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On Scientific Data and Image Compression Based on Adaptive Higher-Order FEM

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Abstract. We present a novel compression algorithm for 2D scientific data and images based on exponentially-convergent adaptive higher-order finite element methods (FEM). So far, FEM has been used mainly for the solution of partial differential equations (PDE), but we show that it can be applied to data and image compression easily. The adaptive compression algorithm is trivial compared to adaptive FEM algorithms for PDE since the error estimation step is not present. The method attains extremely high compression rates and is able to compress a data set or an image with any prescribed error tolerance. Compressed data and images are stored in the standard FEM format, which makes it possible to analyze them using standard PDE visualization software. Numerical examples are shown. The method is presented in such a way that it can be understood by readers who may not be experts of the finite element method.

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Key words: Data compression, image compression, adaptive *hp*-FEM, orthogonal projection, best approximation problem, error control.

1 Introduction

The finite element method (FEM) is the most widely used numerical method for the solution of partial differential equations (PDEs) — see, e.g., [3, 6, 10]. The PDEs describe various natural processes on macroscopic scale, such as fluid flow, elasticity, heat transfer, electromagnetics, etc. The FEM splits the computational domain into a set of geometrically simple subdomains (elements) such as quadrilaterals in 2D or hexahedra in 3D. Inside the elements, the physical fields are approximated by polynomials. The coefficients of the polynomials are called *degrees of freedom (DOF)*. Performance of an adaptive FEM algorithm is the rate at which the approximation error

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decreases. This rate can be measured either in terms of the CPU (computer) time or the number of DOF in the discrete problem.

The *hp*-FEM is a modern version of FEM capable of extremely fast (exponential) convergence [1,2]. In practical computations, adaptive *hp*-FEM routinely outperforms standard adaptive FEM by orders of magnitude in terms of both the number of DOF and CPU time [8,9]. The extremely high efficiency of the *hp*-FEM has its roots in the approximation theory: Very smooth functions with small local changes are approximated optimally using large elements equipped with high-degree polynomials, while small low-degree elements are much more efficient in areas where the solution exhibits important small-scale features.

The outline of this paper is as follows: In Section 2 we describe the main idea of how adaptive *hp*-FEM can be applied to data and image compression. In Sections 3 we illustrate the methodology on three different examples. Conclusions and outlook are drawn in Section 4.

2 Applying FEM to data and image compression

The typical application of FEM is to approximate unknown solutions of PDE. However, FEM can also be applied to data and image compression naturally as follows: In 2D, the computational domain Ω is a rectangle containing the image. Usually, the domain is split into a finite number of pixels, and a greyscale image is represented by a discontinuous, pixel-wise constant function f. A color image consists of three such functions f_r, f_g, f_b for the red, green, and blue components, respectively. Unlike standard image compression algorithms such as JPEG, however, our method is not restricted to pixel-wise constant functions f – the data can be represented by an arbitrary real function f defined in Ω.

2.1 Finite element approximation

Let us explain briefly the way FEM works. The method uses a *finite element mesh*, which is a collection of nonoverlapping convex polygons covering Ω. Given the shape of Ω in our application, it is natural to use rectangular elements. A finite element mesh is said to be *regular* if no vertex of an element lies inside of an edge of another one, and *irregular* otherwise. This is illustrated in Fig. 1.

Most existing finite element codes use regular meshes, since they are easier to implement and analyze mathematically. However, when used with automatic adaptive algorithms, such meshes produce *regularity-enforced refinements* which slow down their convergence [7]. Irregular meshes are much better for adaptive algorithms since element refinement is a local operation, i.e., it does not cause any changes in neighboring elements. Technical details of *hp*-FEM approximation on arbitrarily-irregular meshes lie beyond the scope of this presentation, and we refer to [7].

The mesh over Ω consists of M elements $K_1, K_2, \dots, K_M$ which are equipped with polynomial degrees $1 \le p_i = p(K_i)$. Is standard FEM, the polynomial degree typically