Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 2, pp. 242-256 (2009) April 2009

A Posteriori Error Estimates of Triangular Mixed Finite Element Methods for Semilinear Optimal Control Problems

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Received 31 October 2008; Accepted (in revised version) 24 December 2008

Available online 17 March 2009

Abstract. In this paper, we present an a posteriori error estimates of semilinear quadratic constrained optimal control problems using triangular mixed finite element methods. The state and co-state are approximated by the order $k \leq 1$ Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant element. We derive a posteriori error estimates for the coupled state and control approximations. A numerical example is presented in confirmation of the theory.

AMS subject classifications: 49J20, 65N30

Key words: Semilinear optimal control problems; mixed finite element methods; a posteriori error estimates.

1 Introduction

Optimal control problems have attracted substantial interest in recent years due to their applications in aero-hydrodynamics, combustion, exploration and extraction of oil and gas resources, and engineering. The past decade has seen significant developments in theoretical and computational methods for optimal control problems. The finite element method is a valid numerical method of studying the partial differential equation, but it is not deeply studied in solving optimal control problems. For optimal control problems governed by linear elliptic equations, there are some pioneering

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work on numerical approximation by Falk [9] and Mossino [21]. An optimal control problem for a two-dimensional elliptic equation is investigated with pointwise control constraints in Meyer and Rösch [19]. A systematic introduction of the finite element method for optimal control problems can be found in, for instance, [12, 13, 16] and the references cited therein. Most of these researches have been, however, only for the standard finite element methods for optimal control problems.

In many optimal control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is very important in the numerical discretization of the state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. In [5, 23] the authors presented a priori error estimates and superconvergence of mixed finite element methods for linear optimal control problems. However, there does not seem to exist much work on theoretical estimates of mixed finite element methods for nonlinear optimal control problems.

Adaptive finite element approximation is a most important means to boost accuracy and efficiency of the finite element discretization. Adaptive finite element approximation uses a posteriori error indicator to guide the mesh refinement procedure. In [25], the author proposed a posteriori error estimates of gradient recovery type for linear optimal control problems. Liu and Yan investigated a posteriori error estimates and adaptive finite element approximation for optimal control problems governed by Stokes equations in [18]. In [3, 4, 24], we derived a priori error estimates and superconvergence for linear quadratic optimal control problems using mixed finite element methods. A posteriori error estimates of mixed finite element methods for general convex optimal control problems was addressed in [6–8].

The purpose of this work is to obtain a posteriori error estimates of triangular mixed finite element methods for quadratic optimal control problems governed by semilinear elliptic equations. Compared with the related work [11], the present paper gives the first a posteriori error estimate for semilinear quadratic optimal control problems when they are discretized by Raviart-Thomas mixed finite element methods.

In this paper, we consider the following quadratic optimal control problems governed by semilinear elliptic equations:

$$\min_{u\in K\subset U}\Big\{\frac{1}{2}\parallel \boldsymbol{p}-\boldsymbol{p}_d\parallel^2+\frac{1}{2}\parallel y-y_d\parallel^2+\frac{\nu}{2}\parallel u\parallel^2\Big\}, \tag{1.1}$$

$$\mathrm{div}\boldsymbol{p}+\phi(y)=u, \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{p}=-A\nabla y, \qquad \text{in } \Omega, \tag{1.3}$$

$$y=0, \qquad \text{on } \partial\Omega, \tag{1.4}$$

where the bounded open set $\Omega\subset\mathbb{R}^2$, is a convex polygon with boundary $\partial\Omega$, $f\in L^2(\Omega)$, and K is a closed convex set in $L^2(\Omega)$. For any $R>0$ the function $\phi(\cdot)\in W^{2,\infty}(-R,R)$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'(y)\geq 0$. Furthermore, we assume the coefficient matrix

$$A(x)=(a_{i,j}(x))_{2\times 2}\in (W^{1,\infty}(\Omega))^{2\times 2},$$

is a symmetric 2×2-matrix and there is a constant $c>0$ satisfying for any vector $\mathbf{X}\in\mathbb{R}^2$, $\mathbf{X}^t A\mathbf{X}\geq c\|\mathbf{X}\|^2_{\mathbb{R}^2}$.

In this paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p}=\sum_{|\alpha|\leq m}\|D^\alpha v\|^p_{L^p(\Omega)}$. We set

$$W^{m,p}_0(\Omega)=\{v\in W^{m,p}(\Omega):v\,|_{\partial\Omega}=0\}.$$

For $p=2$, we denote

$$H^m(\Omega)=W^{m,2}(\Omega),\quad H^m_0(\Omega)=W^{m,2}_0(\Omega),\ \text{ and }\|\cdot\|_m=\|\cdot\|_{m,2},\quad \|\cdot\|=\|\cdot\|_{0,2}.$$

In addition C or c denotes a general positive constant independent of h.

The rest of this paper is organized as follows. In section 2, we construct the triangular mixed finite element discretization for quadratic constrained optimal control problems governed by semilinear elliptic equations. In section 3, a posteriori error estimates are derived for semilinear optimal control problems using Raviart-Thomas mixed finite element methods. A numerical example is presented in section 4.

2 Mixed methods for optimal control problems

In this section, we study the mixed finite element discretization of the semilinear quadratic optimal control problems (1.1)-(1.4). Let $\boldsymbol{V}=H(\text{div};\Omega)=\{\boldsymbol{v}\in(L^2(\Omega))^2, \text{div}\boldsymbol{v}\in L^2(\Omega)\}$ endowed with the norm given by $\|\boldsymbol{v}\|_{H(\text{div};\Omega)}=(\|\boldsymbol{v}\|^2_{0,\Omega}+\|\text{div}\boldsymbol{v}\|^2_{0,\Omega})^{1/2}$ and $W=U=L^2(\Omega)$. We recast (1.1)-(1.4) as the following weak form: find $(\mathbf{p},y,u)\in\boldsymbol{V}\times W\times U$ such that

$$\min_{u\in K\subset U}\Big\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\Big\}, \tag{2.1}$$

$$(A^{-1}\boldsymbol{p},\boldsymbol{v})-(y,\text{div}\boldsymbol{v})=0,\qquad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.2}$$

$$(\text{div}\boldsymbol{p},w)+(\phi(y),w)=(u,w),\qquad \forall w\in W. \tag{2.3}$$

Similar to [3], it can be proved that the optimal control problem (2.1)-(2.3) has a unique solution $(\boldsymbol{p},y,u)$, and that a triplet $(\boldsymbol{p},y,u)$ is the solution of (2.1)-(2.3) if and only if there is a co-state $(\boldsymbol{q},z)\in V\times W$ such that $(\boldsymbol{p},y,\boldsymbol{q},z,u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p},\boldsymbol{v})-(y,\text{div}\boldsymbol{v})=0,\qquad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.4}$$

$$(\text{div}\boldsymbol{p},w)+(\phi(y),w)=(u,w),\qquad \forall w\in W, \tag{2.5}$$

$$(A^{-1}\boldsymbol{q},\boldsymbol{v})-(z,\text{div}\boldsymbol{v})=-(\boldsymbol{p}-\boldsymbol{p}_d,\boldsymbol{v}),\qquad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{2.6}$$

$$(\text{div}\boldsymbol{q},w)+(\phi'(y)z,w)=(y-y_d,w),\qquad \forall w\in W, \tag{2.7}$$

$$(z+\nu u,\tilde{u}-u)_U\geq 0,\qquad \forall\tilde{u}\in U, \tag{2.8}$$

where $(\cdot,\cdot)_U$ is the inner product of U. In the rest of the paper, we shall simply write the product as $(\cdot,\cdot)$ whenever no confusion should be caused.

For ease of exposition we will assume that Ω is a convex polygon. Let $\mathcal{T}_h$ be regular triangulation of Ω, where $|\tau|$ is the area of τ, h_τ is the diameter of τ and $h = \max h_\tau$.

Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the order $k \leq 1$ Raviart-Thomas mixed finite element space [22], namely, $V_k(\tau) = P_k^2 + x \cdot P_k$, $W_k(\tau) = P_k$, where P_k denotes the space of polynomials of total degree at most k, $x = (x_1, x_2)$ which treated as a vector, and

$$\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{V}: \ \forall \tau \in \mathcal{T}_h, \ \boldsymbol{v}_h|_\tau \in \boldsymbol{V}_k(\tau)\},\\
W_h &:= \{w_h \in W: \ \forall \tau \in \mathcal{T}_h, \ w_h|_\tau \in W_k(\tau)\},\\
U_h &:= \{\tilde{u}_h \in U: \ \forall \tau \in \mathcal{T}_h, \ \tilde{u}_h|_\tau \in P_0(\tau)\}.
\end{aligned}$$

By the definition of finite element subspace, the mixed finite element discretization of (2.1)-(2.3) is as follows: compute $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \Big\{\frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{v}{2}\|u_h\|^2\Big\}, \tag{2.9}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.10}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \qquad \forall w_h \in W_h. \tag{2.11}$$

Similarly, optimal control problem (2.9)-(2.11) again has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.9)-(2.11) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.12}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \qquad \forall w_h \in W_h, \tag{2.13}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.14}$$

$$(\mathrm{div}\boldsymbol{q}_h, w_h) + (\phi'(y_h) z_h, w_h) = (y_h - y_d, w_h), \qquad \forall w_h \in W_h, \tag{2.15}$$

$$(z_h + v u_h, \tilde{u}_h - u_h) \geq 0, \qquad \forall \tilde{u}_h \in U_h. \tag{2.16}$$

Now, let us give the local definition of these differential operators (understood in the distributional sense), namely, div_h, curl_h: $H^1(\mathcal{T}_h)^2 \to L^2(\Omega)$ and ∇_h, Curl_h: $H^1(\mathcal{T}_h) \to L^2(\Omega)^2$ defined such that for any $\tau \in \mathcal{T}_h$:

$$\begin{aligned}
&\mathrm{div}_h \boldsymbol{v}|_\tau := \mathrm{div}(\boldsymbol{v}|_\tau), \qquad &&\mathrm{curl}_h \boldsymbol{v}|_\tau := \mathrm{curl}(\boldsymbol{v}|_\tau),\\
&\nabla_h \boldsymbol{v}|_\tau := \nabla(\boldsymbol{v}|_\tau), \qquad &&\mathrm{Curl}_h \boldsymbol{v}|_\tau := \mathrm{Curl}(\boldsymbol{v}|_\tau).
\end{aligned}$$

Let $\mathcal{E}_h$ denote the set of element sides in $\mathcal{T}_h$. If there is no risk of confusion the local mesh size h is defines on both $\mathcal{T}_h$ and $\mathcal{E}_h$ by $h|_\tau := h_\tau$ for $\tau \in \mathcal{T}_h$ and $h|_E := h_E$ for $E \in \mathcal{E}_h$, respectively. For all $E \in \mathcal{E}_h$ we fix one direction of a unit normal on E pointing in the outside of Ω in case that $E \subset \partial\Omega$. We define an operator $[v]$: $H^1(\mathcal{T}_h) \to L^2(\mathcal{E}_h)$ is the jump of the function v across the edge E, and $\mathbf{t}$ being the tangential unit vector along E.

Define $S^0(\mathcal{T}_h) \subset L^2(\Omega)$ as the piecewise constant space and $S^1(\mathcal{T}_h) \subset H^1(\Omega)$ or $S^1_0(\mathcal{T}_h) \subset H^1_0(\Omega)$ as continuous and piecewise linear functions, piecewise is understood with respect to $\mathcal{T}_h$. We consider Clement's interpolation operator $\hat{\pi}_h$: $H^1(\Omega)\to S^1(\mathcal{T}_h)$ which satisfies [5]:

$$\|v-\hat{\pi}_h v\|_{0,\tau} \leq Ch_\tau \|v\|_{1,w_\tau}, \qquad \forall v \in H_0^1(\Omega), \tag{2.17}$$

$$\|v-\hat{\pi}_h v\|_{0,E} \leq Ch_E^{1/2} \|v\|_{1,w_E}, \qquad \forall v \in H_0^1(\Omega), \tag{2.18}$$

for each $\tau \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$, $w_\tau = \{\tau' \in \mathcal{T}_h, \bar{\tau} \cap \bar{\tau}' \neq \emptyset\}$, $w_E = \{\tau \in \mathcal{T}_h, E \in \bar{\tau}\}$.

Now, we define the standard $L^2(\Omega)$-orthogonal projection P_h: $W\to W_h$, which satisfies the approximation property [10]:

$$\|h^{-1}\cdot(v-P_h v)\|_{0,\Omega} \leq C\|\nabla_h v\|_{0,\Omega}, \qquad \forall v \in H^1(\mathcal{T}_h). \tag{2.19}$$

Let us define the projection operator Π_h: $\boldsymbol{V}\to\boldsymbol{V}_h$, which satisfies: for any $\mathbf{q}\in\boldsymbol{V}$

$$\int_E w_h(\boldsymbol{q}-\Pi_h\boldsymbol{q})\cdot\nu_E ds = 0, \qquad \forall w_h \in W_h,\ E\in\mathcal{E}_h, \tag{2.20}$$

$$\int_\tau (\boldsymbol{q}-\Pi_h\boldsymbol{q})\cdot\boldsymbol{v}_h dxdy = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\ \tau\in\mathcal{T}_h. \tag{2.21}$$

Then, the interpolation operator Π_h satisfies a local error estimate:

$$\|h^{-1}\cdot(\boldsymbol{q}-\Pi_h\boldsymbol{q})\|_{0,\Omega} \leq C|\boldsymbol{q}|_{1,\mathcal{T}_h}, \qquad \boldsymbol{q}\in H^1(\mathcal{T}_h)\cap\boldsymbol{V}. \tag{2.22}$$

For $\varphi \in W_h$, we let [20]:

$$\phi(\varphi)-\phi(\boldsymbol{p}) = -\tilde{\phi}'(\varphi)(\boldsymbol{p}-\varphi) = -\phi'(\boldsymbol{p})(\boldsymbol{p}-\varphi)+\tilde{\phi}''(\varphi)(\boldsymbol{p}-\varphi)^2, \tag{2.23}$$

where

$$\tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi+s(\boldsymbol{p}-\varphi))ds, \quad \tilde{\phi}''(\varphi) = \int_0^1 (1-s)\phi''(\boldsymbol{p}+s(\varphi-\boldsymbol{p}))ds$$

are bounded functionals in $\bar{\Omega}$.

3 A posteriori error estimates

The constrained optimal control problem normally has singularity. Under the constraint of an obstacle type, typically it has gradient jumps around the free boundary of the contact set. Thus the numerical error of the finite element solution is frequently concentrated around these areas. Adaptive finite element approximation has been found very useful in computing optimal control problems. It uses a posteriori error indicator to guide the mesh refinement procedure. Adaptive finite element approximation refines only the area where the error indicator is larger, so that a higher density of nodes is distributed over the area where the solution is difficult to approximate. In this sense the efficiency and reliability of adaptive finite element approximation very much rely on those of the error indicator used.

We consider the most useful type of constraints:

$$K = \{v \in L^2(\Omega) : \ v \geq d\},$$

where d is a constant. Let $K_h = K \cap U_h$, and assume that U_h is the piecewise constant finite element space. Then, it is easy to see that $K_h \subset K$.

In order to have sharp a posteriori error estimates, we divide Ω into some subsets:

$$\begin{aligned}
\Omega_d^- &= \{x \in \Omega : \ z_h(x) \leq -\nu d\}, \\
\Omega_d &= \{x \in \Omega : \ z_h(x) > -\nu d, u_h = d\}, \\
\Omega_d^+ &= \{x \in \Omega : \ z_h(x) > -\nu d, u_h > d\}.
\end{aligned}$$

Then, it is clear that three subsets do not intersect each other, and $\Omega = \Omega_d^- \cup \Omega_d \cup \Omega_d^+$. Now let us have an intuitive analysis on the approximation error for the control. On Ω_d, asymptotically we can assume that

$$0 < z_h + \nu u_h \to z + \nu u.$$

Hence it follows from the optimality conditions that $u = u_h = d$ on Ω_d. Thus the error on Ω_d may be negligible. We should only need to estimate the error on $\Omega \backslash \Omega_d = \Omega_d^- \cup \Omega_d^+$ in order to avoid over-estimate. As in [4], let

$$J(u) = \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2, \tag{3.1}$$

$$J_h(u_h) = \frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2. \tag{3.2}$$

It can be shown that

$$\begin{aligned}
(J'(u), v) &= (\nu u + z, v), \\
(J'(u_h), v) &= (\nu u_h + z(u_h), v), \\
(J_h'(u_h), v) &= (\nu u_h + z_h, v),
\end{aligned}$$

where $z(u_h)$ is the solution of the equations with $\tilde{u} = u_h$:

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.3}$$

$$(\mathrm{div}\boldsymbol{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (\tilde{u}, w), \qquad \forall w \in W, \tag{3.4}$$

$$(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.5}$$

$$(\mathrm{div}\boldsymbol{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \qquad \forall w \in W. \tag{3.6}$$

In many applications, $J(\cdot)$ is uniform convex near the solution u (see, e.g., [17]). The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. If $J(\cdot)$ is uniformly convex, then there is a $c > 0$, such that

$$(J'(u) - J'(u_h), u - u_h) \geq c\|u - u_h\|^2_{L^2(\Omega)}, \tag{3.7}$$

where u and u_h are the solutions of (2.1) and (2.9), respectively. We will assume the above inequality throughout this paper.

Now we establish the following error estimates, which can be proved similarly to the proofs given in [6].

Lemma 3.1. *Let u and u_h be the solutions of* (2.1) *and* (2.9), *respectively. Assume that $K_h \subset K$. Then*

$$\|u - u_h\|^2_{L^2(\Omega)} \le C\big(\eta_1^2 + \|z_h - z(u_h)\|^2_{L^2(\Omega)}\big), \tag{3.8}$$

where $\eta_1^2 = \int_{\Omega_d^-} |z_h + \nu u_h|^2 dx$.

Fix a function $u_h \in U_h$, let $(\boldsymbol{p}(u_h), y(u_h)) \in \boldsymbol{V} \times W$ be the solution of the equations (3.3)-(3.4).

Let $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U$ and $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ be the solution of (2.1)-(2.3) and (2.9)-(2.11), respectively. Set some intermediate errors:

$$\varepsilon_1 := \boldsymbol{p}(u_h) - \boldsymbol{p}_h, \qquad e_1 := y(u_h) - y_h.$$

To analyze the fixing u_h approach, let us first note the following error equations from (2.10)-(2.11) and (3.3)-(3.4):

$$(A^{-1}\varepsilon_1, \boldsymbol{v}_h) - (e_1, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.9}$$

$$(\mathrm{div}\varepsilon_1, w_h) + (\phi(y(u_h)) - \phi(y_h), w_h) = 0, \qquad \forall w_h \in W_h. \tag{3.10}$$

It follows from the uniqueness of the solutions for (3.3)-(3.4) that $y(u_h) \in H_0^1(\Omega)$,

$$\boldsymbol{p}(u_h) = -A\nabla y(u_h), \qquad \forall x \in \Omega, \tag{3.11}$$

$$\mathrm{div}\boldsymbol{p}(u_h) + \phi(y(u_h)) = u_h, \qquad \forall x \in \Omega. \tag{3.12}$$

In order to estimate $\|y(u_h) - y_h\|_{L^2(\Omega)}$ in L^2 norm, we need a priori regularity estimate for the following auxiliary problems:

$$-\mathrm{div}(A\nabla\xi) + \Phi\xi = F_1, \qquad x \in \Omega, \qquad \xi|_{\partial\Omega} = 0, \tag{3.13}$$

$$-\mathrm{div}(A^*\nabla\zeta) + \phi'(y(u_h))\zeta = F_2, \qquad x \in \Omega, \qquad \zeta|_{\partial\Omega} = 0, \tag{3.14}$$

where

$$\Phi = \begin{cases} \dfrac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \neq y_h, \qquad (3.15) \\ \phi'(y_h), & y(u_h) = y_h. \qquad (3.15') \end{cases}$$

The next lemma gives the desired a priori estimate; see e.g., [17].

Lemma 3.2. *Let ξ and ζ be the solutions of* (3.13) *and* (3.14), *respectively. Assume that Ω is convex, $A \in (W^{1,\infty}(\Omega))^{(2\times 2)}$, $X^tAX \ge c\|X\|^2_{\mathbb{R}^2}$ for all $X \in \mathbb{R}^2$. Then*

$$\|\xi\|_{H^2(\Omega)} \le C\|F_1\|_{L^2(\Omega)}, \tag{3.16}$$

$$\|\zeta\|_{H^2(\Omega)} \le C\|F_2\|_{L^2(\Omega)}. \tag{3.17}$$

Then we can have:

Theorem 3.1. *For the Raviart-Thomas elements, there is a positive constant C which only depends on A, Ω, and the shape of the elements and their maximal polynomial degree k, such that*

$$\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\|_{H(\mathrm{div};\Omega)} + \|y(u_h) - y_h\|_{L^2(\Omega)} \le C\eta_2, \tag{3.18}$$

where

$$\begin{aligned}\eta_2 &:= \Big(\sum_{\tau\in\mathcal{T}_h} \eta_{2\tau}^2\Big)^{1/2}\\ &:= \Big[\sum_{\tau\in\mathcal{T}_h}\Big(\|u_h - \mathrm{div}\boldsymbol{p}_h - \phi(y_h)\|_{0,\tau}^2 + h_\tau^2\cdot\|\mathrm{curl}_h(A^{-1}\boldsymbol{p}_h)\|_{0,\tau}^2\\ &\quad + \|h_E^{1/2}\cdot[A^{-1}\boldsymbol{p}_h\cdot\mathbf{t}]\|_{0,\partial\tau}^2 + h_\tau^2\cdot\min_{w_h\in W_h}\|\nabla_h w_h - A^{-1}\boldsymbol{p}_h\|_{0,\tau}^2\Big)\Big]^{1/2}. \end{aligned}\tag{3.19}$$

Proof. We consider a Helmholtz decomposition [2] of $A^{-1}\boldsymbol{p}_h$ with a fixing $\varphi\in H_0^1(\Omega)$ such that $-\mathrm{div}(A\nabla\varphi)=\mathrm{div}\boldsymbol{p}_h$. Then, there is some $\psi\in H^1(\Omega)$ satisfy $\int_\Omega \psi dx = 0, \mathrm{Curl}\psi \perp\nabla H_0^1(\Omega)$, and

$$\boldsymbol{p}_h = -A\nabla\varphi + \mathrm{Curl}\psi. \tag{3.20}$$

From (3.11) and (3.20) we obtain

$$\varepsilon_1 = A\nabla\chi - \mathrm{Curl}\ \psi, \qquad \text{with} \qquad \chi = \varphi - y(u_h) \in H_0^1(\Omega),$$

and hence the error decomposition

$$(A^{-1}\varepsilon_1, \varepsilon_1) = (A\nabla\chi, \nabla\chi) + (A^{-1}\mathrm{Curl}\psi, \mathrm{Curl}\psi). \tag{3.21}$$

It follows from (2.19), the Poincare's inequality, and ellipticity of A that

$$\begin{aligned}&(A\nabla\chi, \nabla\chi) = (\nabla\chi, \varepsilon_1) = -(\mathrm{div}\varepsilon_1, \chi)\\ =\ & (\mathrm{div}\varepsilon_1, P_h\chi - \chi) - (\mathrm{div}\varepsilon_1, P_h\chi)\\ \le\ & C\|h\cdot\mathrm{div}\varepsilon_1\|_{0,\Omega}\cdot\|A^{1/2}\nabla\chi\|_{0,\Omega} + C\|\mathrm{div}\varepsilon_1\|_{0,\Omega}\cdot\|P_h\chi\|_{0,\Omega}\\ \le\ & C\|h\cdot\mathrm{div}\varepsilon_1\|_{0,\Omega}\cdot\|A^{1/2}\nabla\chi\|_{0,\Omega} + C\|\mathrm{div}\varepsilon_1\|_{0,\Omega}\cdot\|A^{1/2}\nabla\chi\|_{0,\Omega}. \end{aligned}\tag{3.22}$$

To estimate the second contribution to the right-hand side of (3.21), we utilize Clement's operator $\hat{\pi}_h$. By its definition $\hat{\pi}_h\psi \in S^1(\mathcal{T}_h) \subset H^1(\Omega)$, $\mathrm{Curl}\hat{\pi}_h\psi \in S^0(\mathcal{T}_h)^2 \cap H(\mathrm{div};\Omega) \subset V_h$ and $\mathrm{Curl}\hat{\pi}_h\psi\perp\nabla H_0^1(\Omega)$, whence $\mathrm{div}(\mathrm{Curl}\hat{\pi}_h\psi) = 0$. Therefore, we have

$$\begin{aligned}&(A^{-1}\mathrm{Curl}\psi, \mathrm{Curl}\hat{\pi}_h\psi)\\ =\ & -(A^{-1}\varepsilon_1, \mathrm{Curl}\hat{\pi}_h\psi) = -(e_1, \mathrm{div}\mathrm{Curl}\hat{\pi}_h\psi) = 0.\end{aligned}$$

It follows from (3.20) and (2.17)-(2.18) that

$$\begin{aligned}&(A^{-1}\mathrm{Curl}\psi, \mathrm{Curl}\psi)\\ =\ & (A^{-1}\mathrm{Curl}\psi, \mathrm{Curl}(\psi - \hat{\pi}_h\psi)) = (A^{-1}\boldsymbol{p}_h, \mathrm{Curl}(\psi - \hat{\pi}_h\psi))\\ =\ & -(\psi - \hat{\pi}_h\psi, \mathrm{curl}_h(A^{-1}\boldsymbol{p}_h)) + ([A^{-1}\boldsymbol{p}_h\cdot n], \psi - \hat{\pi}_h\psi)_{\mathcal{E}_h}\\ \le\ & C\Big(\|h\cdot\mathrm{curl}_h(A^{-1}\boldsymbol{p}_h)\|_{0,\Omega} + \|h^{1/2}\cdot[A^{-1}\boldsymbol{p}_h\cdot\mathbf{t}]\|_{0,\mathcal{E}_h}\Big)\|\psi\|_{1,\Omega}. \end{aligned}\tag{3.23}$$

With the Poincare's inequality and the ellipticity of A we deduce

$$\|\psi\|_{1,\Omega} \le C\|\nabla\psi\|_{0,\Omega} = C\|\text{Curl}\psi\|_{0,\Omega} \le C\|A^{-1/2}\text{Curl}\psi\|_{0,\Omega}. \tag{3.24}$$

From (3.11) and (2.23) we can obtain that

$$\begin{aligned}\text{div}\varepsilon_1 &= u_h - \text{div}\boldsymbol{p}_h - \phi(y(u_h)) \\ &= u_h - \text{div}\boldsymbol{p}_h - \phi(y_h) - \tilde{\phi}'(y(u_h)) \cdot e_1,\end{aligned}$$

and together with (3.21)-(3.24) we have

$$\begin{aligned}\|\varepsilon_1\|_{H(\text{div};\Omega)} \le C\Big(&\|u_h - \text{div}\boldsymbol{p}_h - \phi(y_h)\|_{0,\Omega} + \|e_1\|_{0,\Omega} \\ &+ h\cdot\|\text{curl}_h(A^{-1}\boldsymbol{p}_h)\|_{0,\Omega} + \|h^{1/2}[A^{-1}\boldsymbol{p}_h\cdot\mathbf{t}]\|_{0,\mathcal{E}_h}\Big).\end{aligned} \tag{3.25}$$

Now, let us estimate $\|e_1\|_{0,\Omega}$. Let ξ be the solution of (3.13) with $F_1 = y(u_h) - y_h$. According to (3.13), we have $\xi \in H_0^1(\Omega)\cap H^2(\Omega)$. Then it follows from (2.12), (3.11), (3.13), and (3.15) that

$$\begin{aligned}&\|e_1\|_{0,\Omega}^2 = (y(u_h) - y_h, -\text{div}(A\nabla\xi) + \Phi\xi) \\ =& -(\boldsymbol{p}(u_h), \nabla\xi) + (y_h, \text{div}\circ\Pi_h(A\nabla\xi)) + (\phi(y(u_h)) - \phi(y_h), \xi) \\ =& (\text{div}\boldsymbol{p}(u_h), \xi) + (\phi(y(u_h)), \xi) + (A^{-1}\boldsymbol{p}_h, \Pi_h(A\nabla\xi)) - (\phi(y_h), \xi) \\ =& (u_h - \text{div}\boldsymbol{p}_h - \phi(y_h), \xi) + (\nabla_h w_h - A^{-1}\boldsymbol{p}_h, (I - \Pi_h)(A\nabla\xi)) \\ \le& C\Big(\|u_h - \text{div}\boldsymbol{p}_h - \phi(y_h)\|_{0,\Omega} + \|h\cdot(\nabla_h w_h - A^{-1}\boldsymbol{p}_h)\|_{0,\Omega}\Big)\cdot\|\xi\|_{2,\Omega} \\ \le& C\Big(\|u_h - \text{div}\boldsymbol{p}_h - \phi(y_h)\|_{0,\Omega}^2 + \|h\cdot(\nabla_h w_h - A^{-1}\boldsymbol{p}_h)\|_{0,\Omega}^2\Big) + \delta\|e_1\|_{0,\Omega}^2,\end{aligned}$$

for any $w_h \in W_h$. Using the triangle inequality, we obtain

$$\|e_1\|_{0,\Omega} \le C\Big(\|u_h - \text{div}\boldsymbol{p}_h - \phi(y_h)\|_{0,\Omega} + \|h\cdot(\nabla_h w_h - A^{-1}\boldsymbol{p}_h)\|_{0,\Omega}\Big). \tag{3.26}$$

Consequently, Theorem 3.1 is proved by combining (3.26) with (3.25). □

Using the argument similar to the proof of Theorem 3.1, we can also derive the following result:

Theorem 3.2. *For the Raviart-Thomas elements, there is a positive constant C which only depends on A, Ω, and the shape of the elements and their maximal polynomial degree k, such that*

$$\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\|_{H(\text{div};\Omega)} + \|z(u_h) - z_h\|_{L^2(\Omega)} \le C(\eta_2 + \eta_3), \tag{3.27}$$

where

$$\begin{aligned}\eta_3 &:= \Big(\sum_{\tau\in\mathcal{T}_h}\eta_{3\tau}^2\Big)^{1/2} \\ &:= \Big[\sum_{\tau\in\mathcal{T}_h}\Big(\|y_h - \text{div}\boldsymbol{q}_h - \phi'(y_h)z_h - y_d\|_{0,\tau}^2 + h_\tau^2\cdot\|\text{curl}_h(A^{-1}\boldsymbol{q}_h)\|_{0,\tau}^2 \\ &\quad + \|h_E^{1/2}\cdot[A^{-1}\boldsymbol{q}_h\cdot\mathbf{t}]\|_{0,\partial\tau}^2 + h_\tau^2\cdot\min_{w_h\in W_h}\|\nabla_h w_h - A^{-1}\boldsymbol{q}_h\|_{0,\tau}^2\Big)\Big]^{1/2}.\end{aligned} \tag{3.28}$$

Let $(\boldsymbol{p},y,\boldsymbol{q},z,u)\in(\boldsymbol{V}\times W)^2\times U$ and $(\boldsymbol{p}_h,y_h,\boldsymbol{q}_h,z_h,u_h)\in(\boldsymbol{V}_h\times W_h)^2\times U_h$ be the solutions of (2.4)-(2.8) and (2.12)-(2.16). By applying the intermediate errors, we can decompose the errors as following

$$\begin{aligned}
\boldsymbol{p}-\boldsymbol{p}_h &= \boldsymbol{p}-\boldsymbol{p}(u_h)+\boldsymbol{p}(u_h)-\boldsymbol{p}_h := \epsilon_1+\varepsilon_1,\\
y-y_h &= y-y(u_h)+y(u_h)-y_h := r_1+e_1,\\
\boldsymbol{q}-\boldsymbol{q}_h &= \boldsymbol{q}-\boldsymbol{q}(u_h)+\boldsymbol{q}(u_h)-\boldsymbol{q}_h := \epsilon_2+\varepsilon_2,\\
z-z_h &= z-z(u_h)+z(u_h)-z_h := r_2+e_2.
\end{aligned}$$

From (2.4)-(2.7), (3.3)-(3.6) and (2.23), we have

$$(A^{-1}\epsilon_1,\boldsymbol{v})-(r_1,\mathrm{div}\boldsymbol{v})=0, \qquad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{3.29}$$

$$(\mathrm{div}\epsilon_1,w)+(\tilde{\phi}'(y)r_1,w)=(u-u_h,w), \qquad \forall w\in W, \tag{3.30}$$

$$(A^{-1}\epsilon_2,\boldsymbol{v})-(r_2,\mathrm{div}\boldsymbol{v})=-(\epsilon_1,\boldsymbol{v}), \qquad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{3.31}$$

$$(\mathrm{div}\epsilon_2,w)+(\phi'(y)r_2,w)=(r_1,w)-(\tilde{\phi}''(y)z(u_h)r_1,w), \qquad \forall w\in W. \tag{3.32}$$

The assumption that $A\in L^\infty(\Omega;\mathbb{R}^{2\times2})$ implies the boundedness of the inverse operator of the map $\{\epsilon_1,r_1\}$: $\mathbb{R}^3\to\boldsymbol{V}\times W$ defined by the saddle-point problem (3.29)-(3.30) [1]:

$$\|\epsilon_1\|_{H(\mathrm{div};\Omega)}+\|r_1\|_{L^2(\Omega)}\le C\|u-u_h\|_{L^2(\Omega)}. \tag{3.33}$$

Similarly, by (3.33), we have

$$\begin{aligned}
&\|\epsilon_2\|_{H(\mathrm{div};\Omega)}+\|r_2\|_{L^2(\Omega)}\\
\le\,& C\big(\|\boldsymbol{p}-\boldsymbol{p}(u_h)\|_{L^2(\Omega)}+\|y-y(u_h)\|_{L^2(\Omega)}\big)\\
\le\,& C\|u-u_h\|_{L^2(\Omega)}.
\end{aligned} \tag{3.34}$$

Using the Lemma 3.1, Theorems 3.1 and 3.2, and (3.33)-(3.34), we can derive the following result:

Theorem 3.3. *Let u and u_h be the solutions of* (2.1) *and* (2.9)*, respectively. Assume that $K_h\subset K$. Then*

$$\begin{aligned}
&\|\boldsymbol{p}-\boldsymbol{p}_h\|^2_{H(\mathrm{div};\Omega)}+\|y-y_h\|^2_{L^2(\Omega)}+\|\boldsymbol{q}-\boldsymbol{q}_h\|^2_{H(\mathrm{div};\Omega)}\\
&\quad+\|z-z_h\|^2_{L^2(\Omega)}+\|u-u_h\|^2_{L^2(\Omega)}\le C\sum_{i=1}^{3}\eta_i^2,
\end{aligned} \tag{3.35}$$

where η_1, η_2, and η_3 are defined in Lemma 3.1*, Theorems* 3.1 *and* 3.2*, respectively.*

4 Numerical tests

In the section, we use a posteriori error estimates presents in our paper as an indicator for the adaptive finite element approximation. There has been immense research on developing fast numerical algorithms for optimal control problems in the scientific literature that it simply impossible to give even a very brief review here. However there seems still some way to go before efficient solvers can be developed even for the constrained quadratic optimal control governed by an elliptic equation. The reason seems that there are so many computational bottlenecks in solving an optimal control problem. It has been recently found that suitable adaptive meshes can greatly reduce discretization errors, see, for example, [15].

For our numerical test, a posteriori error estimators are used as error indicators to guide the mesh refinement in adaptive finite element methods (h-method). The optimal control problem were solved numerically by a preconditioned projection algorithm, with codes developed based on AFEPACK. The package is freely available and the details can be found at [14].

Our numerical example is the following optimal control problem:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u-u_0\|^2\right\}, \tag{4.1}$$

$$\mathrm{div}\boldsymbol{p}+y^3=f+u, \qquad \boldsymbol{p}=-\nabla y, \qquad x\in\Omega, \quad y|_{\partial\Omega}=0, \tag{4.2}$$

$$\mathrm{div}\boldsymbol{q}+3y^2z=y-y_d, \quad \boldsymbol{q}=-(\nabla z+\boldsymbol{p}-\boldsymbol{p}_d), \quad x\in\Omega, \quad z|_{\partial\Omega}=0. \tag{4.3}$$

In our examples, we choose the domain $\Omega=[0,1]\times[0,1]$. Let Ω be partitioned into $\mathcal{T}_h$ as described in section 2. We shall use η_1 as the control mesh refinement indicator, and η_2 and η_3 as the state's and co-state's.

For the constrained optimal control problem:

$$\min_{u\in K\subset U} J(u), \tag{4.4}$$

where $J(u)$ is a uniform convex functional on U and $K=\{u\in L^2(\Omega):u\geq 0\}$, the iterative scheme reads ($n=0,1,2,\cdots$)

$$b(u_{n+\frac{1}{2}},v)=b(u_n,v)-\rho_n(J'(u_n),v), \qquad \forall v\in U, \tag{4.5}$$

$$u_{n+1}=P_K^b(u_{n+\frac{1}{2}}), \tag{4.6}$$

where $b(\cdot,\cdot)$ is a symmetric and positive definite bilinear form such that there exist constant c_0 and c_1 satisfying

$$|b(u,v)|\leq c_1\|u\|_U\|v\|_U, \quad \forall u,\ v\in U, \tag{4.7}$$

$$b(u,u)\geq c_0\|u\|_U^2, \tag{4.8}$$

and the projection operator $P_K^b U\rightarrow K$ is defined: For given $w\in U$ find $P_K^b w\in K$ such that

$$b(P_K^b w-w,P_K^b w-w)=\min_{u\in K} b(u-w,u-w). \tag{4.9}$$

The bilinear form $b(\cdot,\cdot)$ provides suitable preconditioning for the projection algorithm. Otherwise its speed may be slow when h is very small. One can use a fixed step size, or variable ones from a line search procedure. When the step sizes are small enough, its convergence can be shown with the standard techniques. Let $U = U_h$. An application of (4.5)-(4.6) to the discretized semilinear elliptic control problem yields the following algorithm

$$b(u_{n+\frac{1}{2}}, v_h) = b(u_n, v_h) - \rho_n(\nu u_n + z_n, v_h), \qquad \forall v_h \in U_h \tag{4.10}$$

$$(A^{-1}\boldsymbol{p}_n, \boldsymbol{v}_h) - (y_n, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.11}$$

$$(\mathrm{div}\boldsymbol{p}_n, w_h) + (\phi(y_n), w_h) = (f + u_n, w_h), \qquad \forall w_h \in W_h, \tag{4.12}$$

$$(A^{-1}\boldsymbol{q}_n, \boldsymbol{v}_h) - (z_n, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_n - \boldsymbol{p}_d, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.13}$$

$$(\mathrm{div}\boldsymbol{q}_n, w_h) + (\phi'(y_n)z_n, w_h) = (y_n - y_d, w_h), \qquad \forall w_h \in W_h, \tag{4.14}$$

$$u_{n+1} = P_K^b(u_{n+\frac{1}{2}}),\ u_{n+\frac{1}{2}},\ u_n \in U_h. \tag{4.15}$$

The main computational effort is to solve the four state and co-state equations, and to compute the projection $P_K^b u_{n+\frac{1}{2}}$. In this paper we use a fast algebraic multi-grid solver to solve the state and co-state equations. Then it is clear that the key to saving computing time is how to compute $P_K^b u_{n+\frac{1}{2}}$ efficiently. If one uses the C^0 finite elements to approximate to the control, then one has to solve a global variational inequality, via, e.g., semi-smooth Newton method. The computational load is not trivial. For the piecewise constant elements, $K_h = \{u_h : u_h \geq 0\}$ and $b(u, v) = (u, v)_U$, then

$$P_K^b u_{n+\frac{1}{2}}|_\tau = \max\big(0, \mathrm{avg}(u_{n+\frac{1}{2}})|_\tau\big), \tag{4.16}$$

where $\mathrm{avg}(u_{n+\frac{1}{2}})|_\tau$ is the average of $u_{n+\frac{1}{2}}$ over τ.

In solving our discretized optimal control problem, we use the preconditioned projection gradient method (4.10-4.14) with $b(u, v) = (u, v)_{U_h}$ and a fixed step size $\rho = 0.8$. We now briefly describe the solution algorithm to be used for solving the following numerical examples ([15]).

Algorithm 4.1: (Algorithm A)

Step 1: Solve the discritized optimal control problem with the projection gradient method on the current meshes and calculate the error estimators η_i;

Step 2: Adjust the meshes using the estimators and update the solution on new meshes, as described.

Example We set the known functions as follows:

$$\lambda = \begin{cases} 0.5, & x_1 + x_2 > 1.0, \\ 0.0, & x_1 + x_2 \leq 1.0, \end{cases}$$

$$y = \sin 2\pi x_1 \sin 2\pi x_2,$$

$$u_0 = 1 - \sin\frac{\pi x_1}{2} - \sin\frac{\pi x_2}{2} + \lambda,$$

$$z = 2\sin 2\pi x_1 \sin 2\pi x_2, \qquad y_d = (1 - 16\pi^2)y - 3y^2 z,$$

$$u = \max(u_0 - z, 0), \qquad f = \mathrm{div}\boldsymbol{p} + y^3 - u,$$

$$q = -\begin{pmatrix} 4\pi\cos 2\pi x_1 \sin 2\pi x_2 \\ 4\pi \sin 2\pi x_1 \cos 2\pi x_2 \end{pmatrix}, \quad p = p_d = -\begin{pmatrix} 2\pi\cos 2\pi x_1 \sin 2\pi x_2 \\ 2\pi \sin 2\pi x_1 \cos 2\pi x_2 \end{pmatrix}.$$

In Fig. 1, the exact solution u is plotted. The control function u is discretized by piecewise constant functions, whereas the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximation by the RT0 mixed finite element functions. In Table 1, the uniform and adaptive meshes information is displayed with the approximation errors for the control and the states. We provide the comparison of solutions and the numerical errors obtained on uniform meshes and adaptive meshes in Table 1. It is clear that the adaptive meshes generated using our error indicators are able to save substantial computational work, in comparison with the uniform meshes.

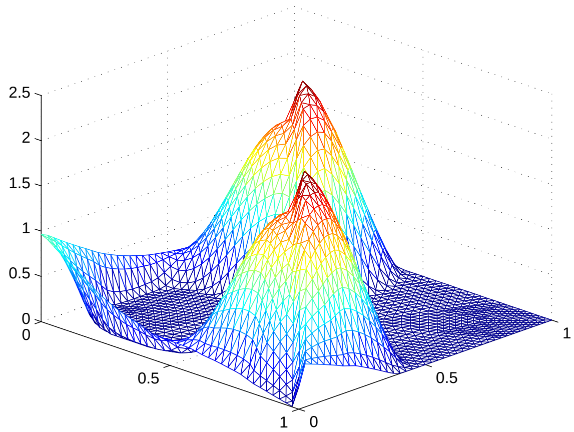

Figure 1: The exact solution of u.

In Fig. 2, the adaptive mesh for u are shown. They are obtained by using the h-method. In the computations, we use η_1 as the control mesh refinement indicator and η_2-η_3 as the state mesh refinement indicator in the adaptive finite element method.

Table 1: Comparison of uniform mesh and adaptive mesh.

mesh information	uniform mesh	adaptive mesh
u_nodes	8097	1266
u_sides	23968	3592
u_elements	15872	2327
yz_nodes	8097	2065
yz_sides	23968	6032
yz_elements	15872	3968
$\|u-u_h\|_{L^2(\Omega)}$	2.74399e-02	2.79115e-02
$\|y-y_h\|_{L^2(\Omega)}$	1.12311e-02	2.24592e-02
$\|z-z_h\|_{L^2(\Omega)}$	2.24620e-02	4.49172e-02
$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{H(\mathrm{div};\Omega)}$	5.38781e-02	7.06471e-02
$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{H(\mathrm{div};\Omega)}$	7.61840e-02	9.99123e-02

It can be observed that the meshes are well adapted well to the neighborhood of the free boundaries and discontinuity, and a higher density of node points are indeed distributed in the desired areas.

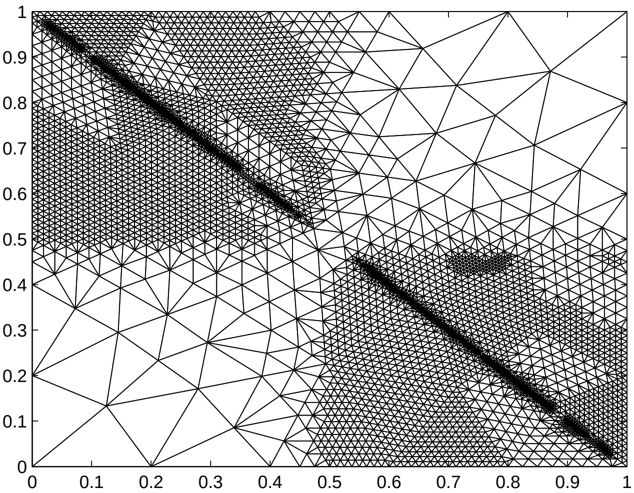

Figure 2: The adaptive mesh of u.

Acknowledgements

This work is supported by Guangdong Provincial 'Zhujiang Scholar Award Project', National Science Foundation of China 10671163, the National Basic Research Program under the Grant 2005CB321703, Scientific Research Fund of Hunan Provincial Education Department 06A069, and Hunan Provincial Innovation Foundation for Postgraduate S2008yjscx04.

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