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Numerical Approximation of a Nonlinear 3D Heat Radiation Problem

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Abstract. In this paper, we are concerned with the numerical approximation of a steady-state heat radiation problem with a nonlinear Stefan-Boltzmann boundary condition in $\mathbb{R}^3$. We first derive an equivalent minimization problem and then present a finite element analysis to the solution of such a minimization problem. Moreover, we apply the Newton iterative method for solving the nonlinear equation resulting from the minimization problem. A numerical example is given to illustrate theoretical results.

AMS subject classifications: 65N30

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1 Introduction

The main purpose of this paper is to study a finite element approximation to the solution of the steady-state heat radiation problem with a nonlinear Stefan-Boltzmann boundary condition in R^3. In particular, we assume that Ω is a bounded domain in R^3 with Lipschitz continuous boundary Γ. Let ν be the outward unit normal to Γ. Consider the following stationary heat conduction equation

$$-\text{div}(A\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

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with mixed Dirichlet-radiation boundary conditions

$$\begin{aligned} u &= \bar{u} \quad \text{on } \Gamma_1, \\ \alpha u + \nu^T A \nabla u + \beta u^4 &= g \quad \text{on } \Gamma_2, \end{aligned} \tag{1.2}$$

where A is a diagonal uniformly positive definite 3×3 matrix of heat conductivities, $f \geq 0$ is the density of body heat sources and $u \geq 0$ is the temperature of the body to be determined. Moreover, Γ_1 and Γ_2 are non-empty disjoint sets relatively open in Γ and satisfying $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\alpha \geq 0$ is the heat transfer coefficient, $\beta = \sigma f_{em}$ with the Stefan-Boltzmann constant $\sigma = 5.669 \times 10^{-8}$ [$\mathrm{Wm^{-2}K^{-4}}$] and the relative emissivity function $0 \leq f_{em} \leq 1$, $\bar{u} \geq 0$ is the prescribed temperature, and $g \geq 0$ is the density of surface heat sources.

Because they are of practical importance, numerical approximations of similar heat radiation problems have been extensively studied (see, e.g., [5, 7, 9, 10, 14]). The case $\Gamma_1 = \emptyset$ in R^2 is investigated in [8]. It is well known that the traces of the variational solution of (1.1) and (1.2) belong to the Lebesgue space $L^5(\partial\Omega)$ due to the nonlinearity in the Stefan-Boltzmann boundary condition. In the two-dimensional case, we may seek the variational solution of (1.1) and (1.2) in the Sobolev space $H^1(\Omega)$ whose functions have traces in $L^5(\partial\Omega)$ by the trace theorem (cf. [8, 9]). However, it is no longer true in the three-dimensional case. Taking this into account, we may define a new function space in which the variational solution of (1.1) and (1.2) uniquely exists. Such a function space is also used to find the minimizing element of the minimization problem in [5], where only axially symmetric domains are treated (see also [13]). In [11, 12], the three-dimensional heat radiation problem is solved on arbitrary geometries by means of a Fredholm boundary integral equation and the boundary element method. Another approach, how to avoid the problem with traces in three-dimensions, is to use a discontinuous Galerkin method from [15].

The paper is organized as follows. In Section 2, we derive a variational formulation of the heat radiation problem. Section 3 is devoted to an analysis of the finite element approximation to the solution of the minimization problem and a discussion of the Newton iterative method for the nonlinear equation arising from the minimization problem. In Section 4 we present a numerical example to illustrate the theoretical analysis.

2 Variational formulation of the radiation problem

Assume that $a_i \in L^\infty(\Omega)$, $i = 1, 2, 3$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_2)$, $\bar{u} \in H^1(\Omega)$ and $\bar{u}|_\Gamma \in L^5(\Gamma_2)$, $\alpha, \beta \in L^\infty(\Gamma_2)$ and $\beta \geq \beta_0$ *a.e.* for some positive constant β_0. For simplicity, we denote by $\|\cdot\|_k$ the norm $\|\cdot\|_{H^k(\Omega)}$ for integer $k \geq 0$. Also, for a relatively open subset D in Γ, we denote by $\|\cdot\|_{q,D}$ the norm $\|\cdot\|_{L^q(D)}$ for $q \geq 1$.

We next define a bilinear form on $H^1(\Omega) \times H^1(\Omega)$ by

$$a(v, w) := \int_\Omega (\nabla v)^T A \nabla w \, dx + \int_{\Gamma_2} \alpha v w \, ds, \quad v, w \in H^1(\Omega), \tag{2.1}$$

and a linear functional on $L^2(\Omega)$ by

$$F(v) := \int_\Omega fv\,dx + \int_{\Gamma_2} gv\,ds, \quad v \in L^2(\Omega). \tag{2.2}$$

Consider the following functional of potential energy

$$J(v) := \frac{1}{2}a(v+\bar{u}, v+\bar{u}) + \frac{1}{5}\int_{\Gamma_2} \beta|v+\bar{u}|^5\,ds - F(v+\bar{u}). \tag{2.3}$$

Note that by the trace theorem (cf. [1], p. 234) in the three-dimensional space we only have that $v|_\Gamma \in L^4(\Gamma)$ for any $v \in H^1(\Omega)$, whereas in two-dimensional space $v|_\Gamma \in L^p(\Gamma)$ for all $p \in [1,\infty)$. To ensure the functional J is well-defined in three dimensions, we introduce the space V by setting

$$V := \big\{v \in H^1(\Omega) :\ v|_{\Gamma_2} \in L^5(\Gamma_2)\big\}. \tag{2.4}$$

Let us equip V with the norm $\|\cdot\|_V$ defined by

$$\|v\|_V := \|v\|_1 + \|v\|_{5,\Gamma_2}, \quad v \in V. \tag{2.5}$$

It is shown in [5,9] that V is a reflexive Banach space. According to the definitions of the first and second Gâteaux derivatives of J, namely,

$$J'(z;v) := \lim_{t\to 0}\frac{1}{t}\big[J(z+tv) - J(z)\big], \quad v,z \in V,$$

and

$$J''(z;v,w) = \lim_{t\to 0}\frac{1}{t}\big[J'(z+tw;v) - J'(z;v)\big], \quad v,w,z \in V,$$

it can be easily verified that

$$J'(z;v) = a(z+\bar{u}, v) + \int_{\Gamma_2} \beta|z+\bar{u}|^3(z+\bar{u})v\,ds - F(v), \quad v,z \in V, \tag{2.6}$$

and

$$J''(z;v,w) = a(w,v) + 4\int_{\Gamma_2} \beta|z+\bar{u}|^3 wv\,ds, \quad v,w,z \in V. \tag{2.7}$$

We next formulate a variational analogue of (1.1) and (1.2). To this end, we define a linear subspace V^0 of V by

$$V^0 := \{v \in V :\ v|_{\Gamma_1} = 0\}. \tag{2.8}$$

By Friedrichs' inequality there exists a positive constant c such that for all $v \in V^0$,

$$a(v,v) \geq c\|v\|_1^2. \tag{2.9}$$

Consider the following minimization problem: Find an element $u \in V^0$ such that

$$J(u) = \inf_{v\in V^0} J(v). \tag{2.10}$$

Theorem 2.1. *The functional J is continuous on V and strictly convex on V^0. Moreover, the functional J is coercive on V^0, that is,*

$$J(v) \to \infty \text{ as } \|v\|_V \to \infty.$$

Therefore, the minimization problem (2.10) *has a unique solution $u \in V^0$. Furthermore, we have that $u + \bar{u} \geq 0$ a.e. on $\overline{\Omega}$.*

Proof. By a direct computation, we find that J is continuous on V. From (2.7) and (2.9) it follows that for any $v, z \in V^0$,

$$J''(z; v, v) \geq c\|v\|_1^2 + 4\int_{\Gamma_2} \beta |z + \bar{u}|^3 v^2 \, ds \geq 0.$$

By the trace theorem, $\|v\|_1 = 0$ implies $v = 0$ almost everywhere on Γ for $v \in H^1(\Omega)$. This means that $J''(z; v, v) = 0$ if and only if $\|v\|_V = 0$ for $v, z \in V^0$. Thus, J is strictly convex on V^0. By (2.3) and (2.9), we have that there exist two positive constants c_1 and c_2 such that for any $v \in V^0$,

$$J(v) \geq c_1(\|v\|_1^2 + \|v\|_{5,\Gamma_2}^2) - c_2(\|v\|_1 + \|\bar{u}\|_1),$$

which implies that $J(v) \to \infty$ as $\|v\|_V \to \infty$, namely, J is coercive on V^0. Summarizing the above properties of J, we conclude that (2.10) has a unique solution $u \in V^0$ (cf. [6], p. 35).

We next prove $u + \bar{u} \geq 0$ *a.e.* on $\overline{\Omega}$. For this we introduce a function

$$\tilde{u}(x) := \max\{u(x), -\bar{u}(x)\}, \quad x \in \Omega \cup \Gamma_2.$$

It is well known that $|v| \in H^1(\Omega)$, when $v \in H^1(\Omega)$. Using the equality

$$\max(u, -\bar{u}) = \frac{1}{2}(u - \bar{u} + |u + \bar{u}|),$$

we find that $\tilde{u} \in H^1(\Omega)$. Since $\bar{u} \in V$ and $u \in V^0$, it follows that $\tilde{u} \in V$. We also observe that $\tilde{u}$ has a zero trace on Γ_1 due to the fact that $\overline{u} = u$ on Γ_1. Now it is easily seen that $\tilde{u} \in V^0$.

Moreover, for any $x \in \Omega \cup \Gamma_2$ we have

$$\tilde{u}(x) + \bar{u}(x) = \begin{cases} 0 & \text{if } u(x) \leq -\bar{u}(x), \\ u(x) + \bar{u}(x) & \text{if } u(x) > -\bar{u}(x). \end{cases}$$

Recalling that $\alpha \geq 0$, $\beta > 0$, and A is a diagonal and uniformly positive matrix, we have that

$$a(\tilde{u} + \bar{u}, \tilde{u} + \bar{u}) \leq a(u + \bar{u}, u + \bar{u}), \tag{2.11}$$

and

$$\int_{\Gamma_2} \beta |\tilde{u} + \bar{u}|^5 \, ds \leq \int_{\Gamma_2} \beta |u + \bar{u}|^5 \, ds. \tag{2.12}$$

Also, since $f \geq 0$ and $g \geq 0$, it follows from (2.2) that $F(\tilde{u}) \geq F(u)$. Combining this with (2.11) and (2.12) yields that $J(\tilde{u}) \leq J(u)$. This implies that $\tilde{u} = u$. By the definition of $\tilde{u}$, we confirm that $u + \bar{u} \geq 0$, which completes the proof. □

The following result establishing a variational equality for solving the minimization problem (2.10) can be found in [5,9].

Remark 2.2 The solution u of (2.10) is characterized by the following variational equation

$$J'(u;v) = 0 \quad \text{for all } v \in V^0. \tag{2.13}$$

Remark 2.3 Applying Remark 2.2 we can show that the solution of the minimization problem (2.10) is "equivalent" to the solution of the radiation problem.

To see this, assume that $u \in V^0$ is sufficiently smooth such that $u + \bar{u}$ satisfies the heat conduction equation (1.1) with boundary conditions (1.2). Then u is also the solution of minimization problem (2.10). Indeed, if a smooth $u \in V$ is the solution of (1.1) with mixed boundary conditions (1.2), then by Green's theorem we have that for any $v \in C^\infty(\overline{\Omega})$,

$$0 = \int_\Omega [\operatorname{div}(A\nabla u) + f] v \, dx = \int_\Omega [-(\nabla u)^T A \nabla v + fv] \, dx + \int_\Gamma (v^T A \nabla u) v \, ds. \tag{2.14}$$

Since $C^\infty(\overline{\Omega})$ is dense in V, (2.14) holds for any $v \in V$. Using (1.2), we have that u is the solution of variational equation (2.13). That is, u is the solution of minimization problem (2.10).

Conversely, let $u \in V^0$ be a sufficiently smooth solution of (2.10). Then $u + \bar{u}$ satisfies (1.1) and (1.2) by a standard argumentation (cf. [3], p. 125).

3 Finite element approximation

In this section, we are concerned with finding a finite element approximation to the solution of the minimization problem (2.10) and then determining the numerical solution of (2.10) by using the Newton iterative method.

Let Ω be a bounded polyhedral domain with Lipschitz boundary. Let $\mathcal{T}_h$ be a standard face-to-face partition of $\overline{\Omega}$ into tetrahedral elements and let $\{V_h^0\}_{h\to 0}$ be a sequence of finite-dimensional subspaces of V^0 on Ω associated with $\mathcal{T}_h$. We assume that V_h^0 satisfies the approximation hypothesis:

For every $v \in V^0$ there exists $v_h \in V_h^0$ such that

$$\|v - v_h\|_V \to 0 \quad \text{as } h \to 0. \tag{H}$$

Consider the discretized minimization problem: Find an element $u_h \in V_h^0$ such that

$$J(u_h) = \inf_{v_h \in V_h^0} J(v_h). \tag{3.1}$$

Following an argument similar to the proof of Theorem 2.1 and Remark 2.2, we have that minimization problem (3.1) has a unique solution $u_h \in V_h^0$, which is completely characterized by the variational equation

$$J'(u_h; v_h) = 0 \ \text{ for all } v_h \in V_h^0. \tag{3.2}$$

The next theorem shows the solution u_h of the discretized minimization problem (3.1) converges to the solution of the minimization problem (2.10) in V.

Theorem 3.1. *Assume that* $\{V_h^0\}_{h\to 0}$ *satisfies hypothesis (H). Let* u *and* u_h *be the solutions of* (2.10) *and* (3.1), *respectively. Then*

$$\|u - u_h\|_V \to 0 \quad \text{as } h \to 0. \tag{3.3}$$

Proof. Recall that $V_h^0 \subset V^0$. It follows from (2.13) and (3.2) that for any $v_h \in V_h^0$,

$$J'(u; u_h - v_h) - J'(v_h; u_h - v_h) = J'(u_h; u_h - v_h) - J'(v_h; u_h - v_h). \tag{3.4}$$

For notational convenience we let

$$b(u, v, w) := \int_{\Gamma_2} \beta\big[|u + \bar{u}|^3 (u + \bar{u}) - |v + \bar{u}|^3 (v + \bar{u})\big] w \, ds, \ \ u, v, w \in V. \tag{3.5}$$

It is shown in [5] that for all $u, v \in V$,

$$\big||u|^3 u - |v|^3 v\big| \le |u - v|\big[|u|^3 + |v|(|u| + |v|)^2\big] \le |u - v|(|u| + |v|)^3,$$

and

$$8 \int_{D'} \big[|u|^3 u - |v|^3 v\big](u - v) \, ds \ge \int_{D'} |u - v|^5 \, ds,$$

where D' is an arbitrary relatively open subset of Γ. Thus, we have

$$\begin{aligned} |b(u, v_h, u_h - v_h)| &= \Big| \int_{\Gamma_2} \beta\big[|u + \bar{u}|^3 (u + \bar{u}) - |v_h + \bar{u}|^3 (v_h + \bar{u})\big](u_h - v_h) \, ds \Big| \\ &\le \beta \int_{\Gamma_2} |(u + \bar{u}) - (v_h + \bar{u})|(|u + \bar{u}| + |v_h + \bar{u}|)^3 |u_h - v_h| \, ds \\ &\le \beta \int_{\Gamma_2} |u - v_h||u_h - v_h|(|u| + |v_h| + 2|\bar{u}|)^3 \, ds, \end{aligned} \tag{3.6}$$

and

$$b(u_h, v_h, u_h - v_h) \ge c_1 \|u_h - v_h\|_{5,\Gamma_2}^5, \tag{3.7}$$

where c_1 is a positive constant. Combining (3.7) with (2.6) and (2.9), we obtain that

$$\begin{aligned} J'(u_h; u_h - v_h) - J'(v_h; u_h - v_h) &= a(u_h - v_h, u_h - v_h) + b(u_h, v_h, u_h - v_h) \\ &\ge c_2(\|u_h - v_h\|_1^2 + \|u_h - v_h\|_{5,\Gamma_2}^5), \end{aligned} \tag{3.8}$$

where c_2 is a positive constant. For the left-hand term of (3.4), it is easily seen from (2.6) that

$$J'(u; u_h - v_h) - J'(v_h; u_h - v_h) = a(u - v_h, u_h - v_h) + b(u, v_h, u_h - v_h). \tag{3.9}$$

Using the Hölder inequality and the trace theorem, we have

$$\begin{aligned} &|a(u - v_h, u_h - v_h)| \\ \le\ & c_3(\|\nabla(u - v_h)\|_0 \|\nabla(u_h - v_h)\|_0 + \|u - v_h\|_{2,\Gamma_2} \|u_h - v_h\|_{2,\Gamma_2}) \\ \le\ & c_4 \|u - v_h\|_1 \|u_h - v_h\|_1, \end{aligned} \tag{3.10}$$

where c_3 and c_4 are positive constants, and by (3.6),

$$\begin{aligned} &|b(u, v_h, u_h - v_h)| \\ \le\ & c_5 \|u - v_h\|_{5,\Gamma_2} \|u_h - v_h\|_{5,\Gamma_2} \|(|u| + |v_h| + 2|\bar{u}|)^3\|_{5/3,\Gamma_2} \\ \le\ & c_6 \|u - v_h\|_{5,\Gamma_2} \|u_h - v_h\|_{5,\Gamma_2} (\|u\|_{5,\Gamma_2} + \|v_h\|_{5,\Gamma_2} + \|\bar{u}\|_{5,\Gamma_2})^3, \end{aligned} \tag{3.11}$$

where c_5 and c_6 are positive constants. Recall that $\{V_h^0\}_{h\to 0}$ satisfies the hypothesis. Therefore, we may choose $v_h \in V_h^0$ such that $\|u - v_h\|_V \to 0$ as $h \to 0$. Then there exist positive constants h_0 and C independent of h such that for all $h \in (0, h_0)$,

$$(\|u\|_{5,\Gamma_2} + \|v_h\|_{5,\Gamma_2} + \|\bar{u}\|_{5,\Gamma_2})^3 \le (\|u\|_{5,\Gamma_2} + \|\bar{u}\|_{5,\Gamma_2} + 1)^3 \le C. \tag{3.12}$$

From (3.8), (3.4) and (3.9)–(3.12) we conclude that

$$\begin{aligned} &\|u_h - v_h\|_1^2 + \|u_h - v_h\|_{5,\Gamma_2}^5 \\ \le\ & c_7(\|u - v_h\|_1 \|u_h - v_h\|_1 + \|u - v_h\|_{5,\Gamma_2} \|u_h - v_h\|_{5,\Gamma_2}), \end{aligned} \tag{3.13}$$

where c_7 is a positive constant. For simplicity, set

$$\gamma := \max\{\|u - v_h\|_1, \|u - v_h\|_{5,\Gamma_2}\} \quad \text{and} \quad \lambda := \|u_h - v_h\|_1 + \|u_h - v_h\|_{5,\Gamma_2}.$$

When $\|u_h - v_h\|_1 \le \|u_h - v_h\|_{5,\Gamma_2}$, we have

$$\lambda^5 \le 32 \|u_h - v_h\|_{5,\Gamma_2}^5.$$

On the other hand, when $\|u_h - v_h\|_1 > \|u_h - v_h\|_{5,\Gamma_2}$, we have

$$\lambda^2 \le 4 \|u_h - v_h\|_1^2.$$

This implies

$$\min\{\lambda^2,\ \lambda^5\} \le 32(\|u_h - v_h\|_1^2 + \|u_h - v_h\|_{5,\Gamma_2}^5). \tag{3.14}$$

From (3.13) and (3.14) it follows that

$$\min\{\lambda^2,\ \lambda^5\} \le c_8 \gamma \lambda, \tag{3.15}$$

where c_8 is a positive constant. Note that there exists a positive constant $h_1 < h_0$ such that for all $h \in (0, h_1)$ we have $c_8\gamma < 1$. Combining this with (3.15) yields $\lambda \leq c_9\gamma^{1/4}$ for all $h \in (0, h_1)$, where c_9 is a positive constant. Therefore, we have for all $h \in (0, h_1)$,

$$\|u - u_h\|_V \leq 2\gamma + \lambda \leq c_{10}(\gamma + \gamma^{1/4}), \tag{3.16}$$

where c_{10} is a positive constant. Recall that $\gamma \to 0$ as $h \to 0$. By (3.16), we confirm the result of the theorem. $\square$

We next derive an error estimate for $u - u_h$ in V under further assumptions on V_h^0. To this end, assume that $V_h^0 \subset H^{7/6}(\Omega) \cap V^0$. Denote by $\mathcal{I}_h$ a standard linear interpolation operator from $C(\overline{\Omega}) \cap V^0$ to V_h^0. Let k be an integer greater than one. We further assume that the following estimate holds for any $v \in H^k(\Omega) \cap V^0$,

$$\|v - \mathcal{I}_h v\|_s \leq ch^{k-s}\|v\|_k, \quad s \in [1, 7/6]. \tag{3.17}$$

where c is a positive constant. Hence, we obtain the following estimate of $\|u - u_h\|_V$.

Theorem 3.2. *Assume that $\bar{u} \in H^1(\Omega)$ and $\bar{u}|_{\Gamma_2} \in L^6(\Gamma_2)$. Let u and u_h be the solutions of* (2.10) *and* (3.1)*, respectively. Suppose that k is an integer greater than one and that* (3.17) *holds. If $u \in H^k(\Omega)$ then there exists a positive constant c independent of h such that*

$$\|u - u_h\|_V \leq ch^{2(k-1)/5} \quad \text{as } h \to 0. \tag{3.18}$$

Proof. We will use some results from the proof of Theorem 3.1. By (3.5) and (3.6) it follows that for any $v_h \in V_h^0$,

$$\begin{aligned} &|b(u, v_h, u_h - v_h)| \\ \leq\, & c_1\|u - v_h\|_{4,\Gamma_2}\|u_h - v_h\|_{4,\Gamma_2}\|(|u| + |v_h| + 2|\bar{u}|)^3\|_{2,\Gamma_2} \\ \leq\, & c_2\|u - v_h\|_1\|u_h - v_h\|_1(\|u\|_{6,\Gamma_2} + \|v_h\|_{6,\Gamma_2} + \|\bar{u}\|_{6,\Gamma_2})^3, \end{aligned} \tag{3.19}$$

where c_1 and c_2 are positive constants. By the Sobolev imbedding theorem $u \in H^2(\Omega)$ is continuous. Let $v_h = \mathcal{I}_h u$. By the trace theorem and (3.17), we have that there exist positive constants h_0 and C' independent of h such that for all $h \in (0, h_2)$,

$$(\|u\|_{6,\Gamma_2} + \|v_h\|_{6,\Gamma_2} + \|\bar{u}\|_{6,\Gamma_2})^3 \leq c(\|u\|_{7/6} + \|u - v_h\|_{7/6} + \|\bar{u}\|_{6,\Gamma_2})^3 \leq C'. \tag{3.20}$$

From (3.4), (3.8), (3.9), (3.10), (3.19), and (3.20) we conclude that

$$\|u_h - v_h\|_1^2 + \|u_h - v_h\|_{5,\Gamma_2}^5 \leq c\|u - v_h\|_1\|u_h - v_h\|_1. \tag{3.21}$$

This implies that

$$\|u_h - v_h\|_1 \leq c\|u - v_h\|_1 \quad \text{and} \quad \|u_h - v_h\|_{5,\Gamma_2}^5 \leq c\|u - v_h\|_1^2. \tag{3.22}$$

Again, by the trace theorem and (3.17), we have that

$$\|u - v_h\|_1 + \|u - v_h\|_{5,\Gamma_2} \leq c(\|u - v_h\|_1 + \|u - v_h\|_{11/10}) \leq ch^{k-11/10}.$$

Combining this with (3.22) and (3.17) proves estimate (3.18). □

We point out that the spaces V_h^0 may be obtained by using the construction of finite elements developed in [4]. Following an argument on the regularity of partitions of $\overline{\Omega}$ in [2], estimate (3.17) is satisfied.

Now let us turn our attention to the Newton iterative method for solving the nonlinear equation (3.2). Choosing an initial guess $u_{h,0} \in V_h^0$, the Newton iterative method for (3.2) is to find $u_{h,n+1} \in V_h^0$, $n \geq 0$, such that

$$J''(u_{h,n}; v_h, u_{h,n+1} - u_{h,n}) = -J'(u_{h,n}; v_h) \text{ for all } v_h \in V_h^0, \tag{3.23}$$

where J' and J'' are given by (2.6) and (2.7), respectively. In order to show that (3.23) is uniquely solvable, we define the functional L_n on V for a given $u_{h,n}$ by

$$\begin{aligned} L_n(v) := & \frac{1}{2}a(v+\bar{u}, v+\bar{u}) + \int_{\Gamma_2} \beta|u_{h,n}+\bar{u}|^3(2v - 3u_{h,n} + \bar{u})v\,ds \\ & -F(v+\bar{u}), \qquad v \in V. \end{aligned} \tag{3.24}$$

By a simple calculation, we obtain the first and second Gâteaux derivatives of L_n, respectively, given by

$$\begin{aligned} L_n'(z;v) = & a(z+\bar{u}, v) + \int_{\Gamma_2} \beta|u_{h,n}+\bar{u}|^3(4z - 3u_{h,n} + \bar{u})v\,ds \\ & -F(v), \qquad v, z \in V, \end{aligned} \tag{3.25}$$

and

$$L_n''(z; v, w) = a(w, v) + 4\int_{\Gamma_2} \beta|u_{h,n}+\bar{u}|^3 wv\,ds, \quad v, w, z \in V. \tag{3.26}$$

Lemma 3.1. *The functional L_n defined by* (3.24) *is continuous on V, strictly convex on V^0, and coercive on V_h^0.*

Proof. By the argument similar to the proof of Theorem 2.1, we have that L_n is continuous on V and strictly convex on V^0. We next prove that L_n is coercive on V_h^0. It follows from (2.9) and (3.24) that for any $v \in V_h^0$,

$$\begin{aligned} L_n(v) & \geq c_1'\|v\|_1^2 - c_2'(\|u_{h,n}\|_{5,\Gamma_2} + \|\bar{u}\|_{5,\Gamma_2})^5\|v\|_{5,\Gamma_2} - c_3'\|v\|_1 \\ & \geq c_2'\|v\|_1^2 - c_4'\|v\|_V. \end{aligned} \tag{3.27}$$

Notice that the norm $\|v\|_V$ is equivalent to $\|v\|_1$ in the finite-dimensional space V_h^0 when h is fixed, that is, there exist two positive constants C_1 and C_2 such that for all $v \in V_h^0$,

$$C_1\|v\|_V \leq \|v\|_1 \leq C_2\|v\|_V. \tag{3.28}$$

Combining this with (3.27) yields

$$L_n(v) \geq c_2' C_1^2 \|v\|_V^2 - c_4'\|v\|_V.$$

This implies that for $v \in V_h^0$,

$$L_n(v) \to \infty \quad \text{as } \|v\|_V \to \infty,$$

which means L_n is coercive on V_h^0. This completes the proof. □

The next theorem shows that the iterative solution $u_{h,n+1}$ of (3.23) uniquely exists for each $n \geq 0$, and converges to the solution u_h of (3.1), provided that $u_{h,0}$ is sufficiently close to u_h. To state this result, we denote by e_n the error $\|u_h - u_{h,n}\|_V$ for $n \geq 0$.

Theorem 3.3. *Let u_h be the solution of* (3.1). *For every given $u_{h,n} \in V_h^0$, $n \geq 0$, equation* (3.18) *has a unique solution $u_{h,n+1} \in V_h^0$. Furthermore, there exist positive constants δ and c independent of n such that for every $e_0 < \delta$, we have*

$$e_{n+1} \leq c e_n^2. \tag{3.29}$$

Proof. By Lemma 3.3, the following minimization problem

$$L_n(w_{h,n}) = \inf_{v_h \in V_h^0} L_n(v_h) \tag{3.30}$$

has a unique solution $w_{h,n} \in V_h^0$ for every given $u_{h,n} \in V_h^0$. Following the argument similar to Remark 2.2, we have that minimization problem (3.30) is completely characterized by the variational equation

$$L_n'(w_h, v_h) = 0 \ \text{ for all } v_h \in V_h^0, \tag{3.31}$$

where $w_h \in V_h^0$ is to be determined. It is straightforward to show that

$$L_n'(w_h, v_h) = J''(u_{h,n}; v_h, w_{h,n} - u_{h,n}) + J'(u_{h,n}; v_h).$$

Thus, equation (3.23) is equivalent to (3.31), which implies that equation (3.23) has a unique solution $u_{h,n+1} \in V_h^0$ for every given $u_{h,n} \in V_h^0$.

For notational convenience, we set

$$\hat{u}_h := u_h + \bar{u}, \ \ \hat{u}_{h,n} := u_{h,n} + \bar{u},$$

and

$$y_{h,n} := |\hat{u}_h|^3 \hat{u}_h + 3|\hat{u}_{h,n}|^3 \hat{u}_{h,n} - 4|\hat{u}_{h,n}|^3 \hat{u}_h.$$

From (2.6), (2.7), (3.2), and (3.23) it follows that for any $v_h \in V_h^0$,

$$\begin{aligned} &J''(u_{h,n}; v_h, u_{h,n+1} - u_h) \\ &= J'(u_h; v_h) - J'(u_{h,n}; v_h) - J''(u_{h,n}; v_h, u_h - u_{h,n}) \\ &= \int_{\Gamma_2} \beta y_{h,n} v_h \, ds. \end{aligned} \tag{3.32}$$

Let $x_{h,n} := \hat{u}_h^2 + 2|\hat{u}_h \hat{u}_{h,n}| + 3\hat{u}_{h,n}^2$. We first prove that

$$|y_{h,n}| \le (\hat{u}_h - \hat{u}_{h,n})^2 x_{h,n}. \tag{3.33}$$

If $\hat{u}_h > 0$ and $\hat{u}_{h,n} > 0$, we have

$$|y_{h,n}| = |(\hat{u}_h - \hat{u}_{h,n})^2(\hat{u}_h^2 + 2\hat{u}_h\hat{u}_{h,n} + 3\hat{u}_{h,n}^2)| = (\hat{u}_h - \hat{u}_{h,n})^2 x_{h,n}.$$

If $\hat{u}_h < 0$ and $\hat{u}_{h,n} > 0$, we obtain

$$|y_{h,n}| \le \hat{u}_h^4 + 3\hat{u}_{h,n}^4 + 8|\hat{u}_h\hat{u}_{h,n}^3| + 8\hat{u}_h^2\hat{u}_{h,n}^2 + 4|\hat{u}_h^3\hat{u}_{h,n}| = (\hat{u}_h - \hat{u}_{h,n})^2 x_{h,n}.$$

In other cases, we can obtain estimate (3.33) by a similar argument. Combining (3.32) with (3.33) verifies that

$$\begin{aligned} |J''(u_{h,n}; v_h, u_{h,n+1} - u_h)| &\le \int_{\Gamma_2} \beta(\hat{u}_h - \hat{u}_{h,n})^2 |v_h| x_{h,n}\, ds \\ &\le c\|(u_h - u_{h,n})^2 x_{h,n}\|_{5/4,\Gamma_2} \|v_h\|_{5,\Gamma_2} \\ &\le c\|u_h - u_{h,n}\|_{5,\Gamma_2}^2 \|x_{h,n}\|_{5/2,\Gamma_2} \|v_h\|_{5,\Gamma_2}. \end{aligned} \tag{3.34}$$

Note that

$$|J''(u_{h,n}; u_{h,n+1} - u_h, u_{h,n+1} - u_h)| \ge c\|u_{h,n+1} - u_h\|_1^2 \ge c e_{n+1}^2.$$

Choosing $v_h = u_{h,n+1} - u_h$ in (3.34), we get that

$$e_{n+1} \le c\|x_{h,n}\|_{5/2,\Gamma_2} e_n^2. \tag{3.35}$$

It is easy to see that

$$\|x_{h,n}\|_{5/2,\Gamma_2} \le c\||\hat{u}_h| + |\hat{u}_{h,n}|\|_{5,\Gamma_2}^2 \le c(\|u_h\|_V + \|\bar{u}\|_V + e_n)^2. \tag{3.36}$$

We conclude from (3.35) and (3.36) that there exist two positive constants M_1 and M_2 independent of n such that

$$e_{n+1} \le (M_1 + M_2 e_n^2) e_n^2. \tag{3.37}$$

Thus, if we pick δ with $(M_1 + M_2\delta^2)\delta < 1$ and $e_0 < \delta$, then we have that $e_{n+1} \le e_n < \delta,\ n \ge 0$, by an induction on n. Therefore, we obtain that for any $n \ge 0$,

$$e_{n+1} \le (M_1 + M_2\delta^2) e_n^2,$$

which confirms estimate (3.29). $\square$

4 Numerical examples

In this section, we report results of numerical experiments to illustrate the theoretical investigations in the previous section.

The numerical experiments were carried out for heat radiation problem (1.1) and (1.2) on the cube $\Omega = (0,1)^3$. The two parts Γ_1 and Γ_2 of Γ are given, respectively, by

$$\Gamma_2 = \{(x_1, x_2, x_3) : \ x_3 = 1 \text{ and } 0 \leq x_1, x_2 \leq 1\} \ \text{ and } \ \Gamma_1 = \Gamma \backslash \Gamma_2.$$

We choose the matrix $A = \text{diag}\{60, 60, 60\}$ and the parameters $\alpha = 90$, $\beta = 0.75 \times 5.669 \times 10^{-8}$ and the right-hand terms

$$f(x_1, x_2, x_3) = 36000\pi^2 x_3 \sin \pi x_1 \sin \pi x_2, \quad \bar{u}(x_1, x_2, x_3) = 300,$$

and

$$g(x_1, x_2, x_3) = 27000 + 45000 \sin \pi x_1 \sin \pi x_2 + 344.39175(1 + \sin \pi x_1 \sin \pi x_2)^4.$$

Thus, the exact solution of (1.1) and (1.2) is

$$\bar{u}(x_1, x_2, x_3) = 300(1 + x_3 \sin \pi x_1 \sin \pi x_2).$$

The cube Ω is first divided into $N \times N \times N$ subcubes along each axis, with each small cube constructed from six pyramidal elements containing a common vertex at the centre of each cube (see Figure 1). Each pyramidal element is then subdivided into four tetrahedra that have a common edge passing through the centre of the square base.

Figure 1: Partition of a cube into 6 pyramids.

In our examples, N is taken to be $4, 8, 16, 32, 64$, and 128, respectively. The corresponding mesh size h is proportional to $1/4, 1/8, 1/16, 1/32, 1/64$, and $1/128$. We employ piecewise linear polynomials to form approximating subspaces which belong to $C^0(\overline{\Omega})$. Thus, equation (3.23) is equivalent to a linear system associated with a set of basis functions. The Gaussian cubature was used for the numerical integration over

Table 1: Number of iterations and computing time.

N	#1*	#2*	Time [s]
4	429	4	0.105
8	2969	4	1.145
16	22065	4	13.287
32	170081	4	253.853
64	1335489	4	4754.96
128	10584449	4	133819.00

(#1*: #1 of unknowns; #2*: #2 of iterations.)

Table 2: Errors and orders of convergence with $k=2$.

N	h^{-1}	$\|u-u_h\|_1$	$\frac{\|u-u_{h,n+1}\|_1}{\|u-u_{h,n}\|_1}$	$\|u-u_h\|_V$	$\frac{\|u-u_{h,n+1}\|_V}{\|u-u_{h,n}\|_V}$
4	4	91.1320	-	97.8978	-
8	8	45.6345	1.99700	47.3420	2.06788
16	16	22.8253	1.99929	23.2551	2.03577
32	32	11.4136	1.99983	11.5215	2.01841
64	64	5.70694	1.99996	5.73395	2.00934
128	128	2.85349	1.99999	2.86024	2.00471

the tetrahedral elements and a modified Newton Method was used to solve the nonlinear system. The stopping criterion of Newton iterations is

$$\frac{\|\mathbf{u}_{h,n+1}-\mathbf{u}_{h,n}\|_E}{\|\mathbf{u}_{h,n}\|_E} < 10^{-10}, \tag{4.1}$$

where $\mathbf{u}_{h,n}$ stands the vector corresponding to $u_{h,n}$ for $n \geq 0$ and $\|\cdot\|_E$ denotes the Euclidean norm of the corresponding vector. Also, the initial iteration $u_{h,0}$ for each h is chosen to be the solution of linear equation (3.2) with $\beta = 0$.

We used a 64-bit, 1300 MHz Itanium 2 Processor sever with a main memory size of 60.10 GB to carry out the computation. Table 1 shows the number of iterations and the computing time for the performance of Newton iterations until the stopping criterion (4.1) is reached. It can be seen from Table 1 that the number of iterations is independent of the mesh size h.

In Table 2, we list the results on H^1-norm and V-norm of the error of the exact solution u and the approximate solution u_h.

We observe that orders of convergence in the numerical experiments are 1 in both norms. The theoretical result in Theorem 3.2 shows the order of convergence is $2(k-1)/5 = 0.4$ for $k=2$ in V-norm. But note that the L^5-norm of the error on Γ_2, $\|u-u_h\|_{5,\Gamma_2} = \|u-u_h\|_V - \|u-u_h\|_1$ is much less than H^1-norm of the error in Table 2. In this case we may improve the order of convergence to $k-11/10 = 0.9$ by following an argument similar to the proof of Theorem 3.2, which implies the computed order is consistent with the theoretical order in V-norm.

Finally, Figure 2 illustrates the numerical solution of (1.1) and (1.2) for $h = 1/32$.

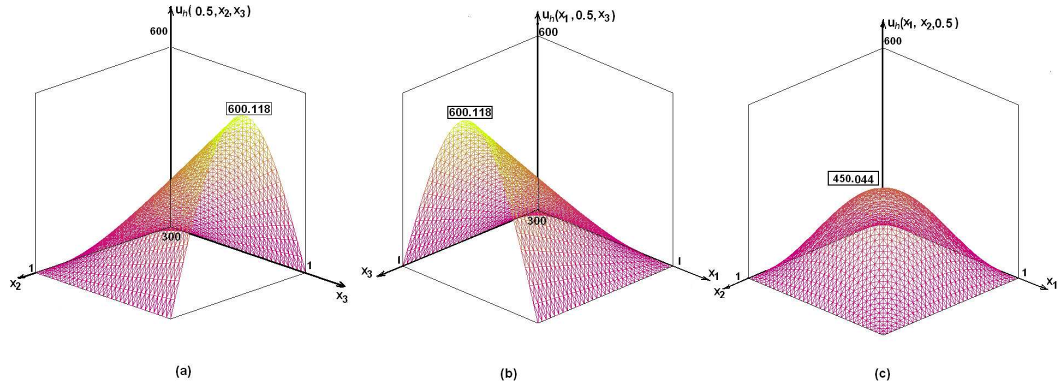

Figure 2: Numerical solution for $h = 1/32$ and (a) $x_1 = 0.5$, (b) $x_2 = 0.5$ and (c) $x_3 = 0.5$.

5 Conclusions

Finite element approximations of an axially symmetric three-dimensional heat radiation problem are studied in [5]. In our paper, we need not assume that Ω is axially symmetric. Then, however, we encounter some trouble with the definition domain of the energy functional J. Namely, the traces of functions from the Sobolev space $H^1(\Omega)$ are not in $L^5(\partial\Omega)$, in general, since Ω is a three-dimensional bounded domain. Therefore, we considered a special reflexive Banach subspace V of $H^1(\Omega)$, where the energy functional is well defined. We proved that finite element solutions converge to the true solution u in the norm of V without any additional regularity assumptions on u. For a sufficiently smooth solution u, we derived the convergence of order $2(k-1)/5$, where $k > 1$ is a given integer. Additionally, we introduced a Newton iterative method for finding finite element approximation of a nonlinear 3D heat radiation problem. Finally, we computed a numerical example with mixed boundary conditions to illustrate our theoretical results and the efficiency of the Newton iterative method. An open problem is how to modify the presented method to anisotropic materials, when the matrix A of heat conductivities in not diagonal. In this case we cannot employ inequality (2.11).

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