Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 1, pp. 140-150 (2009)

February 2009

On the Boundary Integral Equations for a Two-Dimensional Slowly Rotating Highly Viscous Fluid Flow

D. Lesnic[1,*]

[1] *Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK.*

Received 28 October 2008; Accepted (in revised version) 8 December 2008
Available online 16 February 2009

Abstract. In this paper, the two-dimensional slowly rotating highly viscous fluid flow in small cavities is modelled by the triharmonic equation for the streamfunction. The Dirichlet problem for this triharmonic equation is recast as a set of three boundary integral equations which however, do not have a unique solution for three exceptional geometries of the boundary curve surrounding the planar solution domain. This defect can be removed either by using modified fundamental solutions or by adding two supplementary boundary integral conditions which the solution of the boundary integral equations must satisfy. The analysis is further generalized to polyharmonic equations.

AMS subject classifications: 31A30, 31B10
Key words: Boundary integral equations, triharmonic and polyharmonic equations, logarithmic capacity.

1 Introduction

Mathematically, if one considers the incompressible rotating viscous flow [1], at large Ekman numbers, i.e., $E = \nu/(L^2\Omega) \gg 1$, which can be achieved if a highly viscous fluid with dynamic viscosity $\nu \gg 1$ is slowly rotating, i.e. the angular velocity $\Omega \ll 1$, in a small bounded cavity D of characteristic length $L \ll 1$, e.g., a square $[0,L]\times[0,L]$ lid-driven cavity [2], the model presented in [3] can be reduced to the triharmonic equation for the fluid streamfunction ψ, namely,

$$\nabla^6\psi = 0, \quad \text{in } D \subset \mathbb{R}^2. \tag{1.1}$$

In this paper, we analyze the Dirichlet problem in which equation (1.1) has to be solved subject to the essential boundary conditions on the primary variables, namely,

$$\psi = f_0, \quad \frac{\partial\psi}{\partial n} = f_1, \quad \nabla^2\psi = f_2, \quad \text{on } \partial D, \tag{1.2}$$

*Corresponding author.
URL: `http://www.maths.leeds.ac.uk/applied/staff.dir/lesnic/lesnic.html`
Email: `amt5ld@maths.leeds.ac.uk` (D. Lesnic)

where f_0, f_1 and f_2 are prescribed functions, ∂D is the boundary of the fluid domain D, and n is the outward unit normal to ∂D.

A direct boundary integral method for the interior Dirichlet problem for the two-dimensional Laplace equation, namely,

$$\nabla^2\theta = 0, \quad \text{in } D \subset \mathbb{R}^2, \quad \theta = \text{specified on } \partial D,$$

has been investigated in [4–7], whilst for the biharmonic equation, namely,

$$\nabla^4\phi = 0, \quad \text{in } D \subset \mathbb{R}^2, \qquad \phi \text{ and } \frac{\partial\phi}{\partial n} = \text{specified on } \partial D,$$

has been investigated in [8–10].

The purpose of this study is to extend these analyses to the triharmonic case given by equations (1.1) and (1.2), and make a classification for the polyharmonic equation

$$\nabla^{2k}\psi = 0, \quad \text{in } D \subset \mathbb{R}^2, \tag{1.3}$$

where $k \in \mathbb{N}^*$, which has to be solved subject to the boundary conditions

$$\left(\psi, \frac{\partial\psi}{\partial n}, \nabla^2\psi, \frac{\partial(\nabla^2\psi)}{\partial n}, \ldots, \frac{\partial(\nabla^{2p-2}\psi)}{\partial n}, \nabla^{2p}\psi\right) = \text{specified on } \partial D, \quad \text{if } k = 2p+1, \tag{1.4}$$

$$\left(\psi, \frac{\partial\psi}{\partial n}, \nabla^2\psi, \frac{\partial(\nabla^2\psi)}{\partial n}, \ldots, \nabla^{2p-2}\psi, \frac{\partial(\nabla^{2p-2}\psi)}{\partial n}\right) = \text{specified on } \partial D, \quad \text{if } k = 2p. \tag{1.5}$$

2 Boundary integral equations

We assume that the planar domain D is simply connected and bounded by a smooth, simple and closed contour ∂D, and that all the functions occurring in the sequel are as smooth as required by the process of mathematical manipulation in which they are involved.

Among the different methods which may be used for solving problem (1.1)-(1.2), the boundary element method (BEM) plays an important role. Here we are going to study a particular integral equation approach which emerges from the integral representation [3],

$$\begin{aligned}
&\int_{\partial D} \left[G_3(x,y)\frac{\partial(\nabla^4\psi)}{\partial n}(y) - \frac{\partial G_3}{\partial n(y)}(x,y)\nabla^4\psi(y)\right] ds(y) \\
&+ \int_{\partial D} \left[\nabla_y^2 G_3(x,y)\frac{\partial(\nabla^2\psi)}{\partial n}(y) - \frac{\partial(\nabla_y^2 G_3)}{\partial n(y)}(x,y)\nabla^2\psi(y)\right] ds(y) \\
&+ \int_{\partial D} \left[\nabla_y^4 G_3(x,y)\frac{\partial\psi}{\partial n}(y) - \frac{\partial(\nabla_y^4 G_3)}{\partial n(y)}(x,y)\psi(y)\right] ds(y)
\end{aligned}$$

$$= \begin{cases} \psi(x), & x \in D, \\ \frac{1}{2}\psi(x), & x \in \partial D, \\ 0, & x \notin \overline{D}, \end{cases} \tag{2.1}$$

where

$$G_3(x,y) = -\frac{1}{128\pi} r^4 \ln(r), \quad r = |x-y|, \tag{2.2}$$

is the fundamental solution for the triharmonic operator $-\nabla^6$.

Since $\nabla^2\psi$ and $\nabla^4\psi$ are biharmonic and harmonic in D, respectively, we also have the representations [8,10]

$$\begin{aligned}
&\int_{\partial D} \Big[G_2(x,y) \frac{\partial(\nabla^4\psi)}{\partial n}(y) - \frac{\partial G_2}{\partial n(y)}(x,y)\nabla^4\psi(y) \Big] ds(y) \\
&+ \int_{\partial D} \Big[\nabla_y^2 G_2(x,y) \frac{\partial(\nabla^2\psi)}{\partial n}(y) - \frac{\partial(\nabla_y^2 G_2)}{\partial n(y)}(x,y)\nabla^2\psi(y) \Big] ds(y) \\
&= \begin{cases} \nabla^2\psi(x), & x \in D, \\ \frac{1}{2}\nabla^2\psi(x), & x \in \partial D, \\ 0, & x \notin \overline{D}, \end{cases}
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\int_{\partial D} \Big[G_1(x,y) \frac{\partial(\nabla^4\psi)}{\partial n}(y) - \frac{\partial G_1}{\partial n(y)}(x,y)\nabla^4\psi(y) \Big] ds(y) \\
&= \begin{cases} \nabla^4\psi(x), & x \in D, \\ \frac{1}{2}\nabla^4\psi(x), & x \in \partial D, \\ 0, & x \notin \overline{D}, \end{cases}
\end{aligned} \tag{2.4}$$

where

$$G_1(x,y) = -\frac{1}{2\pi}\ln(r), \quad G_2(x,y) = -\frac{1}{8\pi} r^2 \ln(r), \tag{2.5}$$

are the fundamental solutions for the Laplace $-\nabla^2$ and biharmonic $-\nabla^4$ operators, respectively. Let us denote with

$$\frac{\partial(\nabla^2\psi)}{\partial n} = g_0, \quad \nabla^4\psi = g_1, \quad \frac{\partial(\nabla^4\psi)}{\partial n} = g_2, \quad \text{on } \partial D, \tag{2.6}$$

the unknown secondary variables associated to the Dirichlet problem (1.1)-(1.2). Then, from (1.2), (2.1), (2.3) and (2.4), we derive the following set of three boundary integral equations (BIEs):

$$\begin{aligned}
&\int_{\partial D} \Big[G_3(x,y) g_2(y) - \frac{\partial G_3}{\partial n(y)}(x,y) g_1(y) + \nabla_y^2 G_3(x,y) g_0(y) \Big] ds(y) \\
&= \frac{1}{2} f_0(x) + \int_{\partial D} \Big[\frac{\partial(\nabla_y^2 G_3)}{\partial n(y)}(x,y) f_2(y) - \nabla_y^4 G_3(x,y) f_1(y) \\
&\qquad + \frac{\partial(\nabla_y^4 G_3)}{\partial n(y)}(x,y) f_0(y) \Big] ds(y), \qquad x \in \partial D,
\end{aligned} \tag{2.7}$$

$$\int_{\partial D}\Big[G_2(x,y)g_2(y)-\frac{\partial G_2}{\partial n(y)}(x,y)g_1(y)+\nabla_y^2 G_2(x,y)g_0(y)\Big]ds(y)$$
$$=\frac{1}{2}f_2(x)+\int_{\partial D}\frac{\partial(\nabla_y^2 G_2)}{\partial n(y)}(x,y)f_2(y)ds(y),\qquad x\in\partial D, \tag{2.8}$$

$$\frac{1}{2}g_1(x)+\int_{\partial D}\Big[\frac{\partial G_1}{\partial n(y)}(x,y)g_1(y)-G_1(x,y)g_2(y)\Big]ds(y)=0,\ x\in\partial D. \tag{2.9}$$

Eqs.(2.7)-(2.9) constitute three linear BIEs from which the unknowns (2.6) can be determined. Insertion of g_0, g_1 and g_2 into (2.1) for $x \in D$ yields the solution of the original problem (1.1)-(1.2).

The main purpose of our investigation is to find out whether the BIEs (2.7)-(2.9) have a unique solution. It turns out (Section 3) that, in three exceptional cases, the BIEs (2.7)-(2.9) do not have a unique solution, even though the considered boundary value problem (1.1)-(1.2) does. However, the investigation of Section 4 shows that this difficulty can be overcome by introducing two supplementary integral conditions which the solution of the BIEs, i.e. g_0, g_1 and g_2, must satisfy.

Section 5 presents a possible extension to the polyharmonic Dirichlet problem (1.3)-(1.5), whilst Section 6 summarises the conclusions of the paper.

3 Non-uniqueness of the solution

We first investigate a particular case of a special boundary curve ∂D_c given by the circle centred at the origin with radius $c > 0$, namely,

$$\partial D_c:\ x_1=c\cos(\theta),\quad x_2=c\sin(\theta),\quad \theta\in[0,2\pi). \tag{3.1}$$

The integration point y and the parameter point x are now specified by the angular coordinates θ and θ_0, respectively.

By simple geometrical considerations the homogeneous system corresponding to (2.7)-(2.9) can be transformed into

$$\int_0^{2\pi}\Big[g_1(\theta)\rho^4(1+2\ln(\rho^2))-g_2(\theta)c\rho^4\ln(\rho^2)-16g_0(\theta)c\rho^2(1+\ln(\rho^2))\Big]d\theta=0, \tag{3.2}$$
$$\int_0^{2\pi}\big[g_1(\theta)\rho^2(1+\ln(\rho^2))-g_2(\theta)c\rho^2\ln(\rho^2)-4g_0(\theta)c(2+\ln(\rho^2))\big]d\theta=0, \tag{3.3}$$
$$g_1(\theta_0)+\frac{1}{2\pi}\int_0^{2\pi}\big[-g_1(\theta)+g_2(\theta)c\ln(\rho^2)\big]d\theta=0, \tag{3.4}$$

where $\rho^2=4c^2\sin^2\left(\frac{\theta-\theta_0}{2}\right)$. Of course, the system of BIEs (3.2)-(3.4) has the trivial solution $g_0=g_1=g_2=0$. In order to find non-trivial solutions the Fourier series

$$g_i(\theta)=c^{-1-i}\left(a_0^{(i)}+\sum_{n=1}^{\infty}(a_n^{(i)}\cos(n\theta)+b_n^{(i)}\sin(n\theta))\right),\quad i=0,1,2, \tag{3.5}$$

are inserted into (3.2)-(3.4), and by elementary calculations, we obtain for the first terms

$$-64c^2a_0^{(0)}(1+\ln(c))+4c^2a_0^{(1)}(5+6\ln(c))-c^2a_0^{(2)}(7+12\ln(c))=0, \tag{3.6}$$

$$-4a_0^{(0)}(1+\ln(c))+2a_0^{(1)}(1+\ln(c))-a_0^{(2)}(1+2\ln(c))=0, \tag{3.7}$$

$$2c^{-2}a_0^{(2)}\ln(c)=0, \tag{3.8}$$

where we have used that [11]

$$\begin{aligned}
&\int_0^{2\pi}\ln(\rho^2)d\theta=2\pi\ln(c^2),\\
&\int_0^{2\pi}\rho^2\ln(\rho^2)d\theta=4\pi c^2(1+\ln(c^2)),\\
&\int_0^{2\pi}\rho^4\ln(\rho^2)d\theta=2\pi c^4(7+6\ln(c^2)).
\end{aligned}$$

The determinant of this system of algebraic equations is

$$D_0=-32\ln(c)(1+\ln(c))(3+2\ln(c)),$$

which becomes zero when $c\in\{1,e^{-1},e^{-3/2}\}$, and the solutions $a_0^{(i)}$, $i=0,1,2$ are not necessarily equal to zero:

$$\begin{aligned}
&c=1: \ \ a_0^{(i)}\neq 0,\ i=0,1,2,\\
&c=e^{-1}: \ \ a_0^{(2)}=0,\ a_0^{(0)}\neq 0,\ a_0^{(1)}\neq 0,\\
&c=e^{-3/2}: \ \ a_0^{(2)}=0,\ a_0^{(1)}=2a_0^{(0)}\neq 0.
\end{aligned}$$

This result shows that the homogeneous system of equations (3.6)-(3.8) may have a non-trivial solution. Consequently, the inhomogeneous system of BIEs (2.7)-(2.9) may have a non-unique solution. In accordance with [5,8], it can be conjectured that a non-unique solution may appear for a general curve ∂D (not only for the disk), when the exterior mapping radius (or the transfinite diameter, or the logarithmic capacity) of the curve ∂D is 1, e^{-1} or $e^{-3/2}$. These particular boundary contours for which the BIEs do not have a unique solution are sometimes called Γ-contours [4].

This non-uniqueness can be overcome if instead of G_i, $i=1,2,3$, one considers the modified fundamental solutions

$$\left\{\begin{aligned}
G_1^A(r)&=-\tfrac{1}{2\pi}(\ln(r)+A),\\
G_2^B(r)&=-\tfrac{1}{8\pi}r^2(\ln(r)+B),\\
G_3^C(r)&=-\tfrac{1}{128\pi}r^4(\ln(r)+C),
\end{aligned}\right. \tag{3.9}$$

in the BIEs (2.7)-(2.9). Then, using single- and double-layer potential arguments similar to those used in [10], one can show the following uniqueness theorem.

Theorem 3.1. *Let χ and $2\pi\alpha$ be the equilibrium density and Robin's constant, respectively, for ∂D, i.e. the unique density function and number such that* [12]

$$\int_{\partial D} G_1(x,y)\chi(y)ds(y) = \alpha, \quad \int_{\partial D} \chi(y)ds(y) = 1. \tag{3.10}$$

Then, if $A \neq 2\pi\alpha$, $B \neq 2\pi\alpha - 1$ and $C \neq 2\pi\alpha - \frac{3}{2}$, the system of BIEs (2.7)-(2.9) *with G_1, G_2, G_3 replaced by G_1^A, G_2^B, G_3^C, has only the trivial solution $g_0 = g_1 = g_2 = 0$.*

Alternatively, we can obtain the uniqueness by supplying two supplementary conditions, as described in the next section.

4 Supplementary conditions

When the BIEs (2.7)-(2.9) do not have a unique solution, it is necessary to add some supplementary conditions which give additional relations among the known functions f_i and unknown functions $g_i (i = 0, 1, 2)$. Below, nine such conditions are derived in the form of line integrals along a general curve ∂D.

Let $\langle \lambda, \mu \rangle = \lambda_1\mu_1 + \lambda_2\mu_2$ denote the scalar product of two vectors $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ in $\mathbb{R}^2$. It can be shown by direct calculation that for $y \in \partial D$ and x far away from the origin we have the asymptotic expansions

$$\begin{aligned}
r^4\ln(r) &= r^4\ln(|x|) - |x|^2\langle x,y\rangle + 3\Big(\langle x,y\rangle - \frac{1}{2}|y|^2\Big)^2 - \frac{2}{3}\frac{\langle x,y\rangle^4}{|x|^4} \\
&\quad + \frac{2\langle x,y\rangle^2}{|x|^2}\Big(|y|^2 - \frac{2}{3}\langle x,y\rangle\Big) + O(|x|^{-1}), \\
\frac{\partial(r^4\ln(r))}{\partial n(y)} &= 4r^2\langle y-x, n(y)\rangle\ln(|x|) - |x|^2\langle x, n(y)\rangle \\
&\quad + 6\Big(\frac{1}{2}|y|^2 - \langle x,y\rangle\Big)\langle y-x, n(y)\rangle - \frac{8}{3}\frac{\langle x,y\rangle^3\langle x,n(y)\rangle}{|x|^4} \\
&\quad + \frac{4\langle x,y\rangle}{|x|^2}(\langle x,n(y)\rangle|y|^2 + \langle x,y\rangle\langle y-x,n(y)\rangle) + O(|x|^{-1}), \\
\nabla_y^2(r^4\ln(r)) &= 16\Big[\frac{r^2}{2} + r^2\ln(|x|) + \frac{\langle x,y\rangle^2}{|x|^2} - \langle x,y\rangle + \frac{1}{2}|y|^2\Big] + O(|x|^{-1}), \\
\frac{\partial(\nabla_y^2(r^4\ln(r)))}{\partial n(y)} &= 32\Big[\langle y-x,n(y)\rangle(\ln(|x|)+1) + \frac{\langle x,n(y)\rangle\langle x,y\rangle}{|x|^2}\Big] + O(|x|^{-1}), \\
\nabla_y^4(r^4\ln(r)) &= 64(\ln(r) + \frac{3}{2}) = 64(\ln(|x| + \frac{3}{2}) + O(|x|^{-1}), \\
\frac{\partial(\nabla_y^4(r^4\ln(r)))}{\partial n(y)} &= O(|x|^{-1}).
\end{aligned}$$

These expressions are inserted into the integral equation (2.1) whereby we obtain by collecting similar terms:

$$\begin{aligned} 0 =& I_1|x|^4\ln(|x|) + I_2|x|^3\ln(|x|) + I_3|x|^2\ln(|x|) + I_4|x|\ln(|x|) \\ &+I_5\ln(|x|) + I_6|x|^3 + I_7|x|^2 + I_8|x| + I_9 + O(|x|^{-1}), \end{aligned} \tag{4.1}$$

where the right-hand side of (2.1) is equal to zero, since as x is far away from the origin we have that $x \notin \overline{D}$. In the above expression the integrals I_i, $i = \overline{1,9}$, are given by

$$\begin{aligned}
I_1 =& \int_{\partial D} g_2(y)ds(y),\\
I_2 =& 4I_6 = 4\int_{\partial D} \Big[g_1(y)\,\langle \hat{x}, n(y)\rangle - g_2(y)\,\langle \hat{x}, y\rangle\Big]ds(y),\\
I_3 =& \int_{\partial D} \Big\{g_2(y)\,(4\langle \hat{x}, y\rangle^2 + 2|y|^2) - 4g_1(y)\Big(\,\langle y, n(y)\rangle + 2\,\langle \hat{x}, y\rangle\,\langle \hat{x}, n(y)\rangle\,\Big)\\
&+16g_0(y)\Big\}ds(y),\\
I_4 =& 4\int_{\partial D} \Big\{-g_2(y)\,\langle \hat{x}, y\rangle\,|y|^2 + g_1(y)\Big(2\langle y, n(y)\rangle\langle \hat{x}, y\rangle + \langle \hat{x}, n(y)\rangle|y|^2\Big)\\
&+8g_0(y)\langle \hat{x}, y\rangle + 8f_2(y)\langle \hat{x}, n(y)\rangle\Big\}ds(y),\\
I_5 =& \int_{\partial D} \Big\{g_2(y)|y|^4 - 4g_1(y)\langle y, n(y)\rangle|y|^2 + 16g_0(y)|y|^2\\
&-32f_2(y)\langle y, n(y)\rangle + 64f_1(y)\Big\}ds(y),\\
I_7 =& \int_{\partial D} \Big\{3g_2(y)\langle \hat{x}, y\rangle^2 - 6g_1(y)\langle \hat{x}, y\rangle\langle \hat{x}, n(y)\rangle + 8g_0(y)\Big\}ds(y),\\
I_8 =& \int_{\partial D} \Big\{g_1(y)\Big(3|y|^2\langle \hat{x}, n(y)\rangle + 6\langle \hat{x}, y\rangle\langle y, n(y)\rangle - 4\langle \hat{x}, y\rangle^2\langle \hat{x}, n(y)\rangle\Big)\\
&-g_2(y)\,(3\langle \hat{x}, y\rangle|y|^2 + 4\langle \hat{x}, y\rangle^3/3) + 32f_2(y)\langle \hat{x}, n(y)\rangle\Big\}ds(y),\\
I_9 =& \int_{\partial D} \Big\{g_2(y)\Big(3|y|^4/4 - 2\langle \hat{x}, y\rangle^4/3 + 2\langle \hat{x}, y\rangle^2|y|^2\Big) - g_1(y)\Big[3|y|^2\,\langle y, n(y)\rangle\\
&-8\langle \hat{x}, y\rangle^3\langle \hat{x}, n(y)\rangle/3 + 4\langle \hat{x}, y\rangle\langle \hat{x}, n(y)\rangle|y|^2 + 4\langle \hat{x}, y\rangle^2\langle y, n(y)\rangle\Big]\\
&+16g_0(y)(|y|^2 + \langle \hat{x}, y\rangle^2) - 32f_2(y)\,(\langle y, n(y)\rangle + \langle \hat{x}, n(y)\rangle\langle \hat{x}, y\rangle)\\
&+96f_1(y)\Big\}ds(y),
\end{aligned}$$

where $\hat{x} = x/|x|$.

From the identity (4.1), letting $|x| \to \infty$ we obtain $I_i = 0$, $i = \overline{1,9}$. Similarly as in [8], it turns out that only two of them provide the necessary supplementary pieces of information, namely those given as $I_1 = I_5 = 0$, i.e.

$$\int_{\partial D} g_2(y)ds(y) = 0, \tag{4.2}$$

$$\int_{\partial D}\Big\{g_2(y)|y|^2 - 4g_1(y)\langle y, n(y)\rangle + 16g_0(y)\Big\}|y|^2 ds(y)$$
$$= 32\int_{\partial D}\Big[f_2(y)\,\langle y, n(y)\rangle - 2f_1(y)\Big]ds(y). \tag{4.3}$$

Considering again the circular domain ∂D_c given by (3.1), the homogeneous system corresponding to (4.2) and (4.3) yields

$$\int_0^{2\pi} cg_2(\theta)d\theta = 0, \tag{4.4}$$
$$\int_0^{2\pi} c^3\left\{g_2(\theta)c^2 - 4cg_1(\theta) + 16g_0(\theta)\right\}d\theta = 0. \tag{4.5}$$

In these conditions we insert the assumed Fourier series (3.5) and we obtain for the first terms

$$c^{-2}a_0^{(2)} = 0, \tag{4.6}$$
$$c^2(a_0^{(2)} - 4a_0^{(1)} + 16a_0^{(0)}) = 0. \tag{4.7}$$

Condition (4.6) alone gives $a_0^{(2)} = 0$, and then equations (3.6)-(3.8) yield

$$\begin{cases} -16a_0^{(0)}(1+\ln(c)) + a_0^{(1)}(5+6\ln(c)) = 0, \\ -2a_0^{(0)}(1+\ln(c)) + a_0^{(1)}(1+\ln(c)) = 0, \end{cases}$$

with the determinant

$$D_0 = -2(1+\ln(c))(3+2\ln(c)).$$

To summarise, equations (3.6)-(3.8) and condition (4.6) give:

$$\begin{aligned} &c = 1: \ \ a_0^{(0)} = a_0^{(1)} = a_0^{(2)} = 0, \\ &c = e^{-1}: \ \ a_0^{(1)} = a_0^{(2)} = 0, \ a_0^{(0)} \neq 0, \\ &c = e^{-3/2}: \ \ a_0^{(2)} = 0, \ a_0^{(1)} = 2a_0^{(0)} \neq 0. \end{aligned}$$

Also, condition (4.7) alone gives $[a_0^{(2)} - 4a_0^{(1)} + 16a_0^{(0)} = 0]$. If $c \neq 1$, then (3.8) gives $a_0^{(2)} = 0$, and the previous relation, and (3.6) and (3.7) give

$$a_0^{(1)} = 4a_0^{(0)}, \quad 16(1+2\ln(c))a_0^{(0)} = 0, \quad 4(1+\ln(c))a_0^{(0)} = 0,$$

so $a_0^{(0)} = a_0^{(1)} = 0$, and we have uniqueness. However, if $c = 1$, then (4.7) and (3.6)-(3.8) give

$$\begin{cases} 16a_0^{(0)} - 4a_0^{(1)} + a_0^{(2)} = 0, \\ -64a_0^{(0)} + 20a_0^{(1)} - 7a_0^{(2)} = 0, \\ -4a_0^{(0)} + 2a_0^{(1)} - a_0^{(2)} = 0, \end{cases}$$

with the determinant zero, hence a non-unique solution.

By imposing now both conditions (4.6) and (4.7) we obtain $a_0^{(2)} = 0$, $a_0^{(1)} = 4a_0^{(0)}$ and conditions (3.6) and (3.7) give

$$\begin{cases} 4a_0^{(0)}(1+\ln(c)) = 0, \\ 16a_0^{(0)}(1+2\ln(c)) = 0, \end{cases}$$

which gives $a_0^{(0)} = 0$, and hence $a_0^{(1)} = 0$. In conclusion, in accordance with [5,8], it is conjectured that the supplementary conditions (4.2) and (4.3) remove the non-uniqueness of the BIEs (2.7)-(2.9) for any contour ∂D.

Numerically, the BIEs (2.7)-(2.9) and the supplementary conditions (4.2) and (4.3) can be discretised using the BEM, as described in [3], and the resulting overdetermined system of linear algebraic equations solved using an ordinary least-squares procedure, as described in [13]. In the numerical computations [8, 13], the condition number of the resulting BEM-matrix blows up, as the boundary curve ∂D approaches one of its transfinite diameter values, if no supplementary condition is imposed.

5 Extension to polyharmonic equations

We consider the fundamental solution for the two-dimensional operator $-\nabla^{2k}$ in (1.3), given by [14],

$$G_k(x,y) = -\frac{1}{2\pi} r^{2k-2}(a_k \ln(r) - b_k), \quad r = |x-y|, \tag{5.1}$$

where a_k and b_k are constants satisfying the recurrence relations

$$a_k = \frac{a_{k-1}}{4(k-1)^2}, \quad b_k = \frac{1}{4(k-1)^2}\left(\frac{a_{k-1}}{k-1} + b_{k-1}\right), \quad k \geq 2, \tag{5.2}$$

and $a_1 = 1$, $b_1 = 0$. From (5.2) it can be seen that

$$a_k = \frac{1}{4^{k-1}((k-1)!)^2}, \quad k \geq 1, \tag{5.3}$$

and

$$b_k = \frac{b_{k-1}}{4(k-1)^2} + \frac{1}{(k-1)4^{k-1}((k-1)!)^2}, \quad k \geq 2, \tag{5.4}$$

starting with $b_1 = 0$. Solving the recurrence relation (5.4), we obtain

$$b_k = \frac{1}{4^{k-1}((k-1)!)^2} \sum_{l=1}^{k-1} \frac{1}{l}, \quad k \geq 1. \tag{5.5}$$

For engineering purposes, it is useful to list the first few fundamental solutions [15],

$$\begin{aligned} G_1 &= -\frac{1}{2\pi}\ln(r), & G_2 &= -\frac{1}{8\pi} r^2[\ln(r) - 1], \\ G_3 &= -\frac{1}{128\pi} r^4\Big[\ln(r) - \frac{3}{2}\Big], & G_4 &= -\frac{1}{4608\pi} r^4\Big[\ln(r) - \frac{11}{6}\Big], \end{aligned} \tag{5.6}$$

etc. From the previous analysis for the triharmonic Dirichlet problem (1.1) and (1.2), and the above expressions (5.6), looking at the free term, it can be seen that the Γ-contours of logarithmic capacities 1, e^{-1}, $e^{-3/2}$, $e^{-11/6}$, etc. occur for the polyharmonic Dirichlet problem (1.3)-(1.5). More concretely, it is conjectured that these Γ-contours have the logarithmic capacities

$$e^{-b_l/a_l} = exp\left(-\sum_{m=1}^{l-1}\frac{1}{m}\right), \quad \text{for } l = \overline{1,k}. \tag{5.7}$$

Eq.(5.7) gives the complete description of the Γ-contours for the Dirichlet problem for the polyharmonic equation given by (1.3)-(1.5).

6 Conclusions

In this paper, we have investigated the solvability of the BIEs arising from the application of the BEM for the two-dimensional triharmonic equation. In particular, the existence of three Γ-contours of logarithmic capacities 1, e^{-1} and $e^{-3/2}$ has been provided. The possible non-uniqueness for these particular geometries can be removed by using modified fundamental solutions, or more simply by imposing two supplementary conditions. Although the analysis was presented for the Dirichlet problem, it is believed that exactly the same situation occurs for mixed boundary conditions, as shown in [16] for Laplace's equation. Also, as remarked in [9] for the biharmonic equation, the Γ-contours do not exist in three-dimensions. Natural extensions to polyharmonic equations have also been provided in this paper. A recent work [17], showed that similar Γ-contours exists for two-dimensional Stokes equations of slow viscous fluid flows. Further work will concern the calculation of the condition number of the BEM-matrix arising from the triharmonic equation following the harmonic and biharmonic numerical analyses of [13, 16, 18].

References

[1] D. J. ACHESON , *Elementary Fluid Dynamics*, Oxford Univ. Press, Oxford, 1990.

[2] F. PAN AND A. ACRIVOS , *Steady flows in rectangular cavities*, J. Fluid Mech., 28 (1967), pp. 643-655.

[3] B. LONSDALE, M. I. G. BLOOR, AND M. A. KELMANSON, *An iterative integral-equation method for 6th-order inhomogeneous partial differential equations*, Boundary Element Technology XI, (eds. R. C. Ertekin, C. A. Brebbia, M. Tanaka and R. P. Shaw), Comput. Mech. Publ., Southampton, UK, (1996), pp.369-378.

[4] M. A. JASWON, *Integral equation methods in potential theory. I*, Proc. Roy. Soc., A275 (1963), pp. 23-32.

[5] S. CHRISTIANSEN, *Integral equations without a unique solution can be made useful for solving some plane harmonic problems*, J. Inst. Maths. Applics., 16 (1975), pp. 143-159.

[6] C. CONSTANDA, *On the solution of the Dirichlet problem for the two-dimensional Laplace equation*, Proc. Amer. Math. Soc., 119 (1993), pp. 877-884.

[7] G. S. GIPSON AND B. W. YEIGH, *The unit circle trap in boundary elements redux*, Eng. Anal. Boundary Elements, 32 (2008), pp. 431-437.

[8] S. CHRISTIANSEN AND P. HOUGAARD, *An investigation of a pair of integral equations for the biharmonic problem*, J. Inst. Maths. Applics., 22 (1978), pp. 15-27.

[9] B. FUGLEDE, *On a direct method of integral equations for solving the biharmonic Dirichlet problem*, ZAMM, 61 (1981), pp. 449-459.

[10] C. CONSTANDA, *On the Dirichlet problem for the two-dimensional biharmonic equation*, Math. Meth. Appl. Sci., 20 (1997), pp. 885-890.

[11] I. S. GRADSHTEYN AND I. M. RYZHIK, *Tables of Integrals, Series and Products*, Academic Press, London, (1965).

[12] E. HILLE, *Analytic Function Theory*, Ginn and Co., New York, Vol.II (1962).

[13] D. LESNIC, *The condition number of BEM-matrices arising from the biharmonic equation*, Advances in Boundary Integral Methods - Proceedings of the Sixth UK Conference on Boundary Integral Methods, (ed. J.Trevelyan), Durham University, UK, Ch.**30** (2007), pp. 283-290.

[14] C. A. BREBBIA AND A. J. NOWAK, *Treatment of domain integrals by using the dual and multiple reciprocity methods*, Discretisation Methods in Structure Mechanics, (eds. G. Kuhn and H. Mang), pp. 13-28.

[15] A. H. D. CHENG, H. ANTES AND N. ORTNER, *Fundamental solutions of products of Helmholtz and polyharmonic operators*, Eng. Anal. Boundary Elements, 14 (1994), pp. 187-191.

[16] W. DIJKSTRA AND R. M. M. MATTHEIJ, *The condition number of the BEM-matrix arising from Laplace's equation*, Electronic J. Boundary Elements, 4 (2006), pp. 67-81.

[17] W. DIJKSTRA AND R. M. M. MATTHEIJ, *Condition number of the BEM matrix arising from the Stokes equations in 2D*, Eng. Anal. Boundary Elements, 32 (2008), pp. 736-746.

[18] W. DIJKSTRA AND R. M. M. MATTHEIJ, *A relation between the logarithmic capacity and the condition number of the BEM-matrices*, Commun. Numer. Meth. Eng., 23 (2007), pp. 665-680.