Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 3, pp. 301-318 (2009) June 2009

A Discontinuous Galerkin Method Based on a BGK Scheme for the Navier-Stokes Equations on Arbitrary Grids

Hong Luo[1,*], Luqing Luo[1] and Kun Xu[2]

[1] *Department of Mechanical and Aerospace Engineering North Carolina State University, Raleigh, NC, 27695, USA*

[2] *Department of Mathematics Hong Kong University of Science and Technology, Hong Kong, China*

Received 09 December 2008; Accepted (in revised version) 12 March 2009
Available online 22 April 2009

Abstract. A discontinuous Galerkin Method based on a Bhatnagar-Gross-Krook (BGK) formulation is presented for the solution of the compressible Navier-Stokes equations on arbitrary grids. The idea behind this approach is to combine the robustness of the BGK scheme with the accuracy of the DG methods in an effort to develop a more accurate, efficient, and robust method for numerical simulations of viscous flows in a wide range of flow regimes. Unlike the traditional discontinuous Galerkin methods, where a Local Discontinuous Galerkin (LDG) formulation is usually used to discretize the viscous fluxes in the Navier-Stokes equations, this DG method uses a BGK scheme to compute the fluxes which not only couples the convective and dissipative terms together, but also includes both discontinuous and continuous representation in the flux evaluation at a cell interface through a simple hybrid gas distribution function. The developed method is used to compute a variety of viscous flow problems on arbitrary grids. The numerical results obtained by this BGKDG method are extremely promising and encouraging in terms of both accuracy and robustness, indicating its ability and potential to become not just a competitive but simply a superior approach than the current available numerical methods.

AMS subject classifications: 76M10
Key words: Discontinuous Galerkin Methods, BGK scheme, compressible Navier-Stokes equations.

1 Introduction

The accuracy of many finite-volume and finite-element methods currently used in

*Corresponding author.
URL: `http://www.mae.ncsu.edu/directories/faculty/luo.html`
Email: hong_luo@ncsu.edu (H. Luo), lluo2@ncsu.edu (L. Luo), makxu@ust.hk (K. Xu)

computational science and engineering is at best second order. There are a number of situations where these numerical methods do not reliably yield engineering-required accuracy. The development of a practical higher-order (>2nd) solution method could help alleviate this accuracy problem by significantly decreasing time required to achieve an acceptable error level. Unfortunately, numerous reasons exist for why current finite-volume algorithms are not practical at higher order and have remained second-order. The root cause of many of these difficulties lies in the extended stencils that these algorithms employ. By contrast, discontinuous Galerkin (DG) finite element formulation introduces higher-order effects compactly within the element. While DG was originally introduced by Reed and Hill [1] for solving the neutron transport equation back in 1973, major interest did not focus on it until the nineties [2–5]. Nowadays, it is widely used in the computational fluid dynamics, computational aeroacoustics, and computational electromagnetics, to name just a few [6–17]. The discontinuous Galerkin methods (DGM) combine two advantageous features commonly associated with finite element and finite volume methods (FVM). As in classical finite element methods, accuracy is obtained by means of high-order polynomial approximation within an element rather than by wide stencils as in the case of FVM. The physics of wave propagation is, however, accounted for by solving the Riemann problems that arise from the discontinuous representation of the solution at element interfaces. In this respect, the methods are therefore similar to FVM. What is known so far about this method offers a tantalizing glimpse of its full potential. Indeed, what sets this method apart from the crowd is many attractive features it possesses: (1) It has several useful mathematical properties with respect to conservation, stability, and convergence. (2) The method can be easily extended to higher-order (>2nd) approximation. (3) The method is well suited for complex geometries since it can be applied on unstructured grids. In addition, the method can also handle non-conforming elements, where the grids are allowed to have hanging nodes. (4) The method is highly parallelizable, as it is compact and each element is independent. Since the elements are discontinuous, and the inter-element communications are minimal, domain decomposition can be efficiently employed. The compactness also allows for structured and simplified coding for the method. (5) It can easily handle adaptive strategies, since refining or coarsening a grid can be achieved without considering the continuity restriction commonly associated with the conforming elements. The method allows easy implementation of *hp*-refinement, for example, the order of accuracy, or shape, can vary from element to element. (6) It has the ability to compute low Mach number flow problems without recourse to the time-preconditioning techniques normally required for the finite volume methods.

In contrast to the enormous advances in the theoretical and numerical analysis of the DGM, the development of a viable, attractive, competitive, and ultimately superior DG method over the more mature and well-established second order methods is relatively an untouched area. This is mainly due to the fact that DGM have a number of weaknesses that have to be addressed, before they can be applied to flow problems of practical interest in a complex configuration environment. In particular, how to ef-