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Numerical Approximation of a Nonlinear 3D Heat Radiation Problem

Liping Liu[1], Min Huang[2,*], Kewei Yuan[1] and Michal Křížek[3]

[1] *Department of Mathematical Sciences, Lakehead University, 955 Oliver Road, Thunder Bay, Ontario P7B 5E1, Canada.*

[2] *Department of Computer Science, Purdue University, 250 N. University Street, West Lafayette, IN 47907, USA.*

[3] *Institute of Mathematics, Academy of Sciences, Žitná 25, CZ-115 67, Prague 1, Czech Republic.*

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Abstract. In this paper, we are concerned with the numerical approximation of a steady-state heat radiation problem with a nonlinear Stefan-Boltzmann boundary condition in $\mathbb{R}^3$. We first derive an equivalent minimization problem and then present a finite element analysis to the solution of such a minimization problem. Moreover, we apply the Newton iterative method for solving the nonlinear equation resulting from the minimization problem. A numerical example is given to illustrate theoretical results.

AMS subject classifications: 65N30

Key words: Heat radiation problem, Stefan-Boltzmann condition, Newton iterative method.

1 Introduction

The main purpose of this paper is to study a finite element approximation to the solution of the steady-state heat radiation problem with a nonlinear Stefan-Boltzmann boundary condition in R^3. In particular, we assume that Ω is a bounded domain in R^3 with Lipschitz continuous boundary Γ. Let ν be the outward unit normal to Γ. Consider the following stationary heat conduction equation

$$-\mathrm{div}(A\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

*Corresponding author.
URL: `http://flash.lakeheadu.ca/~lipingli/`
Email: liping.liu@lakeheadu.ca (L. Liu), kewei.yuan@lakeheadu.ca (K. Yuan), huang3@purdue.edu (M. Huang), krizek@math.cas.cz (M. Krizek)

with mixed Dirichlet-radiation boundary conditions

$$\begin{aligned} u &= \bar{u} \quad \text{on } \Gamma_1, \\ \alpha u + \nu^T A \nabla u + \beta u^4 &= g \quad \text{on } \Gamma_2, \end{aligned} \tag{1.2}$$

where A is a diagonal uniformly positive definite 3×3 matrix of heat conductivities, $f \geq 0$ is the density of body heat sources and $u \geq 0$ is the temperature of the body to be determined. Moreover, Γ_1 and Γ_2 are non-empty disjoint sets relatively open in Γ and satisfying $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\alpha \geq 0$ is the heat transfer coefficient, $\beta = \sigma f_{em}$ with the Stefan-Boltzmann constant $\sigma = 5.669 \times 10^{-8}$ [$\mathrm{Wm^{-2}K^{-4}}$] and the relative emissivity function $0 \leq f_{em} \leq 1$, $\bar{u} \geq 0$ is the prescribed temperature, and $g \geq 0$ is the density of surface heat sources.

Because they are of practical importance, numerical approximations of similar heat radiation problems have been extensively studied (see, e.g., [5, 7, 9, 10, 14]). The case $\Gamma_1 = \emptyset$ in R^2 is investigated in [8]. It is well known that the traces of the variational solution of (1.1) and (1.2) belong to the Lebesgue space $L^5(\partial\Omega)$ due to the nonlinearity in the Stefan-Boltzmann boundary condition. In the two-dimensional case, we may seek the variational solution of (1.1) and (1.2) in the Sobolev space $H^1(\Omega)$ whose functions have traces in $L^5(\partial\Omega)$ by the trace theorem (cf. [8, 9]). However, it is no longer true in the three-dimensional case. Taking this into account, we may define a new function space in which the variational solution of (1.1) and (1.2) uniquely exists. Such a function space is also used to find the minimizing element of the minimization problem in [5], where only axially symmetric domains are treated (see also [13]). In [11, 12], the three-dimensional heat radiation problem is solved on arbitrary geometries by means of a Fredholm boundary integral equation and the boundary element method. Another approach, how to avoid the problem with traces in three-dimensions, is to use a discontinuous Galerkin method from [15].

The paper is organized as follows. In Section 2, we derive a variational formulation of the heat radiation problem. Section 3 is devoted to an analysis of the finite element approximation to the solution of the minimization problem and a discussion of the Newton iterative method for the nonlinear equation arising from the minimization problem. In Section 4 we present a numerical example to illustrate the theoretical analysis.

2 Variational formulation of the radiation problem

Assume that $a_i \in L^\infty(\Omega)$, $i = 1, 2, 3$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_2)$, $\bar{u} \in H^1(\Omega)$ and $\bar{u}|_\Gamma \in L^5(\Gamma_2)$, $\alpha, \beta \in L^\infty(\Gamma_2)$ and $\beta \geq \beta_0$ *a.e.* for some positive constant β_0. For simplicity, we denote by $\|\cdot\|_k$ the norm $\|\cdot\|_{H^k(\Omega)}$ for integer $k \geq 0$. Also, for a relatively open subset D in Γ, we denote by $\|\cdot\|_{q,D}$ the norm $\|\cdot\|_{L^q(D)}$ for $q \geq 1$.

We next define a bilinear form on $H^1(\Omega) \times H^1(\Omega)$ by

$$a(v, w) := \int_\Omega (\nabla v)^T A \nabla w \, dx + \int_{\Gamma_2} \alpha v w \, ds, \quad v, w \in H^1(\Omega), \tag{2.1}$$