2^2-\omega_1^2$ and that the kernel of the corresponding auxiliary linear operator \mathcal{L} is a vector space of 4 dimension, say,

$$\ker[\mathcal{L}] = \sum_{n=1}^2 [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)], \quad (4.16)$$

where A_n and B_n are arbitrary constants.

Without loss of generality, let us consider the case of $\alpha=1$, $\beta=1$, $\Omega=1/3$ and $\xi=10^{-4}$. According to (4.15), we can choose the same initial guess as (4.8), since they have the same physical parameters. Note that, unlike all other approximation methods, the HAM contains the so-called “convergence-control parameter” c_0 , which has no physical meanings but can guarantee the convergence of the solution series. As shown in Fig. 4, the optimal convergence-control parameter is about $c_0 \approx -1.5$. Using $c_0 = -3/2$ and the initial guess (4.8), the corresponding series solution converges very quickly, from $1.2 \times$

Table 2: Residual error square of $u(t)$ at different order of approximations of the forced Duffing equation (1.1) in case of $\alpha=1$, $\beta=1$, $\Omega=1/3$ and different values of ξ , given by the HAM approach described in Subsection 4.2 when $\lambda=\omega_1^2$ and $\delta=\omega_2^2-\omega_1^2$.

Order of approximation	$\xi=0$ $c_0=-3/2$	$\xi=10^{-4}$ $c_0=-3/2$	$\xi=0.01$ $c_0=-3/2$	$\xi=0.1$ $c_0=-8/5$
0	1.2E-3	1.2E-3	1.2E-3	1.1E-3
1	4.0E-4	4.0E-4	4.0E-4	4.1E-4
3	1.2E-5	1.2E-5	1.2E-5	1.7E-5
5	5.2E-7	5.2E-7	5.2E-7	1.3E-6
10	1.7E-10	1.7E-10	1.7E-10	1.4E-9
15	1.3E-13	1.3E-13	1.3E-13	2.0E-12
20	6.4E-17	6.4E-17	6.4E-17	3.6E-15
25	1.2E-19	1.2E-19	1.2E-19	7.6E-18
30	7.9E-23	7.9E-23	7.9E-23	1.7E-20

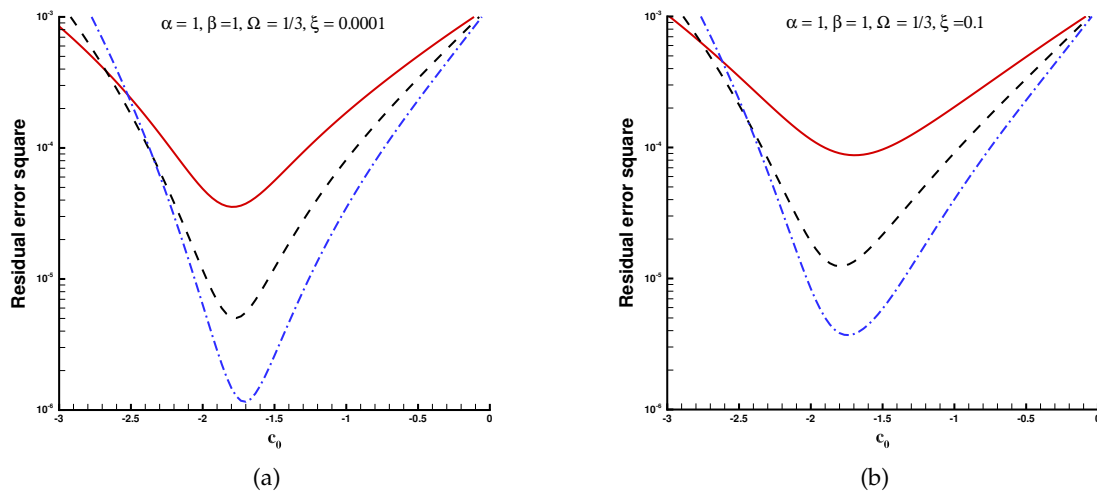


Figure 4: Residual error squares of approximations of $u(t)$ at different-orders versus the convergence-control parameter c_0 in case of $\alpha = 1, \beta = 1, \Omega = 1/3$ with different values of ζ , given by the HAM approach described in Subsection 4.2 when $\lambda = \omega_1$ and $\delta = \omega_2^2 - \omega_1^2$. Solid line: 2nd-order approximation; Dashed-line: 3rd-order approximation; Dash-dotted line: 4th-order approximation. (a) $\zeta = 10^{-4}$ using the initial guess (4.8); (b) $\zeta = 0.1$.

10^{-3} at the beginning to 7.9×10^{-23} at the 30th-order of approximation, about 20 orders of magnitude less, as shown in Table 2. Similarly, the solution series converge very quickly for $\zeta = 0, 0.01$ and 0.1 , as shown in Table 2. It is found that the corresponding limit-cycles given by $\lambda = \omega_1$ and $\delta = \omega_2^2 - \omega_1^2$ in case of $\zeta = 0.0001$ and $\zeta = 0.1$ are exactly the same as (a) and (b) in Fig. 2 given by $\lambda = 2$ and $\delta = 0$, respectively. In addition, the 2nd, 3rd and 4th-order HAM iteration formulas also give convergent series solutions rather quickly, as shown in Fig. 5. It is found again that, the higher the order of iteration formula, the faster the solution series converges.

All of these confirm the validity of the HAM approach described in Subsection 4.2. It is important that, in the case of $\lambda = \omega_1$ and $\delta = \omega_2^2 - \omega_1^2$, our HAM approach has **nothing** to do with the so-called “small denominators”! In other words, the “small denominator problem” never appears from the viewpoint of the HAM.

Note that, according to traditional mathematical theories, a linear differential operator \mathcal{L} , whose kernel is the same as the vector space of 4 dimension defined by (4.16), should correspond to the 4th-order differential equation

$$\mathcal{L}[u] = u^{(4)} + (\omega_1^2 + \omega_2^2)u'' + \omega_1^2\omega_2^2u = 0, \tag{4.17}$$

whose inverse operator reads

$$\mathcal{L}^{-1} \left[A\cos(\omega t) + B\sin(\omega t) \right] = \frac{A\cos(\omega t) + B\sin(\omega t)}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}, \quad \omega \notin \{\omega_1, \omega_2\}, \tag{4.18}$$

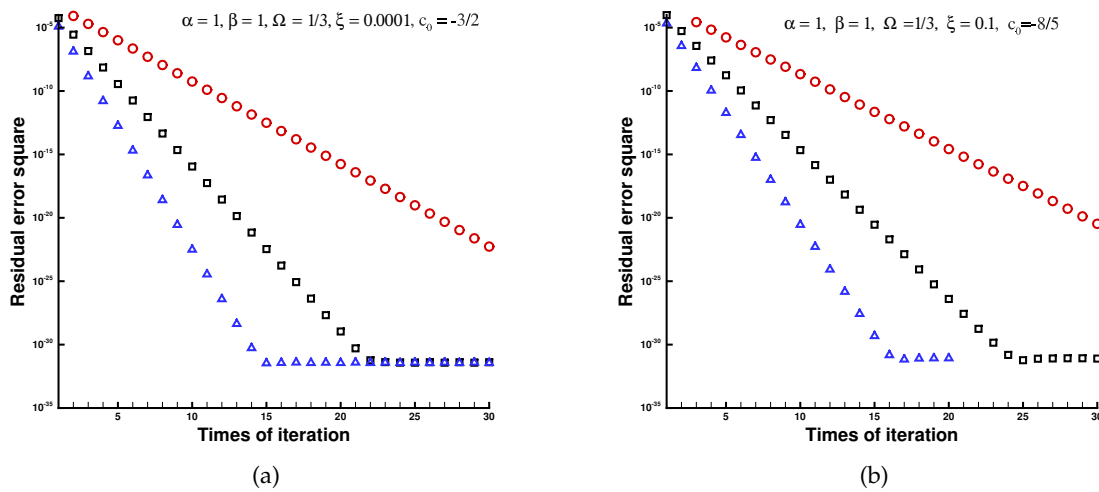


Figure 5: Residual error square of the forced Duffing equation (1.1) versus the times of iteration in case of $\alpha=1, \beta=1, \Omega=1/3$, given by the HAM approach described in Subsection 4.2 when $\lambda=\omega_1$ and $\delta=\omega_2^2-\omega_1^2$. Cycle: the 2nd-order HAM iteration; Square: the 3rd-order HAM iteration; Delta: the 4th-order HAM iteration. (a) $\xi=0.0001$ using the initial guess (4.8) and $c_0=-3/2$; (b) $\xi=0.1$ using $c_0=-8/5$.

for arbitrary constants A and B . However, the above expression is obviously *different* from our inverse operator (4.11) that looks like one for a 2nd-order linear differential equation! In fact, we even do *not* know how to explicitly express the corresponding auxiliary linear operator \mathcal{L} when $\kappa=2$, say, $\lambda=\omega_1$ and $\delta=\omega_2^2-\omega_1^2$ in the HAM approach described in Subsection 4.2, but fortunately it is *unnecessary* to know it in the frame of the HAM! The most important fact is that our HAM-based approach is valid and the corresponding solution series of the limiting cycle converge quickly, as shown in Table 2 and Figs. 4 and 5. This verifies the validity and novelty of our HAM approach mentioned above.

4.2.2 Results when $\kappa=3$

In this case we have $\delta=\omega_3^2-\omega_1^2$ and the kernel of the auxiliary linear operator \mathcal{L} is a vector space of 6 dimensions, say,

$$\ker[\mathcal{L}] = \sum_{n=1}^3 [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)], \tag{4.19}$$

where A_n and B_n are arbitrary constants.

Besides, according to (4.15), the initial guess should be in the form

$$u_0(t) = \sum_{n=1}^3 [a_{0,n} \cos(\omega_n t) + b_{0,n} \sin(\omega_n t)]. \tag{4.20}$$

Substituting it into the forced Duffing equation (1.1) and enforcing the coefficients of the terms $\cos(\omega_n t)$ and $\sin(\omega_n t)$, ($n = 1, 2, 3$) being zero, we have a set of six nonlinear algebraic equations, whose real solutions determine the six unknown constants in (4.20). It is interesting that the set of these six nonlinear algebraic equations has multiple real solutions (complex solutions have no physical meanings here) in many cases, for example, such as $\alpha = 1$, $\zeta = 0$, $\Omega = 1/3$ but a large value of β , i.e., $\beta = 5$:

$$a_{0,1} = 0.333781, \quad a_{0,2} = 0.107352, \quad a_{0,3} = -0.509166, \quad (4.21a)$$

$$a_{0,1} = 0.526136, \quad a_{0,2} = -0.11976, \quad a_{0,3} = 0.181401, \quad (4.21b)$$

$$a_{0,1} = 0.482000, \quad a_{0,2} = 0.0264671, \quad a_{0,3} = -0.200815, \quad (4.21c)$$

with $b_{0,1} = b_{0,2} = b_{0,3} = 0$, respectively, corresponding to the three initial guesses in the form (4.20).

Using the initial guess (4.21a) and the corresponding optimal convergence-control parameter $c_0 = -1$, we gain a convergent series solution, shown as (a) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.11 at the beginning to 4.9×10^{-22} at the 30th iteration.

Using the initial guess (4.21b) and the corresponding optimal convergence-control parameter $c_0 = -3/2$, we gain the convergent series solution, shown as (b) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.11 at the beginning to 2.1×10^{-30} at the 20th iteration.

Using the initial guess (4.21c) and the corresponding optimal convergence-control parameter $c_0 = -1$, we gain the convergent series solution, shown as (c) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.11 at the beginning to 7.7×10^{-30} at the 30th iteration.

It is interesting that we have the three initial guesses (4.21a)-(4.21c) in the case of $\kappa = 3$, which give us three different limit-cycles, as shown in (a), (b) and (c) of Fig. 6, respectively. It should be emphasized that, it is the HAM that provides us such kind of great freedom to choose the initial guess. Note also that, in the frame of the perturbation method, there exists the *unique* initial guess (1.8) only, and thus in theory it is impossible to find these multiple limit-cycles by the perturbation method[†]. This illustrates the advantages and novelty of the HAM beyond perturbation.

Note that the forced Duffing equation (1.1) contains the nonlinear term βu^3 . So, the larger the value of β , the higher the nonlinearity of the Duffing equation. As mentioned in Section 1, the perturbation approach is invalid even for $\beta \geq 0.012$ and small ζ . However, using our HAM approach in a similar way, we can gain convergent series solution even in the cases with rather high nonlinearity, such as $\alpha = 1$, $\Omega = 1/3$, $\zeta = 0$ and $10 \leq \beta \leq 40$, as shown in Fig. 7. It is found that, when $\kappa = 3$, using the approach mentioned in

[†]In fact, as mentioned in Section 1, the unique perturbation series diverges even when $\beta = 0.012$ in case of $\Omega = 1/3$ and $\alpha = 1$, corresponding to a very weak nonlinearity.

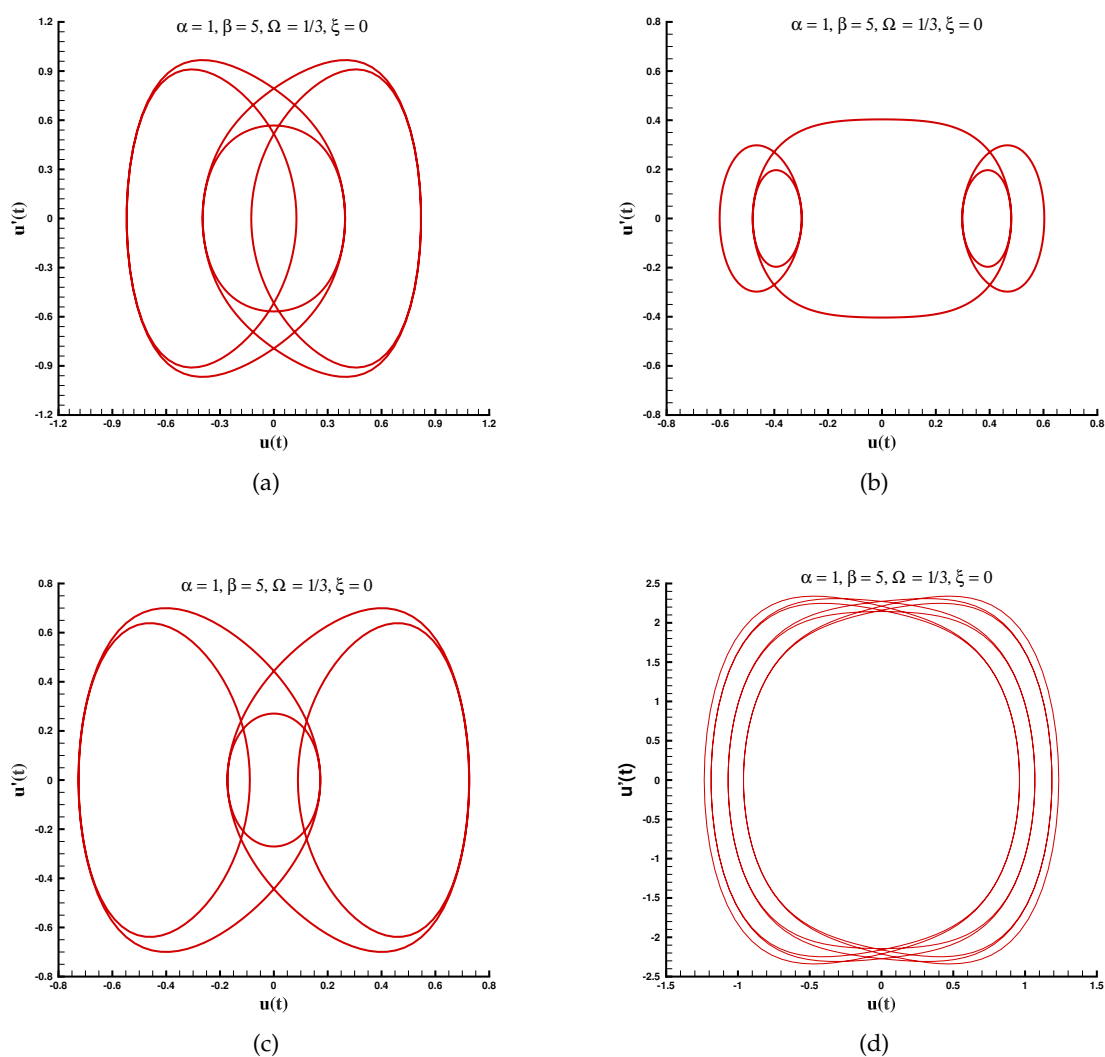


Figure 6: Multiple limit-cycles of the forced Duffing equation (1.1) in cases of $\alpha=1$, $\Omega=1/3$, $\xi=0$ and $\beta=5$, given by the 2nd-order HAM iteration described in Subsection 4.2 when $\lambda=\omega_1$ and $\delta=\omega_\kappa^2-\omega_1^2$. (a) using the initial guess (4.21a), $c_0=-1$ and $\kappa=3$; (b) using the initial guess (4.21b), $c_0=-3/2$ and $\kappa=3$; (c) using the initial guess (4.21c), $c_0=-1$ and $\kappa=3$; (d) using the initial guess (4.27d), $c_0=-1$ and $\kappa=4$.

Subsection 3.2, there exist only one initial guess in the form (4.20) for each $\beta \in [10, 40]$, say,

$$\beta = 10: \quad a_{0,1} = 0.450482, \quad a_{0,2} = -0.0931627, \quad a_{0,3} = 0.0779936, \quad (4.22a)$$

$$\beta = 20: \quad a_{0,1} = 0.371919, \quad a_{0,2} = -0.0737289, \quad a_{0,3} = 0.0477389, \quad (4.22b)$$

$$\beta = 30: \quad a_{0,1} = 0.330788, \quad a_{0,2} = -0.0646278, \quad a_{0,3} = 0.0380275, \quad (4.22c)$$

$$\beta = 40: \quad a_{0,1} = 0.303900, \quad a_{0,2} = -0.0589144, \quad a_{0,3} = 0.0328625, \quad (4.22d)$$

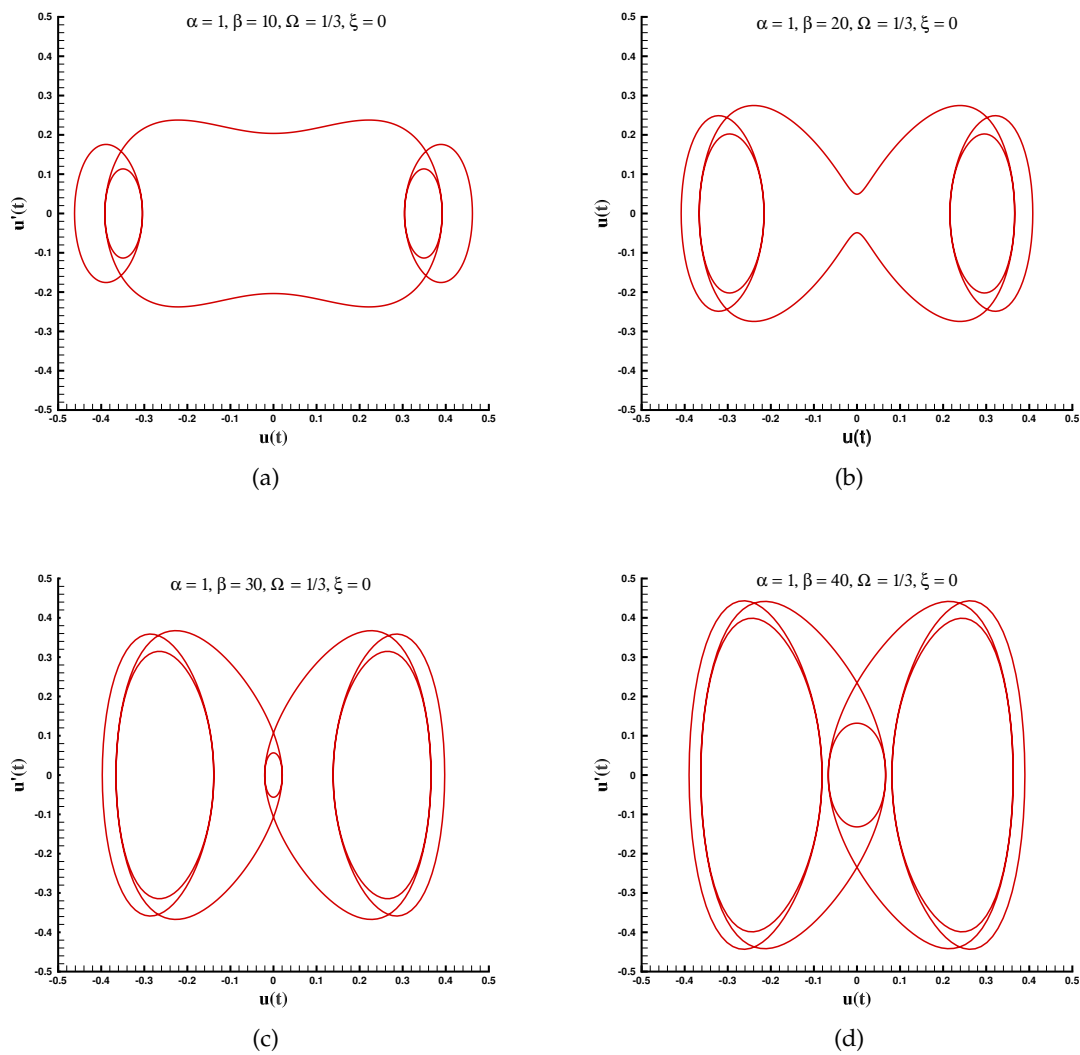


Figure 7: Multiple limit-cycles of the forced Duffing equation (1.1) in cases of $\alpha=1$, $\Omega=1/3$, $\xi=0$ and some large values of β , given by the 2nd-order HAM iteration described in Subsection 4.2 when $\lambda=\omega_1$ and $\delta=\omega_3^2-\omega_1^2$, corresponding to $\kappa=3$. (a) $\beta=10$ using the unique initial guess (4.22a) and $c_0=-3/2$; (b) $\beta=20$ using the unique initial guess (4.22b) and $c_0=-1/2$; (c) $\beta=30$ using the unique initial guess (4.22c) and $c_0=-1/5$; (d) $\beta=40$ using the unique initial guess (4.22d) and $c_0=-1/25$.

where $b_{0,n}=0$ for $n=1,2,3$, corresponding to the four initial guesses in the form (4.20). In all of these cases, the convergent series solutions are obtained by means of the 2nd-order HAM iteration using a proper convergence-control parameter c_0 , as shown in Fig. 7. Thus, our HAM approach is indeed valid for high nonlinearity. Besides, it should be emphasized that the “small denominator problem” never appears in all cases. This illus-

trates the validity of the HAM approach for high nonlinearity in the case of $\lambda = \omega_1$ and $\delta = \omega_3^2 - \omega_1^2$ and its advantages beyond perturbation.

Note that, according to traditional mathematical theories, a linear differential operator \mathcal{L} , whose kernel is the same as the vector space of 6 dimension defined by (4.19), should correspond to the 6th-order differential equation

$$\begin{aligned} \mathcal{L}[u] = & u^{(6)} + (\omega_1^2 + \omega_2^2 + \omega_3^2) u^{(4)} \\ & + (\omega_1^2 \omega_2^2 + \omega_1^2 \omega_3^2 + \omega_2^2 \omega_3^2) u'' + \omega_1^2 \omega_2^2 \omega_3^2 u = 0, \end{aligned} \quad (4.23)$$

whose inverse operator reads

$$\begin{aligned} \mathcal{L}^{-1} [A \cos(\omega t) + B \sin(\omega t)] \\ = \frac{A \cos(\omega t) + B \sin(\omega t)}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)}, \quad \omega \notin \{\omega_1, \omega_2, \omega_3\}, \end{aligned} \quad (4.24)$$

for arbitrary constants A and B . However, the above expression is obviously *different* from our inverse operator (4.11) that looks like one for a 2nd-order linear differential equation! In fact, we even do *not* know how to explicitly express the corresponding auxiliary linear operator \mathcal{L} when $\kappa=3$, say, $\lambda = \omega_1$ and $\delta = \omega_3^2 - \omega_1^2$, but fortunately it is *unnecessary* to know it in the frame of the HAM. The most important fact is that our HAM-based approach is valid and the corresponding solution series converge quickly, as mentioned above, which verifies the validity and novelty of our HAM approach.

4.2.3 Results given by $\kappa=4$

In this case, we have $\lambda = \omega_1$ and $\delta = \omega_4^2 - \omega_1^2$ so that the kernel of the corresponding auxiliary linear operator \mathcal{L} is a vector space of 8 dimensions, say,

$$\ker[\mathcal{L}] = \sum_{n=1}^4 [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)], \quad (4.25)$$

where A_n and B_n are arbitrary constants.

According to (4.15), the initial guess should be in the form

$$u_0(t) = \sum_{n=1}^4 [a_{0,n} \cos(\omega_n t) + b_{0,n} \sin(\omega_n t)]. \quad (4.26)$$

Substituting it into the forced Duffing equation (1.1) and enforcing the coefficients of the terms $\cos(\omega_n t)$ and $\sin(\omega_n t)$, ($n=1,2,3,4$) being zero, we have a set of eight nonlinear algebraic equations, whose real solutions determine the eight unknown constants in (4.26). It is found that the set of these eight nonlinear algebraic equations has **four** real solutions

in the case of $\alpha = 1, \Omega = 1/3, \xi = 0$ and $\beta = 5$:

$$a_{0,1} = 0.261766, \quad a_{0,2} = 0.0766644, \quad a_{0,3} = -0.563565, \quad a_{0,4} = -0.0899237, \quad (4.27a)$$

$$a_{0,1} = 0.524251, \quad a_{0,2} = -0.119311, \quad a_{0,3} = 0.150425, \quad a_{0,4} = 0.0487805, \quad (4.27b)$$

$$a_{0,1} = 0.420975, \quad a_{0,2} = 0.0508348, \quad a_{0,3} = -0.278947, \quad a_{0,4} = -0.0849981, \quad (4.27c)$$

$$a_{0,1} = 0.104111, \quad a_{0,2} = 0.000412556, \quad a_{0,3} = -0.00628524, \quad a_{0,4} = 1.07865, \quad (4.27d)$$

where $b_{0,n} = 0$ for $n = 1, 2, 3, 4$, corresponding to the four initial guesses in the form (4.26).

Using the initial guess (4.27a) and the convergence-control parameter $c_0 = -2/3$, we gain a convergent series solution of the limiting cycle, which is exactly the same as (a) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.14 at the beginning to 3.6×10^{-30} at the 30th iteration.

Using the initial guess (4.27b) and the convergence-control parameter $c_0 = -1$, we gain a convergent series solution of the limiting cycle, which is exactly the same as (b) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.021 at the beginning to 8.4×10^{-30} at the 15th iteration.

Using the initial guess (4.27c) and $c_0 = -3/2$, we gain a convergent series solution of the limiting cycle, which is exactly the same as (c) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.022 at the beginning to 6.6×10^{-30} at the 20th iteration.

Using the initial guess (4.27d) and $c_0 = -1$, we gain a *new* convergent series solution, shown as (d) in Fig. 6, by means of the 2nd-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 1.48 at the beginning to 5.6×10^{-16} at the 30th iteration. It is interesting that this is a *new* solution, which is however not found by means of $\lambda = \omega_1$ and $\kappa = 3$. This is mainly because, when $\kappa = 4$, we should solve two more nonlinear algebraic equations to gain the initial guess than the case of $\kappa = 3$. This leads to one more initial guess that gives one more limit-cycle by means of the HAM approach described in this paper. It seems that, in the frame of the HAM described in Sections 2 and 3, the larger the value of δ , the greater the possibility to find multiple solutions (if they indeed exist). This further shows the validity and novelty of our HAM approach beyond perturbation.

As shown in Fig. 7, for a given $\beta \in [10, 40]$ (corresponding to high nonlinearity), only one limit-cycle is found by the HAM approach using $\lambda = \omega_1$ and $\delta = \omega_3^2 - \omega_1^2$, corresponding to $\kappa = 3$. Do multiple limit-cycles exist in high nonlinearity, say, for a large β ? Without loss of generality, let us consider the case of $\alpha = 1, \Omega = 1/3, \xi = 0$ and $\beta = 40$. It is found that, when $\kappa = 4$, we have the *four* corresponding initial guesses in the form of (4.26):

$$a_{0,1} = 0.131123, \quad a_{0,2} = -0.00645067, \quad a_{0,3} = -0.0312269, \quad a_{0,4} = 0.336815, \quad (4.28a)$$

$$a_{0,1} = 0.287691, \quad a_{0,2} = -0.0231836, \quad a_{0,3} = -0.0268927, \quad a_{0,4} = 0.0697927, \quad (4.28b)$$

$$a_{0,1} = 0.296537, \quad a_{0,2} = -0.0580039, \quad a_{0,3} = 0.0566195, \quad a_{0,4} = -0.0810271, \quad (4.28c)$$

$$a_{0,1} = 0.261690, \quad a_{0,2} = -0.0276672, \quad a_{0,3} = 0.0714539, \quad a_{0,4} = -0.156464, \quad (4.28d)$$

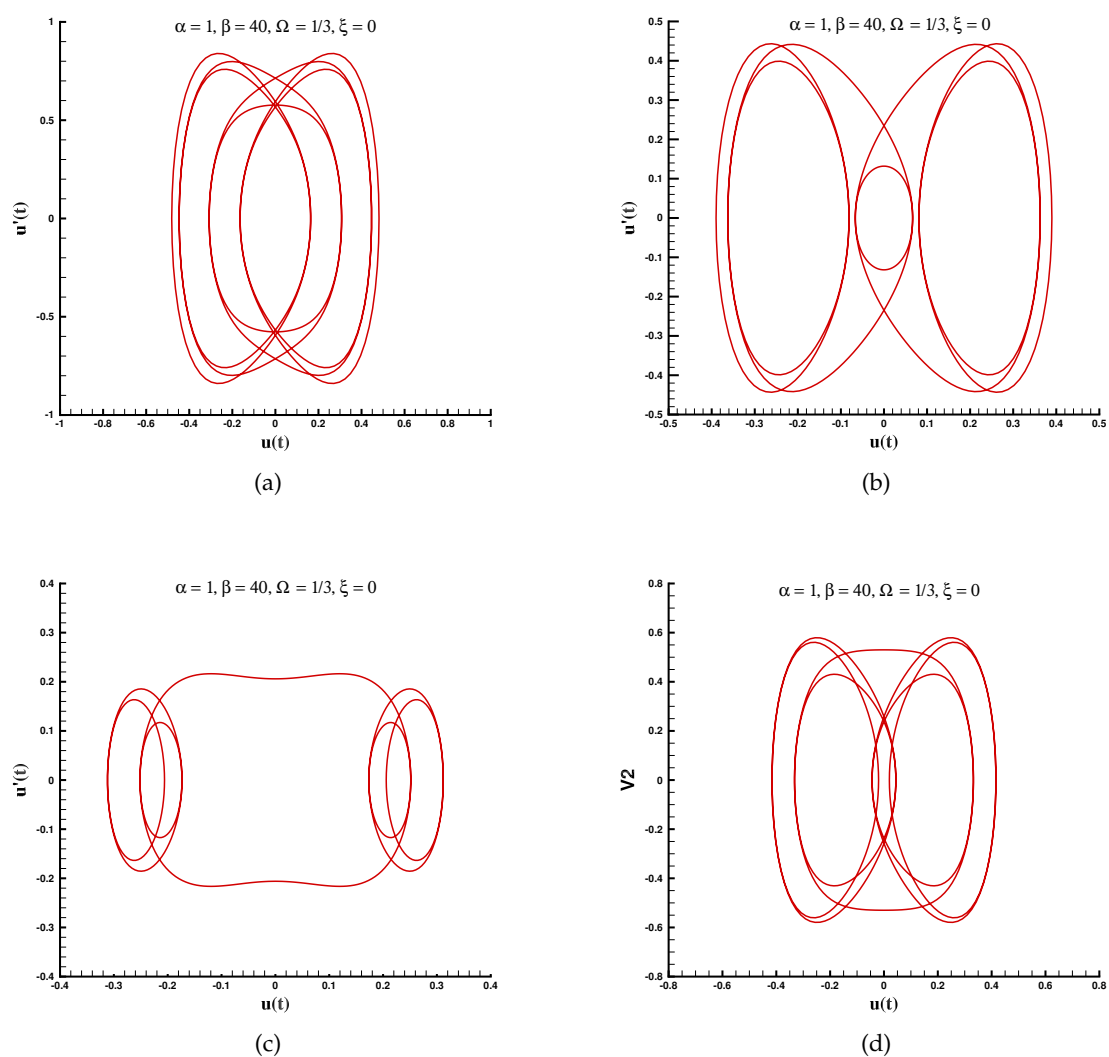


Figure 8: Multiple limit-cycles of the Duffing equation (1.1) in cases of $\alpha=1$, $\Omega=1/3$, $\xi=0$ and $\beta=40$, given by the HAM iteration approach described in Subsection 4.2 when $\lambda=\omega_1$ and $\delta=\omega_4^2-\omega_1^2$. (a) using the initial guess (4.28a) and $c_0=-3/2$; (b) using the initial guess (4.28b) and $c_0=-4/5$; (c) using the initial guess (4.28c) and $c_0=-4/5$; (d) using the initial guess (4.28d) and $c_0=-1$.

where $b_{0,n}=0$, ($n=1,2,3,4$).

Using the initial guess (4.28a) and $c_0=-3/2$, we gain a convergent series solution, shown as (a) in Fig. 8, by means of the 5th-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.26 at the beginning to 10^{-21} at the 10th iterations.

Using the initial guess (4.28b) and $c_0 = -4/5$, we gain a convergent series solution, shown as (b) in Fig. 8, by means of the 5th-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.016 at the beginning to 2.7×10^{-17} at the 10th iterations. This limit-cycle is exactly the same as (d) in Fig. 7, the only one limit-cycle when $\beta = 40$ given by means of $\lambda = \omega_1$ and $\delta = \omega_3^2 - \omega_1^2$, corresponding to $\kappa = 3$.

Using the initial guess (4.28c) and $c_0 = -4/5$, we gain a convergent series solution, shown as (c) in Fig. 8, by means of the 5th-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.013 at the beginning to 8.1×10^{-17} at the 10th iterations.

Using the initial guess (4.28d) and $c_0 = -1$, we gain a convergent series solution, shown as (d) in Fig. 8, by means of the 5th-order HAM iteration: the residual error square of the forced Duffing equation (1.1) decreases from 0.044 at the beginning to 6.4×10^{-21} at the 10th iterations.

Note that when $\kappa = 3$, i.e., $\lambda = \omega_1$ and $\delta = \omega_3^2 - \omega_1^2$, we found only one limit-cycle for $\beta \in [10, 40]$. However, when $\kappa = 4$, i.e., $\lambda = \omega_1$ and $\delta = \omega_4^2 - \omega_1^2$, we successfully gain four limit-cycles in the case of $\beta = 40$, corresponding to a very high nonlinearity. It seems that, the larger the value of κ , say, the larger of δ , more limit-cycles of the forced Duffing equation (1.1) could be found. Note that $\beta = 40$ corresponds to a high nonlinearity: this verifies the validity of our HAM approach for high nonlinearity. This is one of advantages of the HAM, which has been proved in many articles (for example, please refer to Zhong and Liao [26]). In summary, all of these results illustrate the validity and novelty of our HAM approach described in Sections 2 and 3. Note that the so-called "small denominator problem" never appears for the forced Duffing equation (1.1) by means of the HAM approach.

Note that, according to traditional mathematical theories, a linear differential operator \mathcal{L} , whose kernel is the same as the vector space of 8 dimension defined by (4.25), should correspond to the 8th-order differential equation

$$\begin{aligned} \mathcal{L}[u] = & u^{(8)} + (\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2) u^{(6)} \\ & + (\omega_1^2 \omega_2^2 + \omega_1^2 \omega_3^2 + \omega_1^2 \omega_4^2 + \omega_2^2 \omega_3^2 + \omega_2^2 \omega_4^2 + \omega_3^2 \omega_4^2) u^{(4)} \\ & + (\omega_1^2 \omega_2^2 \omega_3^2 + \omega_1^2 \omega_2^2 \omega_4^2 + \omega_1^2 \omega_3^2 \omega_4^2 + \omega_2^2 \omega_3^2 \omega_4^2) u'' \\ & + \omega_1^2 \omega_2^2 \omega_3^2 \omega_4^2 u = 0, \end{aligned} \quad (4.29)$$

whose inverse operator reads

$$\begin{aligned} & \mathcal{L}^{-1} [A \cos(\omega t) + B \sin(\omega t)] \\ & = \frac{A \cos(\omega t) + B \sin(\omega t)}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)}, \quad \omega \notin \{\omega_1, \omega_2, \omega_3, \omega_4\}, \end{aligned} \quad (4.30)$$

for arbitrary constants A and B . However, the above expression is obviously *different* from our inverse operator (4.11) that looks like one for a 2nd-order linear differential

equation! In fact, we even do *not* know how to explicitly express the corresponding auxiliary linear operator \mathcal{L} when $\kappa = 4$, say, $\lambda = \omega_1$ and $\delta = \omega_4^2 - \omega_1^2$, but fortunately it is *unnecessary* to know it in the frame of the HAM. The most important fact is that our HAM-based approach is valid and the corresponding solution series of the limiting cycles converge quickly, as mentioned above, which verifies the validity and novelty of our HAM approach mentioned in Sections 2 and 3.

5 Discussions and concluding remarks

First of all, when perturbation method is used to solve the forced Duffing equation (1.1), the so-called “small denominator problem” is *unavoidable* when $\omega \rightarrow 1$ and $\xi \rightarrow 0$, which leads to the divergence of the perturbative series even for rather small β , corresponding to a very weak nonlinearity. However, for the HAM approach described in Sections 2 and 3, such kind of small denominators **never** appear for arbitrary values of physical parameters α , β , ω and ξ so that the so-called “small denominator problem” **never** occurs! Note that even in the case of large β , corresponding to high nonlinearity, multiple limit-cycles are successfully found by our HAM approach. All of these illustrate the validity and novelty of the HAM approach. Thus, from the viewpoint of the HAM approach described in this paper, the so-called “small denominator problem” does **not** really exist! This suggests that whether or not the so-called “small denominator problem” really exists should highly depend on the used method: it indeed exists for perturbation methods, but **not** for the HAM! Thus, the origin of the so-called “small denominator problem” comes from the limitations and restrictions of perturbation method as a methodology. In other words, the “small denominator problem” is only an *artifact* of perturbation method. Thus, abandoning perturbation method but using the HAM, we can completely avoid the “small denominator problem”. Note that the “small denominator problem” has been regarded as a huge obstacle for many open problems in science. So, the HAM provides us a new way to attack them.

Secondly, unlike all other approximation techniques (including perturbation methods), we can directly **define** the inverse operator \mathcal{L}^{-1} of an undetermined linear operator in the frame of the HAM so as to easily gain the solutions of the linear high-order equations. It should be emphasized that it is the HAM that provides us such kind of great freedom [21, 43]. Using such kind of freedom, the so-called “small denominator problem” can be completely avoided, as illustrated in this paper. Note that, according to traditional mathematical theories, a linear differential operator \mathcal{L} , whose kernel is a vector space of 4 dimension defined by (4.16), should correspond to the 4th-order linear differential equation (4.17), whose inverse operator should be expressed by (4.18). Similarly, a linear differential operator \mathcal{L} , whose kernel is a vector space of 6 dimension defined by (4.19), should correspond to the 6th-order linear differential equation (4.23), whose inverse operator should be expressed by (4.24). In addition, a linear differential operator \mathcal{L} , whose kernel is a vector space of 8 dimension defined by (4.25), should corre-

spond to the 8th-order linear differential equation (4.29), whose inverse operator should be expressed by (4.30). However, when $\lambda = \omega_1$ and $\delta = \omega_\kappa^2 - \omega_1^2$, although its kernel defined by (3.2) is a vector space of 4, 6, 8 dimension for $\kappa = 2, 3, 4$, respectively, its inverse operator defined by (3.1) always looks like that of a second-order linear operator whose kernel should be a vector space of 2 dimension according to the traditional mathematical theorems! Obviously, the inverse operator (3.1), which we directly *define* in the frame of the HAM, is quite *different* from (4.18), (4.24) and (4.30). Not that we even do *not* know how to explicitly express its corresponding auxiliary linear operator \mathcal{L} . Fortunately, it is *unnecessary* to know the undetermined linear operator \mathcal{L} in the frame of the HAM. Thus, to the best of author's knowledge, the auxiliary linear operator defined by (3.1) and (3.2) is fundamentally different from all known traditional linear operators. Note that, in the previous applications of the HAM [14–26, 28–42], one mostly chooses a proper linear auxiliary operator \mathcal{L} and then find its corresponding inverse operator \mathcal{L}^{-1} so as to solve the high-order equations. However, in this paper, we directly *define* the inverse operator \mathcal{L}^{-1} but do *not* care about the explicit expression of the corresponding auxiliary linear operator \mathcal{L} at all. This might be a breakthrough in the field of differential equations. It further illustrates the novelty and great potential of the so-called “method of directly defining inverse mapping” (MDDiM), which was proposed by Liao and Zhao [43] in the frame of the HAM and has been successfully applied to solve many types of nonlinear equations [44–56].

Thirdly, unlike perturbation techniques, the HAM provides us great freedom to choose initial guesses. Using such kind of freedom, we can gain the multiple limit-cycles of the forced Duffing equation (1.1) by means of the HAM. Note that, when $\lambda = \omega_1$ and $\delta = \omega_\kappa^2 - \omega_1^2$ for $\kappa \geq 2$, the larger the value of δ in the definitions (3.1) and (3.2), the greater the probability to find more limit-cycles of the forced Duffing equation (1.1). In contrast, perturbation method provides only one initial guess (1.8) and thus at most one limit-cycle. Thus, this illustrates the novelty of our HAM approach and its advantages beyond perturbation.

Unlike all other approximation techniques (including perturbation methods), the HAM contains the so-called convergence-control parameter c_0 , which provides a simple way to guarantee the convergence of solution series even when the nonlinearity is very high, as illustrated in this paper and also in other publications about the HAM [28–42]. This guarantees that the HAM-based approach is generally valid for high nonlinearity.

As pointed out by Giorgilli [9], Duffing equation “is perhaps the simplest example of a non-integrable system exhibiting **all** problems due to the small denominators”. So, although the forced Duffing equation (1.1) is used here as an example to illustrate the validity and novelty of the HAM approach and its advantages beyond perturbation, most conclusions mentioned above have general meanings.

What will happen if the homotopy analysis method (HAM) instead of perturbation method is first proposed by an intelligent being on a planet in the universe? Certainly, using the HAM-based approach mentioned in this paper, this kind of intelligent being should have *no* ideas of “small denominator problem” at all! Thus, the famous “small

denominator problem" does *not* really exist and should be an *artifact* of perturbation method. Therefore, completely abandoning perturbation methods but using the HAM-based MDDiM, we can thoroughly avoid "small denominator problems" and besides could attack many open problems related to small denominators. In addition, we illustrated here that a nonlinear differential equation can be solved by directly defining a proper inverse operator of an undetermined linear operator. Hopefully, this fact might lead to a breakthrough in the field of differential equations.

In summary, completely abandoning perturbation methods but using the HAM-based MDDiM, one would be never troubled by "small denominator problems"!

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References

- [1] H. POINCARÉ, *Les éthodes Nouvelles de la Mécanique Céleste. Tome II. Méthodes de MM. Newcomb, Gylden, Lindstedt et Bohlin*, Dover Publications, New York, 1957.
- [2] C. E. DELAUNAY, *Théorie du Mouvement de la Lune*, Gauthier-Villars, Paris, 1867.
- [3] R. A. PÉREZ, *A brief but historic article of Siegel*, *Notices of the American Mathematical Society*, 58 (2011), pp. 558–566.
- [4] V. I. ARNOL'D, *Small denominators and problems of stability of motion in classical and celestial mechanics*, *Russ. Math. Surv.*, 18 (1963), 85.
- [5] G. IOOSS, P. I. PLOTNIKOV, AND J. F. TOLAND, *Standing waves on an infinitely deep perfect fluid under gravity*, *Arch. Rational Mech. Anal.*, 177 (2005), pp. 367–478.
- [6] G. IOOSS, *Small divisor problems in fluid mechanics*, *J. Dyn. Differential Equations*, 27 (2015), pp. 787–802.
- [7] S. MARMI, *A method for accurate stability bounds in a small denominator problem*, *J. Phys. A Math. Gen.*, 21 (1988), L961.
- [8] L. CHIERCHIA, AND C. FALCOLINI, *Compensations in small divisor problems*, *Commun. Math. Phys.*, 175 (1996), pp. 135–160.
- [9] A. GIORGILLI, *Small denominators and exponential stability: From Poincaré to the present time*, *Seminario Mat. e. Fis. di Milano*, 68 (1998), 19.
- [10] A. ELIAS-ZÚNIGA, *A general solution of the Duffing equation*, *Nonlinear Dyn.*, 45 (2006), pp. 227–235.
- [11] E. KARTASHOVA, *Nonlinear Resonance Analysis: Theory, Computation, Applications*, Cambridge University Press, 2011.
- [12] J. K. KEVORKIAN, AND J. D. COLE, *Multiple Scale and Singular Perturbation Methods*, Springer New York, 1996.
- [13] R. H. RAND, AND D. ARMBRUSTER, *Perturbation Methods, Bifurcation Theory and Computer Algebra*, 1st Edition, Springer, 1987.

- [14] S. LIAO, *The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems*, Ph.D. thesis, Shanghai Jiaotong University, Shanghai, China, (1992).
- [15] S. LIAO, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall/CRC, 2003.
- [16] S. LIAO, *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer-Verlag, 2012.
- [17] S. LIAO, *An approximate solution technique not depending on small parameters: a special example*, Int. J. Non-Linear Mech., 30(3) (1995), pp. 371–380.
- [18] S. LIAO, *A kind of approximate solution technique which does not depend upon small parameters–II*, Int. J. Non-Linear Mech., 32(5) (1997), pp. 815–822.
- [19] S. LIAO, *An explicit, totally analytic approximate solution for Blasius' viscous flow problems*, Int. J. Non-Linear Mech., 34(4) (1999), pp. 759–778.
- [20] S. LIAO, *On the homotopy analysis method for nonlinear problem*, Appl. Math. Comput., 147 (2004), pp. 499–513.
- [21] S. LIAO, AND Y. TAN, *A general approach to obtain series solutions of nonlinear differential equations*, Stud. Appl. Math., 119 (2007), pp. 297–354.
- [22] S. LIAO, *Notes on the homotopy analysis method: some definitions and theorems*, Commun. Nonlinear Sci. Numer. Simul., 14(4) (2009), pp. 983–997.
- [23] S. LIAO, *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Commun. Nonlinear Sci. Numer. Simul., 15 (2010), pp. 2003–2016.
- [24] S. LIAO, D. XU, AND M. STIASSNIE, *On the steady-state nearly resonant waves*, J. Fluid Mech., 794 (2016), pp. 175–199.
- [25] S. LIAO, *A new non-perturbative approach in quantum mechanics for time-independent Schrödinger equations*, Science China Phys. Mech. Astron., 63(3) (2020), 234612.
- [26] X. ZHONG, AND S. LIAO, *On the limiting Stokes wave of extreme height in arbitrary water depth*, J. Fluid Mech., 843 (2018), pp. 653–679.
- [27] P. J. HILTON, *An Introduction to Homotopy Theory*, Cambridge University Press, 1953.
- [28] S. ZHU, *An exact and explicit solution for the valuation of American put options*, Quantitative Finance, 6(3) (2006), pp. 229–242.
- [29] Y. BOUREMEL, *Explicit series solution for the Glauert-jet problem by means of the homotopy analysis method*, Commun. Nonlinear Sci. Numer. Simul., 12 (2007), pp. 714–724.
- [30] C. J. NASSAR, J. F. REVELLI, AND R. J. BOWMAN, *Application of the homotopy analysis method to the Poisson-Boltzmann equation for semiconductor devices*, Commun. Nonlinear Sci. Numer. Simul., 16(6) (2011), 2501.
- [31] A. KIMIAEIFAR, E. LUND, O. THOMSEN, AND J. SORENSSEN, *Application of the homotopy analysis method to determine the analytical limit state functions and reliability index for large deflection of a cantilever beam subjected to static co-planar loading*, Comput. Math. Appl., 62 (2011), pp. 4646–4655.
- [32] A. R. GHOTBI, M. OMIIDVAR, AND A. BARARI, *Infiltration in unsaturated soils—An analytical approach*, Comput. Geotech., 38 (2011), pp. 777–782.
- [33] J. DUARTE, C. JANUÁRIO, AND N. MARTINS, *Explicit series solution for a glucose-induced electrical activity model of pancreatic beta-cells*, Chaos, Solitons and Fractals, 76 (2015), pp. 1–9.
- [34] J. SARDANYÉS, C. RODRIGUES, C. JANUÁRIO, N. MARTINS, G. GIL-GÓMEZ, AND J. DUARTE, *Activation of effector immune cells promotes tumor stochastic extinction: A homotopy analysis approach*, Appl. Math. Comput., 252 (2015), 484.
- [35] R. A. VAN GORDER, *On the utility of the homotopy analysis method for non-analytic and global solutions to nonlinear differential equations*, Numer. Algorithms, 76(1) (2017), pp. 151–162.

- [36] T. PFEFFER, AND L. POLLET, *A stochastic root finding approach: the homotopy analysis method applied to Dyson–Schwinger equations*, New J. Phys., 19 (2017), 043005.
- [37] A. C. CULLEN, AND S. R. CLARKE, *A fast, spectrally accurate homotopy based numerical method for solving nonlinear differential equations*, J. Comput. Phys., 385 (2019), pp. 106–118.
- [38] J. SULTANA, *Obtaining analytical approximations to black hole solutions in higher-derivative gravity using the homotopy analysis method*, Euro. Phys. J. Plus, 134 (2019), 111.
- [39] L. MASSA, AND J. A. SCHETZ, *Hypersonic heat flux reconstruction with distributed temperature sensors*, J. Thermophys. Heat Transf., 34(2) (2020), pp. 331–346.
- [40] G. KAUR, R. SINGH, AND H. BRIESEN, *Approximate solutions of aggregation and breakage population balance equations*, J. Math. Anal. Appl., 512 (2022), 126166.
- [41] P. K. MASJEDI, AND P. M. WEAVER, *Analytical solution for arbitrary large deflection of geometrically exact beams using the homotopy analysis method*, Appl. Math. Model., 103 (2022), pp. 516–542.
- [42] E. D. BOTTON, J. B. GREENBERG, A. ARAD, D. KATOSHEVSKI, V. VAIKUNTANATHAN, M. IBACH, AND B. WEIGAND, *An investigation of grouping of two falling dissimilar droplets using the homotopy analysis method*, Appl. Math. Model., 104 (2022), pp. 486–498.
- [43] S. LIAO, AND Y. ZHAO, *On the method of directly defining inverse mapping for nonlinear differential equations*, Numer. Algor., 72 (2016), pp. 989–1020.
- [44] M. BAXTER, M. DEWASURENDRA, AND K. VAJRARELU, *A method of directly defining the inverse mapping for solutions of coupled systems of nonlinear differential equations*, Numer. Algor., 77 (2018), pp. 1199–1211.
- [45] M. DEWASURENDRA, M. BAXTER, AND K. VAJRARELU, *A method of directly defining the inverse mapping for solutions of non-linear coupled systems arising in convection heat transfer in a second grade fluid*, Appl. Math. Comput., 339 (2018), pp. 758–767.
- [46] K. VAJRARELU, M. DEWASURENDRA, AND Y. ZHANG, *A method of directly defining the inverse mapping (MDDiM) for solutions of non-linear coupled systems arising in SIR and SIS epidemic models*, Commun. Numer. Anal. 2019(2) (2019), pp. 64–77.
- [47] M. DEWASURENDRA, *Semi-Analytical Solutions of Non-Linear Differential Equations Arising in Science and Engineering*, Ph.D. thesis, University of Central Florida, Florida, USA, (2019).
- [48] M. DEWASURENDRA, Y. ZHANG, N. BOYETTE, I. ISLAM, AND K. VAJRARELU, *A method of directly defining the inverse mapping for a HIV infection of CD4 + T-cells model*, Appl. Math. Nonlinear Sci., 6(2) (2021), pp. 469–482.
- [49] C. W. SAHABANDU, D. KARUNARATHNA, P. SEWVANDI, Z. A. M. S. JUMAN, M. DEWASURENDRA, AND K. VAJRARELU, *A method of directly defining the inverse mapping for a nonlinear partial differential equation and for systems of nonlinear partial differential equations*, Comput. Appl. Math., 40 (2021), 234.
- [50] C. SAHABANDU, M. DEWASURENDRA, Z. JUMAN, K. VAJRARELU, AND A. J. CHAMKHA, *Semi-analytical method for propagation of harmonic waves in nonlinear magneto-thermo-elasticity*, Comput. Math. Appl., 105 (2022), pp. 107–111.
- [51] O. NAVE, AND M. ELBAZ, *Combination of singularly perturbed vector field method and method of directly defining the inverse mapping applied to complex ODE system prostate cancer model*, J. Biolog. Dyn., 12(1) (2018), pp. 961–986.
- [52] O. NAVE, *A new method to find the base functions for the method of directly defining the inverse mapping (MDDiM)*, J. Multiscale Model., 9(4) (2018), 1850008.
- [53] O. NAVE, *A semi-analytical method for solving problems on the role of prey taxis in a biological control-mathematical model*, J. Multiscale Model., 10(2) (2019), 1850009.

- [54] O. NAVE, M. ELBAZ, AND S. B. MENDRAZITSKY, *Analysis of a breast cancer mathematical model by a new method to find an optimal protocol for HER2-positive cancer*, *BioSystems*, 197 (2020), 104191.
- [55] K. GANGADHAR, M. V. S. RAO, S. KUMAR, S. SHARMA, AND S. R. MUNJAM, *A series-form solution of the coupled nonlinear equations by the method of directly defined inverse mapping and SRM*, *Int. J. Ambient Energy*, 43(1) (2022), pp. 1345–1354.
- [56] S. R. MUNJAM, K. GANGADHAR, R. SESHADRI, AND M. RAJESWAR, *Novel technique MD-DiM solutions of MHD flow and radiative Prandtl–Eyring fluid over a stretching sheet with convective heating*, *Int. J. Ambient Energy*, (published online).