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An Inf-Sup Stabilized Finite Element Method by Multiscale Functions for the Stokes Equations

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Abstract. In the paper, an inf-sup stabilized finite element method by multiscale functions for the Stokes equations is discussed. The key idea is to use a Petrov-Galerkin approach based on the enrichment of the standard polynomial space for the velocity component with multiscale functions. The inf-sup condition for P_1-P_0 triangular element (or Q_1-P_0 quadrilateral element) is established. The optimal error estimates of the stabilized finite element method for the Stokes equations are obtained.

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Key words: stabilized finite element method; multiscale functions; Petrov-Galerkin approach; inf-sup condition.

1 Introduction

In fluid dynamics, the Stokes equations model the slow flows of incompressible fluids or alternatively isotropic incompressible elastic materials. The Stokes equations have also become an important model for designing and analyzing finite element algorithms because some of the problems encountered for solving the Navier-Stokes equations already appear in the Stokes equations which are of simpler form. In particular, it gives the right setting for studying the stability problems connected with the choice of finite element spaces for the velocity and the pressure. It is well known that the finite element spaces cannot be chosen independently when the discretization

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is based on the Galerkin variational form, because it is very important to ensure the compatibility of the approximations of velocity and pressure (see, e.g., [19]).

It is well known that the simplest conforming low order elements like the $P_1(Q_1)-P_0$ (linear(bilinear) velocity, constant pressure) element is not stable. To overcome the limitation, many kinds of stabilized finite element methods have been proposed for the Stokes or Navier-Stokes equations. Brezzi and Pitkäranta in [3] firstly proposed the stabilized finite element method for P_1-P_1 triangular element. Later, many stabilized methods have been prooposed by relaxing the incompressibility constraint (i.e., modifying the second equation of (2.3)), see, e.g., [1–3, 6, 14–16, 22]. Furthermore, a general locally stabilized mixed finite element method was provided by Kechkar and Silvester in [20]. In [12], a new locally stabilized method based on the idea of [20] containing the jump terms across the inter-element boundaries of the macro elements was derived, which is called bubble condensation procedure. A particular kind of bubble functions of the velocity space is obtained by the residual free bubble method (RFBM) (see, e.g., [1, 9]), in which the bubble functions are the solutions of a problem containing the residual of the continuous equation at the element level. At the same time, the stabilized finite element method by multiscale functions was derived by [11], and a priori error analysis can be found in [10]. A main characteristic of the above methods is to use the Petrov-Galerkin approach to split the solution into two parts, i.e., the trial function space is enriched with the bubble functions which are the solutions to a local problem containing the residual of the momentum equation and special boundary conditions so that the local problem can be solved analytically.

In the paper, we use the Petrov-Galerkin approach based on the enrichment of the standard polynomial space for the velocity component with multiscale functions to propose a new stabilized finite element method for the Stokes equations. Although the main idea is derived from [10] and [11], our method is different from the one in [11] because the multiscale functions are new and the jump term which is introduced to the Galerkin variational formulation can be calculated no longer on the element boundary.

The remaining part of this paper is organized as follows. In the next section, we present the general framework and derive a stabilized finite element method for the Stokes equations. We then analyze the inf-sup stable condition for $P_1(Q_1)-P_0$ element and obtain the optimal error estimate.

2 Stabilized FEM by multiscale functions

Let Ω be an open bounded domain in $\mathbb{R}^d$ $(d{=}2$ or $3)$ with Lipschitz boundary $\partial\Omega$. We consider the following Stokes equations:

$$-\nu\Delta u + \nabla p = f, \qquad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot u = 0, \qquad \text{in } \Omega, \tag{2.2}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{2.3}$$

where $u=\big(u_1(x),\cdots,u_d(x)\big)$ represents the velocity vector, $p=p(x)$ the pressure, $f=f(x)\in L^2(\Omega)^d$ the prescribed body force, and ν the viscosity coefficient.

For the mathematical setting of the problem (2.1)-(2.3), we introduce Hilbert spaces:

$$X = H_0^1(\Omega)^d, \quad M = L_0^2(\Omega) \triangleq \Big\{q \in L^2(\Omega) : \int_\Omega q dx = 0\Big\}.$$

Furthermore, the space $L^2(\Omega)^d$ are endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_{0,\Omega}$. The space X are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v)_\Omega, \quad |u|_{1,\Omega} = ((u,u))^{1/2}.$$

Define Laplace operator A by

$$Au = -\Delta u, \quad \forall u \in D(A) = H^2(\Omega)^d \cap X.$$

Let

$$B_0\left((u,p);(v,q)\right) = \nu(\nabla u, \nabla v)_\Omega - (p, \nabla\cdot v)_\Omega + (q, \nabla \cdot u)_\Omega.$$

It is easy to check that B_0 satisfies the following important properties (see, e.g., [5,13, 20]):

$$\nu|u|_{1,\Omega}^2 = B_0((u,p);(u,p)), \tag{2.4a}$$

$$|B_0((u,p);(v,q))| \le \gamma\big(|u|_{1,\Omega} + \|p\|_{0,\Omega}\big)\big(|v|_{1,\Omega} + \|q\|_{0,\Omega}\big), \tag{2.4b}$$

$$\alpha_0\big(|u|_{1,\Omega} + \|p\|_{0,\Omega}\big) \le \sup_{(v,\, q)\in(X,\, M)} \frac{B_0((u,p);(v,q))}{|v|_{1,\Omega} + \|q\|_{0,\Omega}}, \tag{2.4c}$$

where $\gamma>0$ and $\alpha_0>0$.

Under the above notations, the standard Galerkin variational formulation of the problem (2.3) reads as follows: find $(u,p)\in(X,M)$ such that

$$B_0((u,p);(v,q)) = (f,v)_\Omega, \quad \forall (v,q) \in (X,M). \tag{2.5}$$

As for the existence and uniqueness of the solution of Stokes equations, we have the classical results as follows (see [13] Chapter IV and [23] Chapter II):

Theorem 2.1. *Assume that Ω is an open bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. Then the problem* (2.3) *admits a unique solution (u,p) such that*

$$|u|_{2,\Omega} + |p|_{1,\Omega} \le C(\nu)\|f\|_{0,\Omega}. \tag{2.6}$$

Remarks 2.1. The validity of Theorem 2.1 is known (see [17,21]) if $\partial\Omega$ is of C^2, or if Ω is a two-dimensional convex polygon.

Let Ω be characterized by $\{\mathcal{T}_h\}_{h>0}$ into triangles (quadrilaterals) in the usual sense (see [19], [20]), i.e., for some σ and λ with $\sigma>1$ and $0<\lambda<1$,

$$\begin{aligned}&h_K \le \sigma\rho_K, && \forall K \in \mathcal{T}_h,\\ &|\cos\theta_{iK}| \le \lambda, && i = 1,2,3,4, \quad \forall K \in \mathcal{T}_h,\end{aligned}$$

where h_K is the diameter of element K, ρ_K is the diameter of the inscribed circle of element K, and θ_{iK} are the angles of K in the case of a quadrilateral partitioning. The mesh parameter h is given by $h=\max\{h_K : K\in\mathcal{T}_h\}$, and the set of inter-element boundaries is denoted by Γ_h.

The finite element subspaces in this paper are defined by

$$\begin{aligned}&X_h = \{v \in C^0(\bar{\Omega})^d \cap X : v_i\big|_K \in R_1(K)^d, \forall K \in \mathcal{T}_h\},\\ &M_h = \{q \in M : q\big|_K \in P_0(K), \forall K \in \mathcal{T}_h\},\end{aligned}$$

where $R_1(K)$ is defined by

$$R_1(K) = \begin{cases} P_1(K) & \text{if K is triangular,}\\ Q_1(K) & \text{if K is quadrilateral.}\end{cases}$$

Here $P_n(K)$ and $Q_n(K)$ are the set of all polynomials on K of degree less than n and $n = 1, 2, \cdots$. Let E_h be a finite dimensional space, called multiscale space, such that

$$E_h \subset H^1(\mathcal{T}_h)^d, \quad E_h \cap X_h = \{0\},$$

where

$$H^1(\mathcal{T}_h)^d = \{v \in L^2(\Omega)^d : v|_K \in H^1(K)^d\}.$$

Under the above notations, we have the Petrov-Galerkin variational formulation of the Stokes equations: find $u_h + u_e\in X_h \oplus E_h$ and $p_h\in M_h$ such that

$$\nu(\nabla(u_h + u_e), \nabla v)_\Omega - (p_h, \nabla \cdot v)_\Omega + (q_h, \nabla \cdot (u_h + u_e))_\Omega = (f, v)_\Omega, \tag{2.7}$$

for all $v\in X_h \oplus E^0_h$ and $q_h\in M_h$, where

$$E^0_h = \{v \in H^1(\mathcal{T}_h)^d : v|_K \in H^1_0(K)^d\}.$$

It follows from (2.7) that

$$\begin{aligned}&\nu\big(\nabla(u_h + u_e), \nabla v_h\big)_\Omega - (p_h, \nabla \cdot v_h)_\Omega + \big(q_h, \nabla \cdot (u_h + u_e)\big)_\Omega\\ &\qquad = (f, v_h)_\Omega, \qquad \forall(v_h, q_h) \in X_h \times M_h, &(2.8)\\ &\nu\big(\nabla(u_h + u_e), \nabla v_K\big)_K - (p_h, \nabla \cdot v_K)_K\\ &\qquad = (f, v_K)_K, \qquad \forall v_K \in H^1_0(K)^d,\ \forall K \in \mathcal{T}_h. &(2.9)\end{aligned}$$

From (2.9), it follows that

$$(-\nu\Delta u_e, v_K)_K = (f + \nu\Delta u_h, v_K)_K, \quad \forall v_K \in H^1_0(K)^d, \quad \forall K \in \mathcal{T}_h, \tag{2.10}$$

which implies that u_e is the strong solution of the local problem

$$-\nu\Delta u_e = f + \nu\Delta u_h, \quad \text{in } K, \tag{2.11}$$

for all $(u_h, p_h) \in X_h \times M_h$.

Denote the mean value operator by $\langle\cdot\rangle$, i.e., if v is a vector field experiencing a discontinuity across an element boundary, then

$$\langle v\cdot n\rangle\big|_{\partial K} = \frac{v^+\cdot n^+ - v^-\cdot n^-}{2} = \frac{v^+\cdot n^+ + v^-\cdot n^+}{2} = n\cdot\Big(\frac{v^+ + v^-}{2}\Big),$$

where $n = n^+ = -n^-, v^+ = v\big|_{\partial K^+}$ and $v^- = v\big|_{\partial K^-}$.

In order to give an expression for u_e in term of u_h, p_h and f on each element K, we need to impose some special boundary condition and obtain the following local problem

$$\begin{aligned}
&-\nu\Delta u_e^K = f + \nu\Delta u_h, && \text{in } K, && (2.12a)\\
&-\nu\Delta u_e^{\partial K} = 0, && \text{in } K, && (2.12b)\\
&u_e^K|_{\partial K} = 0, \quad u_e^{\partial K} = g_e, && \text{on } \partial K, && (2.12c)\\
&-\nu\partial_{ss} g_e = \frac{1}{h_e}\langle \nu\partial_n u_h + p_h I\cdot n\rangle, && \forall s\in T, && (2.12d)\\
&g_e = 0, && \text{at the nodes}, && (2.12e)
\end{aligned}$$

where $u_e\big|_K = u_e^K + u_e^{\partial K}, h_e \triangleq mes T$ denotes the length of the edge T, n is the normal outward vector on ∂K, ∂_s, ∂_n are the tangential and normal derivative operators, respectively, and I is the $\mathbb{R}^{d\times d}$ identity matrix.

It is easy to check that problem (2.12) is well posed, i.e., u_e can be solved in term of u_h, p_h and f on each element K.

For convenience, we define two operators

$$\mathcal{H}_K : L^2(K)^d \to H_0^1(K)^d, \quad \mathcal{J}_K : L^2(\partial K)^d \to H_0^1(\partial K)^d,$$

by

$$\begin{aligned}
&u_e^K = \frac{1}{\nu}\mathcal{H}_K(f + \nu\Delta u_h), && \forall K\in\mathcal{T}_h, && (2.13a)\\
&u_e^{\partial K} = \frac{1}{h_e\nu}\mathcal{J}_K\Big(\langle \nu\partial_n u_h + p_h I\cdot n\rangle\Big), && \forall K\in\mathcal{T}_h. && (2.13b)
\end{aligned}$$

Integrating by parts on $K\in\mathcal{T}_h$, we have

$$\begin{aligned}
&\nu(\nabla u_e, \nabla v_h)_K = -\nu(u_e, \Delta v_h)_K + (u_e, \nu\partial_n v_h)_{\partial K}, && \forall v_h\in X_h, && (2.14a)\\
&(q_h, \nabla\cdot u_e)_K = -(u_e, \nabla q_h)_K + (u_e, q_h I\cdot n)_{\partial K}, && \forall q_h\in M_h. && (2.14b)
\end{aligned}$$

Substituting (2.14) into (2.8) yields

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\Big[\nu(\nabla u_h,\nabla v_h)_K-(p_h,\nabla\cdot v_h)_K+(q_h,\nabla\cdot u_h)_K-(u_e,\nu\Delta v_h)_K\\&+(u_e,\nu\partial_n v_h)_{\partial K}\Big]+\sum_{K\in\mathcal{T}_h}\Big[-(u_e,\nabla q_h)_K+(u_e,q_hI\cdot n)_{\partial K}\Big]\\=&\sum_{K\in\mathcal{T}_h}(f,v_h)_K,\end{aligned}\tag{2.15}$$

which implies that

$$\begin{aligned}&\nu(\nabla u_h,\nabla v_h)_K-(p_h,\nabla\cdot v_h)_K+(q_h,\nabla\cdot u_h)_K-(u_e,\nu\Delta v_h+\nabla q_h)_K\\&\qquad+(u_e^{\partial K},\nu\partial_n v_h+q_hI\cdot n)_{\partial K}=(f,v_h)_K,\end{aligned}\tag{2.16}$$

for $(u_h,p_h),(v_h,q_h)\in X_h\times M_h$, $u_e\in E_h$ and for each $K\in\mathcal{T}_h$. Combining (2.13) with (2.16) yields

$$\begin{aligned}&\nu(\nabla u_h,\nabla v_h)_K-(p_h,\nabla\cdot v_h)_K+(q_h,\nabla\cdot u_h)_K\\&+\frac{1}{\nu}\Big(\mathcal{H}_K(-\nu\Delta u_h)-\frac{1}{h_e}\mathcal{J}_K\big(\langle\nu\partial_n u_h+p_hI\cdot n\rangle\big),\nu\Delta v_h+\nabla q_h\Big)_K\\&+\frac{1}{h_e\nu}\Big(\mathcal{J}_K(\langle\nu\partial_n u_h+p_hI\cdot n\rangle),\nu\partial_n v_h+q_hI\cdot n\Big)_{\partial K}\\=&(f,v_h)_K+\frac{1}{\nu}\big(\mathcal{H}_K(f),\nu\Delta v_h+\nabla q_h\big)_K.\end{aligned}\tag{2.17}$$

From (2.17), we propose the following approach: find $(u_h,p_h)\in X_h\times M_h$ such that

$$B\big((u_h,p_h);(v_h,q_h)\big)=(f,v_h)_\Omega,\quad\forall(v_h,q_h)\in X_h\times M_h,\tag{2.18}$$

where

$$\begin{aligned}B\big((u_h,p_h);(v_h,q_h)\big)=&B_0\left((u_h,p_h);(v_h,q_h)\right)\\&+\sum_{T\in\Gamma_h}\beta_T\Big(\langle\nu\partial_n u_h+p_hI\cdot n\rangle,\langle\nu\partial_n v_h+q_hI\cdot n\rangle\Big)_T.\end{aligned}\tag{2.19}$$

Next, we calculate the parameter β_T. Firstly, we define the matrix function $\mathcal{A}_K$ by

$$\mathcal{A}_K=\big(\mathcal{J}_K(\phi_1)\big|\mathcal{J}_K(\phi_2)\big|\cdots\big|\mathcal{J}_K(\phi_d)\big),$$

where $\phi_i(i=1,\cdots,d)$ is a group of basis of $\mathbb{R}^d$. From the definition, we have $\mathcal{A}_K=a_KI$, where a_K is the solution of

$$-\Delta a_K=0,\ \text{ in } K,\quad a_K=g(s),\ \text{ on each } T\subset\partial K,\tag{2.20}$$

where g=0 if $T\subset\partial\Omega$, and in the internal edges g satisfies

$$-\partial_{ss}g(s)=\frac{1}{h_e},\ \text{ on } T,\quad g=0,\ \text{ at the nodes.}\tag{2.21}$$

Since $(u_h, p_h), (v_h, q_h) \in X_h \times M_h$, (2.17) reduces to

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\Big[\nu(\nabla u_h,\nabla v_h)_K-(p_h,\nabla\cdot v_h)_K+(q_h,\nabla\cdot u_h)_K\Big]\\&\qquad+\sum_{T\in\Gamma_h}\frac{1}{h_e\nu}\Big(\mathcal{J}_K(\langle\nu\partial_n u_h+p_hI\cdot n\rangle),\nu\partial_n v_h+q_hI\cdot n\Big)_T\\&=\sum_{K\in\mathcal{T}_h}(f,v_h)_K.\end{aligned}\tag{2.22}$$

Secondly, we claim that $\langle\nu\partial_n u_h+p_hI\cdot n\rangle\big|_{T\subset\partial K}$ is a constant vector function on every T. By using (2.20) and (2.21), we have

$$\frac{(a_K,1)_T}{h_e}=\frac{h_e}{6}.$$

It is easy to check that

$$\begin{aligned}&\frac{1}{h_e}(\mathcal{J}_K(\langle\nu\partial_n u_h+p_hI\cdot n\rangle),\langle\nu\partial_n v_h+q_hI\cdot n\rangle)_T\\&=\Big(\int_T\mathcal{A}_K\Big)\langle\nu\partial_n u_h+p_hI\cdot n\rangle\big|_T\langle\nu\partial_n v_h+q_hI\cdot n\rangle\big|_T\\&=\frac{h_e}{6}\big(\langle\nu\partial_n u_h+p_hI\cdot n\rangle,\langle\nu\partial_n v_h+q_hI\cdot n\rangle\big).\end{aligned}\tag{2.23}$$

Comparing (2.23) with (2.19) gives $\beta_T=h_e/(6\nu)$.

3 Error estimates

To derive the error estimates of the approximate solution (u_h, p_h), we use the canonical interpolation operator $\mathcal{I}_h$: $X\rightarrow X_h$ defined by

$$\int_T(\mathcal{I}_h u-u)ds=0,\quad\forall T\in\Gamma_h,$$

and the L^2-projection operator $\mathcal{P}_h$: $L^2(\Omega)\rightarrow M_h$. Then the following approximation estimates hold (see, e.g., [5,7,13]):

$$|v-\mathcal{I}_hv|_{m,K}\le C_1h_K^{2-m}|v|_{2,K},\qquad\forall v\in H^2(K),\tag{3.1}$$
$$|v-\mathcal{I}_hv|_{m,T}\le C_2h_e^{2-m-1/2}|v|_{2,\partial K},\qquad\forall v\in H^2(\partial K),\tag{3.2}$$
$$\|v\|^2_{0,\partial K}\le C_3(h_K^{-1}\|v\|^2_{0,K}+h_K|v|^2_{1,K}),\qquad\forall v\in H^1(K),\tag{3.3}$$
$$\|q-\mathcal{P}_hq\|_{0,\Omega}\le C_4h|q|_{1,\Omega},\qquad\forall q\in H^1(\Omega),\tag{3.4}$$

where m=0,1 and C_i (i=1,$\cdots$,4) are positive constants independent of h.

Remarks 3.1. The inequality (3.3) is very important which is called by the local trace theorem.

Remarks 3.2. From (3.3) and (3.4), we deduce that

$$
\begin{aligned}
&h_e\|\langle q-\mathcal{P}_h q\rangle\|_{0,T}^2\\
\le\, &C_3\Big(\|\langle q-\mathcal{P}_h q\rangle\|_{0,K}^2+h_K^2|\langle q-\mathcal{P}_h q\rangle|_{1,K}^2\Big)\le Ch^2|q|_{1,K}^2,
\end{aligned}
$$

which leads to

$$\Big(\sum_{T\in\Gamma_h} h_e\|\langle q-\mathcal{P}_h q\rangle\|_{0,T}^2\Big)^{1/2}\le C_5 h|q|_{1,\Omega},\quad \forall q\in H^1(\Omega). \tag{3.5}$$

Define the mesh-dependent norm

$$|(u,p)|_h=\Big(\nu|u|_{1,\Omega}^2+\sum_{T\in\Gamma_h}\beta_T\|\langle\nu\partial_n u+pI\cdot n\rangle\|_{0,T}^2\Big)^{1/2}. \tag{3.6}$$

Next, we give the inf-sup stable result for P_1-P_0 element as follows:

Lemma 3.1. *There exists $\beta_1>0$ such that*

$$\sup_{0\neq(v_h,q_h)\in X_h\times M_h}\frac{B((u_h,p_h);(v_h,q_h))}{\big(|v_h|_{1,\Omega}^2+\|q_h\|_{0,\Omega}^2\big)^{1/2}}\ge\beta_1\Big(|u_h|_{1,\Omega}^2+\|p_h\|_{0,\Omega}^2\Big)^{1/2}, \tag{3.7}$$

for all $(u_h,p_h)\in X_h\times M_h$.

Proof. From the continuous version of inf-sup condition (see [13]), we know that for each $p_h\in M_h\subset M$ there exist a function $w\in X$ and a finite element approximation $w_h\in X_h$ of w such that $|w|_{1,\Omega}=\|p_h\|_{0,\Omega}$ and

$$\big(\nabla\cdot w,p_h\big)_\Omega\ge C_6\|p_h\|_{0,\Omega}|w|_{1,\Omega}, \tag{3.8}$$

$$\big(\sum_{K\in\mathcal{T}_h}h_K^{-2}\|w-w_h\|_{0,K}^2\big)^{1/2}\le C|w|_{1,\Omega},\quad |w_h|_{1,\Omega}\le C|w|_{1,\Omega}. \tag{3.9}$$

Using the Cauchy-Schwartz inequality and (3.8), we have

$$
\begin{aligned}
&B\big((u_h,p_h);(-w,0)\big)\\
=\,&-\nu(\nabla u_h,\nabla w)-\sum_{K\in\mathcal{T}_h}\beta_T\big(\langle\nu\partial_n u_h\rangle,\langle\nu\partial_n w\rangle\big)_{\partial K}\\
&+(\nabla\cdot w,p_h)_\Omega-\sum_{K\in\mathcal{T}_h}\beta_T\big(\langle p_hI\cdot n\rangle,\langle\nu\partial_n w\rangle\big)_{\partial K}\\
\ge\,&-\nu|u_h|_{1,\Omega}|w|_{1,\Omega}+C_6\|p_h\|_{0,\Omega}|w|_{1,\Omega}\\
&-\sum_{K\in\mathcal{T}_h}\beta_T\|\langle\nu\partial_n u_h+p_hI\cdot n\rangle\|_{0,T}\|\langle\nu\partial_n w\rangle\|_{0,T}\\
\ge\,&-\Big(\nu|u_h|_{1,\Omega}^2+\sum_{T\in\Gamma_h}\beta_T\|\langle\nu\partial_n u_h+p_hI\cdot n\rangle\|_{0,T}^2\Big)^{1/2}\Big(\nu|w|_{1,\Omega}^2\\
&+\sum_{T\in\Gamma_h}\beta_T\|\langle\nu\partial_n w\rangle\|_{0,T}^2\Big)^{1/2}+C_6\|p_h\|_{0,\Omega}|w|_{1,\Omega}.
\end{aligned} \tag{3.10}
$$

Using (3.3) and the inverse inequality, we obtain

$$\begin{aligned}\beta_T\|\langle\nu\partial_n w\rangle\|_{0,T}^2 &\le \frac{h_e}{6\nu}\Big(h_K^{-1}\|\nu\nabla w\cdot n\|_{0,K}^2+h_K|\nu\nabla w\cdot n|_{1,K}^2\Big)\\ &\le \frac{\nu h_e}{6h_K}|w|_{1,K}^2+\frac{\nu h_e}{6}C_K h_K^{-1}|w|_{1,K}^2\\ &\le \frac{\nu(1+C_K)}{6}|w|_{1,K}^2. \end{aligned}\tag{3.11}$$

Combining (3.10) with (3.11) yields

$$\begin{aligned} &B\big((u_h,p_h);(-w,0)\big)\ge -\sqrt{C\nu}|w|_{1,\Omega}\Big(\nu|u_h|_{1,\Omega}^2\\ &\quad+\sum_{K\in\mathcal{T}_h}\beta_T\|\langle\partial_n u_h+p_hI\cdot n\rangle\|_{0,T}^2\Big)^{1/2}+C_6\|p_h\|_{0,\Omega}|w|_{1,\Omega}\\ &=-\sqrt{C\nu}|w|_{1,\Omega}|(u_h,p_h)|_h+C_6\|p_h\|_{0,\Omega}|w|_{1,\Omega}\\ &=-\sqrt{C\nu}\|p_h\|_{0,\Omega}|(u_h,p_h)|_h+C_6\|p_h\|_{0,\Omega}^2\\ &\ge -C\nu\gamma_1^{-1}|(u_h,p_h)|_h^2+[C_6-\gamma_1]\|p_h\|_{0,\Omega}^2, \end{aligned}\tag{3.12}$$

where $C=(7+C_0)/6$ with $C_0=\max_{K\in\mathcal{T}_h}C_K$, and γ_1 is chosen small enough. Let

$$(v_h,q_h)=(u_h-\delta w,p_h),\quad \delta>0.$$

Using (3.12) we have

$$\begin{aligned} &B((u_h,p_h);(v_h,q_h))\\ &=B((u_h,p_h);(u_h,p_h))+\delta B((u_h,p_h);(-w,0))\\ &\ge|(u_h,p_h)|_h^2+\delta\Big(-C\nu\gamma_1^{-1}|(u_h,p_h)|_h^2+[C_6-\gamma_1]\|p_h\|_{0,\Omega}^2\Big)\\ &\ge(1-\delta C\nu\gamma_1^{-1})|(u_h,p_h)|_h^2+\delta(C_6-\gamma_1)\|p_h\|_{0,\Omega}^2\\ &\ge(1-\delta C\nu\gamma_1^{-1})|u_h|_{1,\Omega}^2+\delta(C_6-\gamma_1)\|p_h\|_{0,\Omega}^2, \end{aligned}\tag{3.13}$$

provided that $0<\delta<\gamma_1/(C\nu)$ and $0<\gamma_1<C_6$. Denote

$$C(\nu)\triangleq\min\{(1-\delta C\nu\gamma_1^{-1}),\ \delta(C_6-\gamma_1)\},\quad C(\delta)\triangleq\max\{2,\ 1+2\delta^2\}.$$

Then we have

$$\begin{aligned} &|v_h|_{1,\Omega}^2+\|q_h\|_{0,\Omega}^2=|u_h-\delta w|_{1,\Omega}^2+\|p_h\|_{0,\Omega}^2\\ &\le 2|u_h|_{1,\Omega}^2+(2\delta^2+1)\|p_h\|_{0,\Omega}^2\le C(\delta)\left(|u_h|_{1,\Omega}^2+\|p_h\|_{0,\Omega}^2\right). \end{aligned}\tag{3.14}$$

Taking $\beta_1=C(\nu)/C(\delta)$ ends the proof. □

Lemma 3.2. *Let $(u,p)\in[H^2(\Omega)^d\cap X]\times[H^1(\Omega)\cap M]$ be the solution of* (2.5) *and assume that $(u_h,p_h)\in X_h\times M_h$ satisfy* (2.18). *Then there exists a positive constant C independent of ν and h such that*

$$|B((u-u_h,p-p_h);(v_h,q_h))|\le Ch\Big(\sqrt{\nu}|u|_{2,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big)|(v_h,q_h)|_h. \tag{3.15}$$

Proof. Since $(u,p)\in[H^2(\Omega)^d\cap X]\times[H^1(\Omega)\cap M]$ is the solution of (2.5) and (u_h,p_h) satisfies (2.18), we have

$$\begin{aligned}
&B\big((u-u_h,p-p_h);(v_h,q_h)\big)=B_0\big((u-u_h,p-p_h);(v_h,q_h)\big)\\
&\quad+\sum_{T\in\Gamma_h}\beta_T\Big(\langle\nu\partial_n(u-u_h)+(p-p_h)I\cdot n\rangle,\langle\nu\partial_n v_h+q_hI\cdot n\rangle\Big)_T\\
&=\sum_{T\in\Gamma_h}\beta_T\Big(\langle\nu\partial_n u+pI\cdot n\rangle,\langle\nu\partial_n v_h+q_hI\cdot n\rangle\Big)_T\\
&=\sum_{T\in\Gamma_h}\beta_T\Big(\nu\partial_n u+pI\cdot n,\langle\nu\partial_n v_h+q_hI\cdot n\rangle\Big)_T.
\end{aligned} \tag{3.16}$$

Using (3.16), (3.3), the Cauchy-Schwartz inequality and Lemma 3.1, we obtain

$$\begin{aligned}
&B\big((u-u_h,p-p_h);(v_h,q_h)\big)\\
\le&\sum_{T\in\Gamma_h}\beta_T^{1/2}\|\nu\partial_n u+pI\cdot n\|_{0,T}\beta_T^{1/2}\|\langle\nu\partial_n v_h+q_hI\cdot n\rangle\|_{0,T}\\
\le&\sum_{T\in\Gamma_h}\beta_T^{1/2}\|\nu\partial_n u+pI\cdot n\|_{0,T}|(v_h,q_h)|_h\\
\le&\sum_{T\in\Gamma_h}\beta_T^{1/2}\Big(\|\nu\partial_n u\|_{0,T}+\|pI\cdot n\|_{0,T}\Big)|(v_h,q_h)|_h\\
\le&Ch\Big(\sqrt{\nu}|u|_{2,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big)|(v_h,q_h)|_h,
\end{aligned} \tag{3.17}$$

where C is a positive constant independent of ν and h. □

Lemma 3.3. *Let $(u,p)\in[H^2(\Omega)^d\cap X]\times[H^1(\Omega)\cap M]$ be the solution of* (2.5). *Then there exists a positive constant C independent of ν and h such that*

$$|B\big((u-\mathcal{I}_hu,p-\mathcal{P}_hp);(v_h,q_h)\big)|\le Ch\Big(\sqrt{\nu}|u|_{2,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big)|(v_h,q_h)|_h, \tag{3.18}$$

for all $(v_h,q_h)\in X_h\times M_h$.

Proof. By using the orthogonality of the L^2-projection $\mathcal{P}_h$, we have

$$(p-\mathcal{P}_hp,\nabla\cdot v_h)=0,\quad\forall v_h\in X_h. \tag{3.19}$$

From the definition of the canonical interpolation operator $\mathcal{I}_h$ and the fact that q_h is piecewise constant, it follows that

$$\big(q_h, \nabla\cdot(u-\mathcal{I}_h u)\big)_K = \big(q_h, (u-\mathcal{I}_h u)\cdot n\big)_{\partial K} = 0. \tag{3.20}$$

Using Cauchy-Schwarz inequality and (3.1), we have

$$\begin{aligned} &\nu\big(\nabla(u-\mathcal{I}_h u), \nabla v_h\big)_K \\ \le& \sqrt{\nu}\|\nabla(u-\mathcal{I}_h u)\|_{0,K}\sqrt{\nu}\|\nabla v_h\|_{0,K} \\ \le& C\sqrt{\nu}h_K|u|_{2,K}|(v_h,q_h)|_h. \end{aligned} \tag{3.21}$$

Again, applying Cauchy-Schwarz inequality and using (3.3) and (3.4), we get

$$\begin{aligned} &\sum_{T\in\Gamma_h}\beta_T\Big(\langle\nu\partial_n(u-\mathcal{I}_h u)+(p-\mathcal{P}_h p)I\cdot n\rangle, \langle\nu\partial_n v_h + q_h I\cdot n\rangle\Big)_T \\ \le& \sum_{T\in\Gamma_h}\beta_T^{1/2}\parallel\langle\nu\partial_n(u-\mathcal{I}_h u)+(p-\mathcal{P}_h p)I\cdot n\rangle\parallel_{0,T}|(v_h,q_h)|_h \\ \le& \sum_{T\in\Gamma_h}\beta_T^{1/2}(\parallel\langle\nu\partial_n(u-\mathcal{I}_h u)\parallel_{0,T}+\parallel(p-\mathcal{P}_h p)I\cdot n\rangle\parallel_{0,T})|(v_h,q_h)|_h \\ \le& \sum_{K\in\mathcal{T}_h}Ch\big(\sqrt{\nu}|u|_{2,K}+\frac{1}{\sqrt{\nu}}|p|_{1,K}\big)|(v_h,q_h)|_h. \end{aligned} \tag{3.22}$$

From (3.19)-(3.22), it follows that

$$\begin{aligned} &B\big((u-\mathcal{I}_h u, p-\mathcal{P}_h p);(v_h,q_h)\big) \\ =& \nu(\nabla(u-\mathcal{I}_h u),\nabla v_h)_\Omega + \sum_{T\in\Gamma_h}\beta_T\Big(\langle\nu\partial_n(u-\mathcal{I}_h u) \\ &+(p-\mathcal{P}_h p)I\cdot n\rangle, \langle\nu\partial_n v_h+q_h I\cdot n\rangle\Big)_T \\ \le& Ch\big(\sqrt{\nu}|u|_{2,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\big)|(v_h,q_h)|_h. \end{aligned}$$

□

Lemma 3.4. *There exists a positive constant β_2 independent of h and ν such that*

$$\sup_{0\neq(v_h,q_h)\in X_h\times M_h}\frac{B((u_h,p_h);(v_h,q_h))}{|(v_h,q_h)|_h} \ge \beta_2|(u_h,p_h)|_h, \tag{3.23}$$

for all $(u_h,p_h)\in X_h\times M_h$.

Proof. Using (3.10), we have

$$B\big((u_h,p_h);(-w,0)\big) \ge -|(u_h,p_h)|_h|(v_h,q_h)|_h + C_6\|p_h\|^2_{0,\Omega}, \tag{3.24}$$

which by using the Young's inequality implies that

$$B((u_h,p_h);(-w,0)) \ge -\frac{1}{2\gamma_2}|(u_h,p_h)|_h^2 - \frac{\gamma_2}{2}|(v_h,q_h)|_h^2 + C_6\|p_h\|^2_{0,\Omega}, \tag{3.25}$$

provided that γ_2 is chosen sufficiently small. Denote

$$(v_h, q_h) = (u_h - \delta w, p_h), \quad \delta > 0.$$

Using (3.25) gives

$$\begin{aligned}
&B\big((u_h, p_h); (v_h, q_h)\big) \\
&= B\big((u_h, p_h); (u_h, p_h)\big) + \delta B\big((u_h, p_h); (-w, 0)\big) \\
&\geq |(u_h, p_h)|_h^2 + \delta\Big(-\frac{1}{2\gamma_2}|(u_h, p_h)|_h^2 - \frac{\gamma_2}{2}|(v_h, q_h)|_h^2 + C_6\|p_h\|_{0,\Omega}^2\Big) \\
&\geq (1 - \frac{\delta}{2\gamma_2})|(u_h, p_h)|_h^2 + \delta\Big(C_6\|p_h\|_{0,\Omega}^2 - \frac{\gamma_2}{2}|(v_h, q_h)|_h^2\Big), \qquad (3.26)
\end{aligned}$$

where δ and γ_2 are chosen to satisfy $0<\delta<2\gamma_2$ and γ_2 is small enough. Note that

$$|(v_h, q_h)|_h^2 = |(u_h - \delta w, p_h)|_h^2 \leq |(u_h, p_h)|_h^2. \qquad (3.27)$$

Taking $\beta_2 = 1 - \delta/(2\gamma_2)$ ends the proof. $\square$

Theorem 3.1. *Let $(u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]$ be the solution of* (2.5) *and $(u_h, p_h) \in X_h \times M_h$ be the solution of* (2.18). *Then the following error estimate holds:*

$$|(u - u_h, p - p_h)|_h \leq Ch\Big(\sqrt{\nu}|u|_{2,\Omega} + \frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big). \qquad (3.28)$$

Proof. Starting with Lemma 3.4 we have

$$\begin{aligned}
&|(u_h - \mathcal{I}_h u, p_h - \mathcal{P}_h p)|_h \\
&\leq \frac{1}{\beta_2} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B\big((u_h - \mathcal{I}_h u, p_h - \mathcal{P}_h p); (v_h, q_h)\big)}{|(v_h, q_h)|_h} \\
&\leq \frac{1}{\beta_2} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B\big((u_h - u, p_h - p); (v_h, q_h)\big)}{|(v_h, q_h)|_h} \\
&\quad + \frac{1}{\beta_2} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B((u - \mathcal{I}_h u, p - \mathcal{P}_h p); (v_h, q_h))}{|(v_h, q_h)|_h}. \qquad (3.29)
\end{aligned}$$

Combining (3.15) with (3.18) and (3.29) yields

$$|(u_h - \mathcal{I}_h u, p_h - \mathcal{P}_h p)|_h \leq Ch\Big(\sqrt{\nu}|u|_{2,\Omega} + \frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big), \qquad (3.30)$$

which implies that

$$\begin{aligned}
|(u - u_h, p - p_h)|_h &\leq |(u - \mathcal{I}_h u, p - \mathcal{P}_h p)|_h + |(u_h - \mathcal{I}_h u, p_h - \mathcal{P}_h p)|_h \\
&\leq Ch\Big(\sqrt{\nu}|u|_{2,\Omega} + \frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big).
\end{aligned}$$

This completes the proof of the theorem. $\square$

Theorem 3.2. *Let* $(u,p)\in[H^2(\Omega)^d\cap X]\times[H^1(\Omega)\cap M]$ *be the solution of* (2.5) *and* $(u_h,p_h)\in X_h\times M_h$ *be the solution of* (2.18). *Then the following error estimate holds*

$$\|p-p_h\|_{0,\Omega}\le Ch. \tag{3.31}$$

Proof. By using Lemma 3.1, we have

$$\begin{aligned}
&\|p_h-\mathcal{P}_h(p)\|_{0,\Omega}\\
\le&\frac{1}{\beta_1}\sup_{(v_h,q_h)\in X_h\times M_h}\frac{B\big((u_h-\mathcal{I}_hu,p_h-\mathcal{P}_hp);(v_h,q_h)\big)}{|v_h|_{1,\Omega}+\|q_h\|_{0,\Omega}}\\
\le&\frac{1}{\beta_1}\sup_{(v_h,q_h)\in X_h\times M_h}\frac{B\big((u_h-u,p_h-p);(v_h,q_h)\big)}{|v_h|_{1,\Omega}+\|q_h\|_{0,\Omega}}\\
&+\frac{1}{\beta_1}\sup_{(v_h,q_h)\in X_h\times M_h}\frac{B\big((u-\mathcal{I}_hu,p-\mathcal{P}_hp);(v_h,q_h)\big)}{|v_h|_{1,\Omega}+\|q_h\|_{0,\Omega}}.
\end{aligned} \tag{3.32}$$

Since $(v_h,q_h)\in X_h\times M_h$, we conclude that there exists $C>0$ such that

$$|v_h|^2_{1,\Omega}+\|q_h\|^2_{0,\Omega}\le C,\quad |(v_h,q_h)|_h\le C. \tag{3.33}$$

Combining (3.15) with (3.18) and (3.32)-(3.33) yields

$$\|p_h-\mathcal{P}_hp\|_{0,\Omega}\le Ch. \tag{3.34}$$

Therefore, by (3.4) and the triangle inequality we have

$$\|p-p_h\|_{0,\Omega}\le\|p-\mathcal{P}_hp\|_{0,\Omega}+\|p_h-\mathcal{P}_hp\|_{0,\Omega}\le Ch. \qquad\square$$

Theorem 3.3. *Let* $(u,p)\in[H^2(\Omega)^d\cap X]\times[H^1(\Omega)\cap M]$ *be the solution of* (2.5) *and* $(u_h,p_h)\in X_h\times M_h$ *satisfy* (2.18). *Then the following error estimate holds*

$$\|u-u_h\|_{0,\Omega}\le Ch^2\Big(\sqrt{\nu}|u|_{2,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big). \tag{3.35}$$

Proof. Firstly, we consider the following duality Stokes problem:

$$\begin{cases}-\nu\Delta v-\nabla q=u_h-u, & \nabla\cdot v=0,\quad \text{in }\Omega,\\ v=0, & \text{on }\partial\Omega.\end{cases} \tag{3.36}$$

By Theorem 2.1, we have

$$\nu\|v\|_{2,\Omega}+\|q\|_{1,\Omega}\le C\|u-u_h\|_{0,\Omega}. \tag{3.37}$$

Multiplying (3.36) by $(u_h - u)$ gives

$$\begin{aligned}
&\|u-u_h\|_{0,\Omega}^2\\
=&\nu\big(\nabla v,\nabla(u_h-u)\big)_\Omega+\big(q,\nabla\cdot(u_h-u)\big)_\Omega-(p_h-p,\nabla\cdot v)_\Omega\\
\le&\big|B\big((u_h-u,p_h-p);(v-\mathcal{I}_hv,q-\mathcal{P}_hq)\big)\big|+\big|B\big((u_h-u,p_h-p);(\mathcal{I}_hv,\mathcal{P}_hq)\big)\big|\\
\le&C\Big[|(u-u_h,p-p_h)|_h|(v-\mathcal{I}_hv,q-\mathcal{P}_hq)|_h+(p-p_h,\nabla\cdot(v-\mathcal{I}_hv))\Big]. \qquad (3.38)
\end{aligned}$$

Using (3.38) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\|u-u_h\|_{0,\Omega}^2\\
\le&C\Big[|(u-u_h,p-p_h)|_h|(v-\mathcal{I}_hv,q-\mathcal{P}_hq)|_h\\
&+\|p-p_h\|_{0,\Omega}\|\nabla\cdot(v-\mathcal{I}_hv)\|_{0,\Omega}\Big]\\
\le&C\Big[|(u-u_h,p-p_h)|_h^2+\frac{1}{\nu}\|p-p_h\|_{0,\Omega}^2\Big]^{1/2}\\
&\cdot\Big[|(v-\mathcal{I}_hv,q-\mathcal{P}_hq)|_h^2+\nu\|\nabla\cdot(v-\mathcal{I}_hv)\|_{0,\Omega}^2\Big]^{1/2}. \qquad (3.39)
\end{aligned}$$

Using Theorems 3.1 and 3.2, (3.30), (3.4) and (3.37), we obtain

$$\begin{aligned}
\|u-u_h\|_{0,\Omega}^2&\le Ch^2\Big(\nu|u|_{2,\Omega}^2+\frac{1}{\nu}|p|_{1,\Omega}^2\Big)^{1/2}\Big(\nu|v|_{2,\Omega}^2+\frac{1}{\nu}|q|_{1,\Omega}^2\Big)^{1/2}\\
&\le Ch^2\Big(\sqrt{\nu}|u|_{2,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{1,\Omega}\Big)\|u-u_h\|_{0,\Omega}. \qquad (3.40)
\end{aligned}$$

This completes the proof. □

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