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Discrete Maximum Principle for Poisson Equation with Mixed Boundary Conditions Solved by hp-FEM

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Abstract. We present a proof of the discrete maximum principle (DMP) for the 1D Poisson equation $-u''=f$ equipped with mixed Dirichlet-Neumann boundary conditions. The problem is discretized using finite elements of arbitrary lengths and polynomial degrees (hp-FEM). We show that the DMP holds on all meshes with no limitations to the sizes and polynomial degrees of the elements.

AMS subject classifications: 65N30

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1 Introduction

It is well known that the finite element solutions to elliptic and parabolic PDEs sometimes exhibit behavior which is incompatible with the corresponding maximum principles and, consequently, incompatible with the underlying physics. Most frequently this happens when a finite element mesh contains large dihedral angles, but also in other situations. Discrete maximum principles (DMP) provide additional restrictions on finite element meshes under which the maximum principles are preserved on the discrete level.

Up to our knowledge the first DMP were introduced in the 1960s [16]. In the 1970s

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DMP were used to prove the convergence of finite differences and lowest-order finite element methods (see, e.g., [3,4]). Nowadays the DMP play an important role in computational PDEs by guaranteeing that approximation of physically nonnegative quantities such as the density, temperature, concentration, or electric charge remains nonnegative. Due to the difficulty of the topic, current research in the area of DMP almost exclusively deals with lowest-order elements (see, e.g., [2,7–10,17,18,20]). However, in the last decades, significant progress has been made in the development of the hp-FEM (finite element methods with variable size and polynomial degree of elements) and their applications to challenging large-scale problems in computational science and engineering (see, e.g., [1,11,12,15]). These methods are substantially more efficient compared to standard lowest-order schemes, and an increasing demand for them implies a need for the corresponding generalizations of the DMP.

However, the generalization of the DMP to higher-order approximations is quite demanding and there only are a few known results in this direction. We mention paper [21] concerning the high-order collocation method and a negative result [6] showing that a nonstandard version of DMP is not valid for quadratic and higher-order FEM in 2D.

It was shown in [14] that the DMP cannot be extended from the lowest-order FEM to hp-FEM in a straightforward manner, and a weak DMP was introduced. Recently, a maximum principle for one-dimensional Poisson equation equipped with Dirichlet boundary conditions and discretized by hp-FEM was presented in [19]. The result was proved under a mild sufficient condition stating that the length of the longest element in the mesh must be less than 90% of the length of the entire domain. In this paper we investigate the case of mixed Neumann-Dirichlet boundary conditions. using different analytical methods. Interestingly, it turns out that in this case, the DMP holds true with no restrictions.

In general, the analysis of the DMP for mixed boundary conditions follows the same steps as the analysis for the Dirichlet conditions presented in [19]. Nevertheless, the stiffness matrices in both cases differ. Fortunately, even in the case of the mixed boundary conditions there exists an explicit formula for entries of the inverse stiffness matrix, see Lemma 4.1. Naturally, this formula differs from the case of the pure Dirichlet conditions. Consequently, the corresponding discrete Green's functions differ and, hence, we had to develop a new proof of its nonnegativity in the case of the mixed boundary conditions, see Section 5. Interestingly, the same quantity $H^*_{\mathrm{rel}}(p)$, where p stands for the polynomial degree, plays the crucial role in both cases. However, this role differs. While in the case of Dirichlet conditions the DMP is satisfied if the relative length of all elements is at most $H^*_{\mathrm{rel}}(p)$, in the case of mixed conditions it suffices for the validity of DMP to have $H^*_{\mathrm{rel}}(p)\geq 0$.

Furthermore, the nature of the maximum principle for the Dirichlet and for the mixed boundary conditions differs. In both cases the maximum principle is equivalent to the conservation of nonnegativity, see Definitions 2.1-2.3. However, in the case of Dirichlet conditions this equivalence is trivial and in the case of the mixed conditions the maximum principle implies the conservation of nonnegativity in a nontrivial way.

2 The model problem and its discretization

We solve the one dimensional Poisson equation with mixed Dirichlet-Neumann boundary conditions,

$$
\begin{aligned}
-u'' &= f, && \text{in } \Omega, \\
u(\alpha) &= 0, && u'(\beta) = g(\beta).
\end{aligned}
$$

Here, $\Omega=(\alpha,\beta)\subset\mathbb{R}$ is an interval.

The corresponding weak formulation reads: Find $u\in V$ such that

$$a(u,v) = (f,v) + g(\beta)v(\beta), \qquad \forall v \in V, \tag{2.1}$$

where $V=\{v\in H^1(\Omega); v(\alpha)=0\}$, $f\in L^2(\Omega)$ is a right-hand side, $g(\beta)\in\mathbb{R}$, $(\cdot,\cdot)$ stands for an $L^2(\Omega)$ inner product, and $a(u,v)=(u',v')$.

In a standard way we create a partition $\alpha=x_0<x_1<\ldots<x_M=\beta$ of the domain Ω consisting of M elements $K_i=[x_{i-1},x_i]$, $i=1,2,\ldots,M$. Every element K_i is assigned an arbitrary polynomial degree $p_i\geq 1$. The corresponding finite element space $V_h\subset V$ of piecewise-polynomial and continuous functions has the form

$$V_{hp} = \left\{ v_{hp} \in V;\ v_{hp}|_{K_i} \in P^{p_i}(K_i),\ i = 1,2,\ldots,M \right\}.$$

Here $P^{p_i}(K_i)$ stands for the space of polynomials of degree at most p_i on the element K_i. The space V_{hp} has the dimension $N = \sum_{i=1}^{M} p_i$. There exists a unique finite element solution $u_{hp}\in V_{hp}$ satisfying

$$a(u_{hp},v_{hp}) = (f,v_{hp}) + g(\beta)v(\beta), \qquad \forall v_{hp} \in V_{hp}. \tag{2.2}$$

Definition 2.1. *Problem* (2.2) *satisfies the* discrete maximum principle (DMP) *if*

$$f \leq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \leq 0, \quad \Rightarrow \quad \max_{\overline{\Omega}} u_{hp} = \max_{\partial\Omega} u_{hp},$$

where $\partial\Omega$ is the boundary of the domain Ω.

Definition 2.2. *Problem* (2.2) *satisfies the* discrete minimum principle *if*

$$f \geq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \geq 0, \quad \Rightarrow \quad \min_{\overline{\Omega}} u_{hp} = \min_{\partial\Omega} u_{hp}.$$

Definition 2.3. *Problem* (2.2) conserves nonnegativity *if*

$$f \geq 0 \text{ a.e. in } \Omega \text{ and } g(\beta) \geq 0, \quad \Rightarrow \quad u_{hp} \geq 0 \text{ in } \Omega.$$

Clearly, the discrete maximum and minimum principles are equivalent for problem (2.2). We will use this equivalence and the following lemma to prove the DMP via conservation of nonnegativity.

Lemma 2.1. *If problem* (2.2) *conserves nonnegativity then it satisfies the discrete minimum principle.*

Proof. Since $u_{hp} \geq 0$ in Ω and $u_{hp}(\alpha)=0$, we conclude $\min_{\partial\Omega} u_{hp}=0=\min_{\overline{\Omega}} u_{hp}$. □

Remark 2.1. For the sake of simplicity, we formulated problem (2.2) with a homogeneous Dirichlet boundary condition $u(\alpha)=0$. However, all results of this study hold for a nonhomogeneous condition of the form $u(\alpha)=u_\alpha$. Indeed, the Dirichlet lift is constant in this case and every solution $\hat{u}_{hp}$ to problem (2.2) with nonhomogeneous condition $u(\alpha)=u_\alpha$ can be decomposed to

$$\hat{u}_{hp} = u_\alpha + u_{hp},$$

where u_{hp} vanishes at the endpoint α.

Remark 2.2. The Neumann boundary condition at the point β can be replaced by the more general Robin's boundary condition

$$u'(\beta) + \gamma u(\beta) = g(\beta), \quad \text{with } \gamma \geq 0.$$

The presented analysis can be generalized to this case as well.[†]

3 Discrete Green's function

The discrete Green's function (DGF) is defined in analogy to the standard Green's function:

Definition 3.1. *For an arbitrary $z \in \overline{\Omega}$, the unique solution $G_{hp,z} \in V_{hp}$ to the problem*

$$a(v_{hp}, G_{hp,z}) = v_{hp}(z), \qquad \forall v_{hp} \in V_{hp}, \tag{3.1}$$

is called the discrete Green's function (DGF) *corresponding to the point z.*

In the following, we will use the notation

$$G_{hp}(x,z) = G_{hp,z}(x), \quad \text{for } (x,z) \in \overline{\Omega}^2,$$

where $\overline{\Omega}^2 = \overline{\Omega} \times \overline{\Omega}$. A combination of (2.2) and (3.1) yields the so-called Kirchhoff-Helmholtz representation

$$u_{hp}(z) = \int_\Omega G_{hp}(x,z) f(x)\,\mathrm{d}x + g(\beta) G_{hp}(\beta,z), \qquad \forall z \in \overline{\Omega}. \tag{3.2}$$

The following lemma shows that the DGF can easily be expressed using any basis of V_{hp}, cf. [5]. We use the Kronecker symbol

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases}$$

[†]We thank Sergey Korotov from the Helsinki University of Technology for pointing this out.

Lemma 3.1. *Let* $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ *be a basis of* V_{hp}. *If the stiffness matrix* $A_{ij}=a(\varphi_j, \varphi_i)$, $1\leq i, j\leq N$ *is nonsingular, then*

$$G_{hp}(x,z) = \sum_{j=1}^{N}\sum_{k=1}^{N} A^{-1}_{jk}\varphi_k(x)\varphi_j(z). \tag{3.3}$$

Here, A^{-1}_{jk} *are the entries of the inverse stiffness matrix, i.e.,* $\sum\limits_{j=1}^{N} A_{ij}A^{-1}_{jk} = \delta_{ik}$, $1 \leq i,k \leq N$.

Proof. Substitute

$$G_{hp}(x,z) = \sum_{i=1}^{N} c_i(z)\varphi_i(x), \tag{3.4}$$

into (3.1) with $v_{hp}=\varphi_j$. It follows that

$$\sum_{i=1}^{N} c_i(z)\underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \varphi_j(z).$$

The coefficients $c_i(z)$ are expressed as $c_k(z)=\sum_{j=1}^{N}\varphi_j(z)A^{-1}_{jk}$ in terms of the inverse matrix, and they are substituted back into (3.4). This finishes the proof. □

Theorem 3.1. *Problem* (2.2) *conserves nonnegativity if and only if the corresponding discrete Green's function* $G_{hp}(x,z)=G_{hp,z}(x)$ *defined by* (3.1) *is nonnegative in* $\overline{\Omega}^2$.

Proof. By (3.3), the discrete Green's function $G_{hp}(x,z)$ is continuous up to the boundary of Ω. The rest follows immediately from (3.2). □

This theorem is a useful tool for the analysis of discrete maximum principles. In the rest of this paper we will show that the discrete Green's function corresponding to the problem (2.2) is nonnegative.

4 DGF for the model problem

4.1 Lowest-order case

In this section we will construct the DGF for problem (2.2). We begin with the case $p_1=p_2=\ldots=p_M=1$. Let us define $h_i=x_i - x_{i-1}$. By $\mathcal{B}^L=\{\phi_1, \phi_2, \ldots, \phi_M\}$ we denote the standard lowest-order basis consisting of the piecewise-linear "hat functions" such that $\phi_j(x_i)=\delta_{ij}$, $1 \leq i,j \leq M$. In this case the stiffness matrix $A^L\in\mathbb{R}^{M\times M}$ is tridiagonal,

$$A^L_{ij} = \begin{cases} 1/h_i + 1/h_{i+1}, & \text{for } i=j<M, \\ 1/h_M, & \text{for } i=j=M, \\ -1/h_{i+1}, & \text{for } i=j-1, \\ -1/h_{i-1}, & \text{for } i=j+1, \\ 0, & \text{otherwise}, \end{cases}$$

for $i,j = 1,2,\ldots,M$.

Lemma 4.1. *The inverse matrix* $(A^L)^{-1} \in \mathbb{R}^{M\times M}$ *has the form*

$$(A^L)^{-1} = \begin{pmatrix} x_1-\alpha & x_1-\alpha & x_1-\alpha & \dots & x_1-\alpha \\ x_1-\alpha & x_2-\alpha & x_2-\alpha & \dots & x_2-\alpha \\ x_1-\alpha & x_2-\alpha & x_3-\alpha & \dots & x_3-\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1-\alpha & x_2-\alpha & x_3-\alpha & \dots & x_M-\alpha \end{pmatrix},$$

i.e., $(A^L)^{-1}_{ij}=x_i-\alpha$ *for* $1\le i\le j\le M$ *and* $(A^L)^{-1}_{ij}=x_j-\alpha$ *for* $1\le j<i\le M$.

Proof. We want to show that $z_{ij}=\delta_{ij}$, where

$$z_{ij} = \sum_{k=1}^{M} (A^L)^{-1}_{ik} A^L_{kj} = \sum_{k=1}^{M} (A^L)^{-1}_{ik} a(\phi_j,\phi_k),$$

for all $i,j=1,2,\dots,M$. We fix i and j, and consider the bilinear forms

$$a_1(u,v) = \int_\alpha^{x_i} u'v'\,\mathrm{d}x \quad \text{and} \quad a_2(u,v) = \int_{x_i}^{\beta} u'v'\,\mathrm{d}x.$$

We use the explicit formulae for $(A^L)^{-1}_{ik}$ to get

$$z_{ij} = a\Big(\phi_j, \sum_{k=1}^{i-1}(x_k-\alpha)\phi_k\Big) + (x_i-\alpha)a(\phi_j,\phi_i) + (x_i-\alpha)a\Big(\phi_j, \sum_{k=i+1}^{M}\phi_k\Big).$$

Now, we split the term $a(\phi_j,\phi_i)=a_1(\phi_j,\phi_i)+a_2(\phi_j,\phi_i)$ to obtain

$$z_{ij} = a_1(\phi_j, x-\alpha) + (x_i-\alpha)a_2(\phi_j,1) = a_1(\phi_j,x-\alpha) = \delta_{ij},$$

where the last equality follows from a straightforward simple computation. □

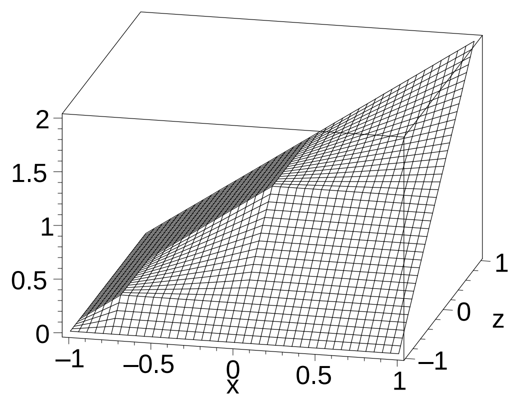

Figure 1: The lowest-order part $G^L_{hp}(x,z)$ of the discrete Green's function $G_{hp}(x,z)$ for the Poisson equation with homogeneous mixed boundary conditions in $\Omega=(-1,1)$ on a mesh with three elements $[-1,-3/4]$, $[-3/4,0]$, and $[0,1]$.

Using Lemma 4.1 and identity (3.3), we can write the DGF in the form

$$G_{hp}^{L}(x,z) = \sum_{i=1}^{M}(x_i - \alpha)\phi_i(x)\phi_i(z) + \sum_{i=1}^{M-1}\sum_{j=i+1}^{M}(x_i - \alpha)[\phi_i(x)\phi_j(z) + \phi_j(x)\phi_i(z)]. \tag{4.1}$$

In particular, we see immediately that

$$G_{hp}^{L}(x,z) \geq 0, \quad \forall (x,z) \in \overline{\Omega}^2. \tag{4.2}$$

The situation is illustrated in Fig. 1.

4.2 Higher-order case

In this paragraph we return to the original setting with arbitrary polynomial degrees $p_i \geq 1$. In order to facilitate the construction of higher-order basis functions of the space V_{hp}, let us introduce the Lobatto shape functions $l_0, l_1, l_2, \ldots$ on a reference interval $\hat{K}=[-1,1]$, see, e.g., [12,15] and (7.1) in Appendix.

The lowest-order Lobatto shape functions l_0 and l_1 have the form $l_0(\xi)=(1-\xi)/2$, $l_1(\xi)=(1+\xi)/2$, $\xi \in \hat{K}$. The higher-order shape functions $l_2, l_3, \ldots$ are defined as antiderivatives to the Legendre polynomials. Therefore, they satisfy

$$\int_{-1}^{1} l_k'(\xi) l_m'(\xi)\,\mathrm{d}\xi = \delta_{km}, \quad k,m = 2,3,\ldots.$$

Every Lobatto shape function l_k, $k=2,3,\ldots$, is a polynomial of degree k and it vanishes at ± 1. Thus it can be expressed as

$$l_{k+2}(\xi) = l_0(\xi) l_1(\xi) \kappa_k(\xi), \quad k = 0,1,2,\ldots,$$

where κ_k is a polynomial of degree k. For reference, a first few kernels κ_k are listed in Appendix.

The basis $\mathcal{B}=\{\phi_1, \phi_2, \ldots, \phi_N\}$ of V_{hp} can be written as $\mathcal{B}=\mathcal{B}^L \cup \mathcal{B}^B$, where $\mathcal{B}^L$ was defined above and $\mathcal{B}^B$ is the higher-order part of the basis comprising functions ϕ_M, $\phi_{M+1}, \ldots, \phi_N$. These are defined in a standard way as follows:

Consider the standard affine transformations of the reference element $\hat{K}$ to an element $K_i=[x_{i-1}, x_i]$, $i=1,2,\ldots,M$,

$$\chi_{K_i}(\xi) = \frac{(x_i - x_{i-1})\xi + (x_i + x_{i-1})}{2}. \tag{4.3}$$

On an element K_i of the polynomial degree p_i, there are $p_i - 1$ higher-order basis functions. These vanish outside of K_i and in K_i they are defined as the Lobatto shape functions $l_2, l_3, \ldots, l_{p_i}$ composed with the inverse map $\chi_{K_i}^{-1}(x)$.

Lemma 4.2. *We have the following orthogonality relations:*

$$a(\phi^L, \phi^B) = 0, \qquad \forall \phi^L \in \mathcal{B}^L,\ \forall \phi^B \in \mathcal{B}^B,$$
$$a(\phi^B, \psi^B) = 0, \qquad \forall \phi^B \in \mathcal{B}^B,\ \forall \psi^B \in \mathcal{B}^B,\ \phi^B \neq \psi^B.$$

Proof. The proof is straightforward, based on the L^2-orthogonality of the Legendre polynomials. □

By Lemma 4.2, both the stiffness matrix A and its inverse have the following block structure:

$$A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix},$$

with

$$D = \operatorname{diag}\Big(\underbrace{\frac{2}{h_1}, \ldots, \frac{2}{h_1}}_{(p_1-1)\text{ times}}, \underbrace{\frac{2}{h_2}, \ldots, \frac{2}{h_2}}_{(p_2-1)\text{ times}}, \ldots, \underbrace{\frac{2}{h_M}, \ldots, \frac{2}{h_M}}_{(p_M-1)\text{ times}}\Big). \tag{4.4}$$

By (3.3), the DGF can be written as

$$G_{hp}(x,z) = G^L_{hp}(x,z) + G^B_{hp}(x,z), \tag{4.5}$$

where $G^L_{hp}(x,z)$ corresponds to (4.1) and

$$G^B_{hp}(x,z) = \sum_{k=M}^{N} D^{-1}_{jj} \phi_j(x) \phi_j(z), \qquad \forall (x,z) \in \overline{\Omega}^2. \tag{4.6}$$

Unfortunately, $G^B_{hp}(x,z)$ defined by (4.6) is not nonnegative in the entire $\overline{\Omega}^2$ in general. For instance, in the example shown in Fig. 2, there are small regions near the points $(1,0)$ and $(0,1)$, where the function $G^B_{hp}(x,z)$ is negative.

Notice that any partition of $\overline{\Omega}$ produces a rectangular grid on $\overline{\Omega}^2$, and that $G^B_{hp}(x,z)$ can be nonzero within the diagonal squares of this grid only. In other words,

$$\operatorname{supp} G^B_{hp} \subset \bigcup_{i=1}^{M} K_i^2. \tag{4.7}$$

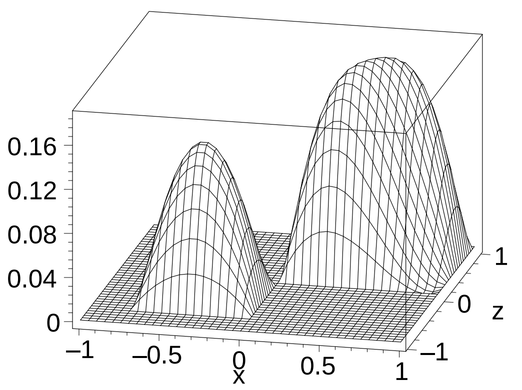

Figure 2: The higher-order part $G^B_{hp}(x,z)$ of the discrete Green's function $G_{hp}(x,z)$ for the Poisson equation with homogeneous mixed boundary conditions in $\Omega = (-1,1)$, on a mesh with three elements $[-1,-3/4]$, $[-3/4,0]$, and $[0,1]$ of the polynomial degrees p_1=1, p_2=2, p_3=3.

Lemma 4.3. *The discrete Green's function G_{hp} defined by* (4.5) *is nonnegative in* $\overline{\Omega}^2 \setminus \bigcup_{i=1}^M K_i^2$.

Proof. Considering (4.7) together with (4.2) leads to the conclusion. $\square$

5 The DGF on K_i^2

As justified by Lemma 4.3, we only need to continue with the study of the discrete Green's function $G_{hp}(x,z)$ in the union of the diagonal squares $\bigcup_{i=1}^M K_i^2$. Without loss of generality, let us restrict ourselves to only one square K_i^2, $1 \le i \le M$. Let $p = p_i$ be the polynomial degree assigned to K_i. Notice that only a few terms in (4.1) and (4.6) are nonzero in K_i^2. Hence, by (4.1), (4.4), and (4.6) we obtain

$$\begin{aligned} G_{hp}(x,z)\big|_{K_i^2} = & (x_i - \alpha)\phi_i(x)\phi_i(z) + (x_{i-1} - \alpha)\phi_{i-1}(x)\phi_{i-1}(z) \\ & + (x_{i-1} - \alpha)\left[\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)\right] \\ & + \frac{x_i - x_{i-1}}{2} G_{hp}^B(x,z)|_{K_i^2}, \end{aligned} \tag{5.1}$$

for $(x,z) \in K_i^2$, $1 \le i \le M$. It is convenient to introduce the notation $K_i = [x_{i-1}, x_i] = [L, R]$.

We transform the function G_{hp} from K_i^2 to the reference square $\hat{K}^2 = [-1,1]^2$ using the linear transformation (4.3) with $x = \chi_{K_i}(\xi)$ and $z = \chi_{K_i}(\eta)$,

$$\begin{aligned} & G_{hp}(x,z)\big|_{K_i^2} = \hat{G}_{hp}(\xi,\eta) \\ = & (R-\alpha) l_1(\xi) l_1(\eta) + (L-\alpha) l_0(\xi) l_0(\eta) \\ & + (L-\alpha)\left[l_1(\xi) l_0(\eta) + l_0(\xi) l_1(\eta)\right] + \frac{R-L}{2} \hat{G}_{hp}^{p,B}(\xi,\eta), \end{aligned} \tag{5.2}$$

for $(\xi,\eta) \in \hat{K}^2$. Here $l_0(\xi)$ and $l_1(\xi)$ are the above-defined lowest-order shape functions on $\hat{K}$ and

$$\hat{G}_{hp}^{p,B}(\xi,\eta) = \sum_{m=2}^{p} l_m(\xi) l_m(\eta) = l_0(\xi) l_0(\eta) l_1(\xi) l_1(\eta) \sum_{k=0}^{p-2} \kappa_k(\xi) \kappa_k(\eta), \tag{5.3}$$

is the higher-order part.

Let us modify formula (5.2) in the following way: Divide (5.2) by $R - L > 0$ and use the identities

$$\frac{R-\alpha}{R-L} = \frac{L-\alpha}{R-L} + 1,$$

and

$$l_0(\xi) l_0(\eta) + l_1(\xi) l_1(\eta) + l_0(\xi) l_1(\eta) + l_1(\xi) l_0(\eta) = 1, \qquad \forall (\xi,\eta) \in \hat{K}^2.$$

We obtain

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{L-\alpha}{R-L} + l_1(\xi) l_1(\eta) + \frac{1}{2} \hat{G}_{hp}^{p,B}(\xi,\eta). \tag{5.4}$$

Table 1: The quantity $H^*_{\text{rel}}(p)$ for $p = 1, 2, 3, \ldots, 20$.

p	$H^*_{\text{rel}}(p)$	p	$H^*_{\text{rel}}(p)$	p	$H^*_{\text{rel}}(p)$	p	$H^*_{\text{rel}}(p)$
1	1	6	1	11	0.953759	16	0.968695
2	1	7	0.935127	12	0.969485	17	0.967874
3	9/10	8	0.987060	13	0.959646	18	0.969629
4	1	9	0.945933	14	0.968378	19	0.970855
5	0.919731	10	0.973952	15	0.964221	20	0.970814

Using (5.3), this formula can be reshaped into

$$\frac{\hat{G}_{hp}(\xi,\eta)}{R-L} = \frac{L-\alpha}{R-L} + l_1(\xi) l_1(\eta)\Big[1 + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=0}^{p-2} \kappa_k(\xi)\kappa_k(\eta)\Big]. \tag{5.5}$$

Clearly, $(L-\alpha)/(R-L) \geq 0$ and $l_1(\xi)l_1(\eta) \geq 0$ in $\hat{K}^2$. It remains to verify nonnegativity of the expression in the square brackets. For this reason we define

$$\begin{aligned} H^*_{\text{rel}}(p) &= 1, \qquad \text{for } p = 1, \\ H^*_{\text{rel}}(p) &= 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=0}^{p-2} \kappa_k(\xi)\kappa_k(\eta), \quad \text{for } p \geq 2. \end{aligned}$$

Hence, if $H^*_{\text{rel}}(p) \geq 0$ then $\hat{G}_{hp}(\xi,\eta) \geq 0$ in $\hat{K}^2$ by (5.5). Transforming (ξ,η) back to (x,z) by (4.3), we obtain nonnegativity of $G_{hp}(x,z)$ in K_i^2, cf. (5.2), for all $i=1,2,\ldots,M$. Thus, in view of Lemma 4.3 we showed that the discrete Green's function $G_{hp}(x,z) \geq 0$ in $\overline{\Omega}^2$, provided $H^*_{\text{rel}}(p_i) \geq 0$ for all $i=1,2,\ldots,M$.

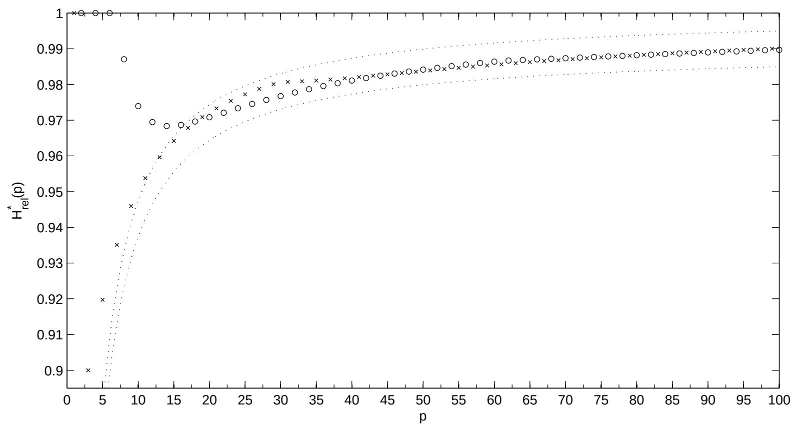

Figure 3: The values $H^*_{\text{rel}}(p)$ for $p = 1, 2, \ldots, 104$. Circles indicate the values for p odd and crosses for p even. The upper dotted line is a graph of $1 + 0.5\ln(1 - 1/x)$ and the bottom line is a shift of this graph by -0.01.

In [19] it was verified that $H^*_{\text{rel}}(p) \geq 0$ for $1 \leq p \leq 100$. More precisely, the value of $H^*_{\text{rel}}(p)$ can be found analytically for $2 \leq p \leq 4$. For $5 \leq p \leq 100$, it needs to be computed numerically. As it is seen from Table 1 and Fig. 3, the smallest value of $H^*_{\text{rel}}(p)$ is for p=3 and it equals to 9/10. Thus, the crucial quantity $H^*_{\text{rel}}(p)$ was checked to be nonnegative for $1 \le p \le 100$. This, consequently, shows nonnegativity of the discrete Green's function in $\overline{\Omega}^2$ and validity of the discrete maximum principle.

6 Main result

Let us summarize the conclusions of the previous analysis:

Theorem 6.1. *Let $\alpha = x_0 < x_1 < \ldots < x_M = \beta$ be a partition of the domain $\Omega = (\alpha, \beta)$ and let $p_i \ge 1$ be a polynomial degree assigned to the element $K_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, M$. If*

$$H^*_{\text{rel}}(p_i) \ge 0 \quad \textit{for all } i = 1, 2, \ldots, M, \tag{6.1}$$

then problem (2.2) *satisfies the discrete maximum principle*

Proof. Let K_i be an element. By (5.2), (5.5), and (6.1) it holds

$$G_{hp}(x,z)|_{K_i^2} = \hat{G}_{hp}(\xi, \eta) \ge 0$$

for all $(x,z) \in K_i^2$ with $\xi = \chi^{-1}_{K_i}(x)$ and $\eta = \chi^{-1}_{K_i}(z)$. Thus, $G_{hp}(x,z) \ge 0$ in $\bigcup_{i=1}^M K_i^2$. Lemma 4.3 implies that $G_{hp}(x,z) \ge 0$ also in $\overline{\Omega}^2 \setminus \bigcup_{i=1}^M K_i^2$. Theorem 3.1 and Lemma 2.1 finish the proof. $\square$

The crucial condition (6.1) was verified analytically for $p \le 4$, therefore Theorem 6.1 proves the discrete maximum principle for problem (2.2) for all meshes and arbitrary polynomial degrees not exceeding 4. However, numerical calculations of $H^*_{\text{rel}}(p)$ show that the condition (6.1) is satisfied for $5 \le p \le 100$ as well. Moreover, the steadily growing trend in H^*_{rel} for $p \ge 50$ observed in Fig. 3 motivates the following conjecture:

Conjecture 1. *The problem* (2.2) *satisfies the discrete maximum principle for arbitrary partition of the domain $\Omega = (\alpha, \beta)$ and for arbitrary distribution of polynomial degrees.*

7 Conclusions and further generalizations

We proved the DMP for the 1D Poisson problem solved by the sophisticated *hp*-version of the FEM. The next natural step is to generalize this result for more general problems in two (or more) dimensions.

Since the key ingredients (Lemma 3.1 and Theorem 3.1) are valid for arbitrary elliptic operator in arbitrary dimension, the presented approach can be, in principle, extended to prove the DMP even in more general settings. However, the conditions for the mesh and polynomial degrees which would guarantee the DMP are then more difficult to find.

More general operators, for example the diffusion-reaction operator, bring difficulties such as (i) the non-existence of a simple formula for the inverse of the stiffness matrix, cf. Lemma 4.1, and (ii) non-orthogonality of the bubble functions to the vertex ones, cf. Lemma 4.2. These difficulties can be treated for instance in the following way. In case (i) we have to find suitable lower bounds for the entries of the inverse stiffness matrix. This can be done by analysing simplified meshes with a few elements and showing that their refinement leads to an increase of nodal values of the discrete Green's function. Difficulty (ii) is not fundamental and it can be treated by orthogonalization of the vertex functions with respect to bubbles (the concept of the discrete minimum energy extensions).

With no doubts, the significance of the hp-FEM lies in 2D and 3D problems. When extending the DMP results to higher-order methods in higher spatial dimensions, one has to overcome not only the two difficulties mentioned above but also (iii) the presence of the edge (and face) basis functions. These basis functions make the process of orthogonalization of the vertex functions to the other basis functions non-local which makes the analysis more demanding but treatable.

The search for suitable conditions for more general and higher dimensional problems is a challenging task of high practical significance. Generalizations of the presented results are desirable because conditions guaranteeing the physical admissibility of hp-FEM approximations are valuable from the practical point of view, and they are demanded from the engineering community.

Appendix

The Lobatto shape functions are defined by

$$l_m(\xi) = \sqrt{\frac{2m-1}{2}} \int_{-1}^{\xi} P_{m-1}(x)\,\mathrm{d}x, \qquad m = 2, 3, \dots, \tag{7.1}$$

where

$$P_m(x) = \mathrm{d}^m/\mathrm{d}x^m (x^2-1)^m/(2^m m!),$$

stands for the mth-degree Legendre polynomial. The kernels are defined by

$$\kappa_k(\xi) = l_{k+2}(\xi)/(l_0(\xi) l_1(\xi)),\ k = 0, 1, 2, \dots,$$

where

$$l_0(\xi) = (1-\xi)/2, l_1(\xi) = (1+\xi)/2, \xi \in [-1, 1].$$

These kernels can be generated by the recurrence

$$\frac{k+4}{\sqrt{2k+7}} \kappa_{k+2}(\xi) = \sqrt{2k+5}\,\xi \kappa_{k+1}(\xi) - \frac{k+1}{\sqrt{2k+3}} \kappa_k(\xi), \qquad k = 0, 1, 2, \dots.$$

Interesting observation is that these kernels are scaled derivatives of Legendre polynomials

$$\kappa_k(\xi) = -\frac{\sqrt{8(2k+3)}}{(k+2)(k+1)} P'_{k+1}(\xi), \qquad k = 0, 1, 2, \ldots.$$

Hence, they form a system of orthogonal polynomials with weight $1 - \xi^2 = 4l_0(\xi)l_1(\xi)$. For reference, we list several kernel functions κ_k (see, e.g., Section 3.1 in [15] or Section 1.2 in [13]):

$$\begin{aligned}
\kappa_0(\xi) &= -\sqrt{6}, \quad \kappa_1(\xi) = -\sqrt{10}\xi, \\
\kappa_2(\xi) &= -\frac{1}{4}\sqrt{14}(5\xi^2 - 1), \\
\kappa_3(\xi) &= -\frac{3}{4}\sqrt{2}(7\xi^2 - 3)\xi, \\
\kappa_4(\xi) &= -\frac{1}{8}\sqrt{22}(21\xi^4 - 14\xi^2 + 1), \\
\kappa_5(\xi) &= -\frac{1}{8}\sqrt{26}(33\xi^4 - 30\xi^2 + 5)\xi, \\
\kappa_6(\xi) &= -\frac{1}{64}\sqrt{30}(429\xi^6 - 495\xi^4 + 135\xi^2 - 5), \\
\kappa_7(\xi) &= -\frac{1}{64}\sqrt{34}(715\xi^6 - 1001\xi^4 + 385\xi^2 - 35)\xi, \\
\kappa_8(\xi) &= -\frac{1}{128}\sqrt{38}(2431\xi^8 - 4004\xi^6 + 2002\xi^4 - 308\xi^2 + 7).
\end{aligned}$$

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