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Discrete Maximum Principle for Poisson Equation with Mixed Boundary Conditions Solved by hp-FEM

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Abstract. We present a proof of the discrete maximum principle (DMP) for the 1D Poisson equation $-u''=f$ equipped with mixed Dirichlet-Neumann boundary conditions. The problem is discretized using finite elements of arbitrary lengths and polynomial degrees (hp-FEM). We show that the DMP holds on all meshes with no limitations to the sizes and polynomial degrees of the elements.

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1 Introduction

It is well known that the finite element solutions to elliptic and parabolic PDEs sometimes exhibit behavior which is incompatible with the corresponding maximum principles and, consequently, incompatible with the underlying physics. Most frequently this happens when a finite element mesh contains large dihedral angles, but also in other situations. Discrete maximum principles (DMP) provide additional restrictions on finite element meshes under which the maximum principles are preserved on the discrete level.

Up to our knowledge the first DMP were introduced in the 1960s [16]. In the 1970s

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DMP were used to prove the convergence of finite differences and lowest-order finite element methods (see, e.g., [3,4]). Nowadays the DMP play an important role in computational PDEs by guaranteeing that approximation of physically nonnegative quantities such as the density, temperature, concentration, or electric charge remains nonnegative. Due to the difficulty of the topic, current research in the area of DMP almost exclusively deals with lowest-order elements (see, e.g., [2,7–10,17,18,20]). However, in the last decades, significant progress has been made in the development of the hp-FEM (finite element methods with variable size and polynomial degree of elements) and their applications to challenging large-scale problems in computational science and engineering (see, e.g., [1,11,12,15]). These methods are substantially more efficient compared to standard lowest-order schemes, and an increasing demand for them implies a need for the corresponding generalizations of the DMP.

However, the generalization of the DMP to higher-order approximations is quite demanding and there only are a few known results in this direction. We mention paper [21] concerning the high-order collocation method and a negative result [6] showing that a nonstandard version of DMP is not valid for quadratic and higher-order FEM in 2D.

It was shown in [14] that the DMP cannot be extended from the lowest-order FEM to hp-FEM in a straightforward manner, and a weak DMP was introduced. Recently, a maximum principle for one-dimensional Poisson equation equipped with Dirichlet boundary conditions and discretized by hp-FEM was presented in [19]. The result was proved under a mild sufficient condition stating that the length of the longest element in the mesh must be less than 90% of the length of the entire domain. In this paper we investigate the case of mixed Neumann-Dirichlet boundary conditions. using different analytical methods. Interestingly, it turns out that in this case, the DMP holds true with no restrictions.

In general, the analysis of the DMP for mixed boundary conditions follows the same steps as the analysis for the Dirichlet conditions presented in [19]. Nevertheless, the stiffness matrices in both cases differ. Fortunately, even in the case of the mixed boundary conditions there exists an explicit formula for entries of the inverse stiffness matrix, see Lemma 4.1. Naturally, this formula differs from the case of the pure Dirichlet conditions. Consequently, the corresponding discrete Green's functions differ and, hence, we had to develop a new proof of its nonnegativity in the case of the mixed boundary conditions, see Section 5. Interestingly, the same quantity $H^*_{\mathrm{rel}}(p)$, where p stands for the polynomial degree, plays the crucial role in both cases. However, this role differs. While in the case of Dirichlet conditions the DMP is satisfied if the relative length of all elements is at most $H^*_{\mathrm{rel}}(p)$, in the case of mixed conditions it suffices for the validity of DMP to have $H^*_{\mathrm{rel}}(p) \geq 0$.

Furthermore, the nature of the maximum principle for the Dirichlet and for the mixed boundary conditions differs. In both cases the maximum principle is equivalent to the conservation of nonnegativity, see Definitions 2.1-2.3. However, in the case of Dirichlet conditions this equivalence is trivial and in the case of the mixed conditions the maximum principle implies the conservation of nonnegativity in a nontrivial way.