On Convergence of a Least-Squares Kansa's Method for the Modified Helmholtz Equations

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Abstract. We analyze a least-squares asymmetric radial basis function collocation method for solving the modified Helmholtz equations. In the theoretical part, we proved the convergence of the proposed method providing that the collocation points are sufficiently dense. For numerical verification, direct solver and a sub-space selection process for the trial space (the so-called adaptive greedy algorithm) is employed, respectively, for small and large scale problems.

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1 Introduction

The unsymmetric RBF collocation method was first proposed by Kansa [6,7]. Since then, many successful applications, from linear partial differential equations [17] to nonlinear shallow-water model [24], of recently developed mesh-free methods can be found in different Mathematics, Physics and Engineering journals.

From the theoretical point of view, the original unsymmetric RBF collocation formulation has neither error bounds nor convergence proofs. In the original formulation proposed by Kansa, the trial and test spaces were closely related; e.g. the set of collocation points and RBF centers coincide. This formulation may fail because the method results in singular systems in some specially constructed situations [5].

In order to carry out some mathematical analysis, it is necessary to make further assumptions and modify the formulation. In [11], we show that solvability can be guaranteed if the Kansa's method was modified in such a way that the test space and

trial space are de-linked. In particular, we show that sets of *proper* RBF centers exist so that the Kansa's resultant matrix is of full rank.

Later, we proposed in [13, 21] another variant of the method so that error bounds for the Poisson problems become possible. Convergence results and error bounds with respect to the $L^\infty(\Omega)$-norm are derived. A direct translation of theories to numerical algorithm results in solving an overdetermined resultant system with linear optimization whose implementation is not at all trivial; see [13] for an adaptive on-the-fly algorithm.

In [8], different formulations of the unsymmetric meshless collocation methods for solving the Poisson problems are compared in exact arithmetics. The numerical solution of convergent unsymmetric collocation method in [13] converges faster than the interpolant with respect to the residual norm. Most importantly, the numerical results in [8] suggests that, if the resulting overdetermined Kansa's system is solved by the least-squares minimization, the accuracy of the approximate PDE solution improves. This motivates the presented research.

In this paper, we are interested in the convergence theories of a radial basis function (RBF) method for solving the modified Helmholtz equation in strong form. In Section 2, we present the methodology of our proposed method. Section 3 devotes to the convergence proof of the proposed method that is done in three main parts. First, we give a brief overview of RBF *interpolation/approximation* theories. Next, a continuous dependency of the modified Helmholtz equation in a special form, which suits the least-squares approach in our formulation, is derived. Then, the denseness requirements of collocation points needed for our convergence results are studied. Finally, we put all the ideas together and show the convergence and error bounds. In Section 4, some numerical examples are given to conclude the work.

2   Overdetermined least-squares Kansa's method

Let $L := \Delta - k^2$, $k \in \mathbb{R}$, denote the modified Helmholtz operator and Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with boundary $\partial\Omega$. Moreover, suppose f is continuous in $\overline{\Omega}$ and g is continuous on $\partial\Omega$. We consider the modified Hemholtz equation with Dirichlet boundary conditions

$$Lu = f \quad \text{in } \Omega, \tag{2.1a}$$
$$u = g \quad \text{on } \partial\Omega. \tag{2.1b}$$

We assume that (2.1) has the exact solution u^* lying in some infinite dimensional trial spaces $\mathcal{U}$; we postpone the precise definition of $\mathcal{U}$ to Section 3.4 in which we prove the convergence of the proposed method.

To obtain a numerical formulation, we need to discretize $\mathcal{U}$ by some finite dimensional subspaces $\mathcal{U}_N$. The overdetermined least-squares based Kansa's method can be initialized by a user-defined set of N scattered RBF centers

$$\Xi_N := \{\xi_i\}_{i=1}^N.$$

For any radial basis kernel Φ, the numerical solutions are expanded in the form of

$$u_N(x) := \sum_{\xi_i \in \Xi_N} \lambda_i \Phi(x, \xi_i) \in \mathcal{U}_N := \left\{ v \; : \; v(x) = \sum_{\xi_i \in \Xi_N} \lambda_i \Phi(x, \xi_i), \; \lambda \in \mathbb{R}^N \right\},$$

where $\lambda := (\lambda_1, \ldots, \lambda_N)^T$ are the unknown coefficients to be solved. We define the *continuous* residual norm or Λ-norm on $\mathcal{U}$ as

$$\|u\|_\Lambda^2 := \|Lu\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2. \tag{2.2}$$

Definition 2.2 will always provide a seminorm. By the continuous dependency of the problem on data (shown later in Section 3.2), we know (2.2) is in fact a norm.

To make the problem numerically accessible, the test space, that is the set of infinitely many collocation conditions, has to be discretized. At the same time, we need to replace the continuous residual norm by a discretized one.

Suppose a sequence of finite number of *quasi-uniform* collocation conditions

$$Lu(x_i) = f(x_i), \quad \text{for} \quad x_i \in \Omega, \quad i = 1, \ldots, M,$$
$$u(\bar{x}_i) = g(\bar{x}_i), \quad \text{for} \quad \bar{x}_i \in \partial\Omega, \quad i = 1, \ldots, m,$$

are imposed at the M and m chosen collocation points in Ω and on $\partial\Omega$, respectively. For convenience, we denote these sets of collocations points by

$$(X_M, \overline{X}_m) := \left(\{x_i\}_{i=1}^M, \{\bar{x}_i\}_{i=1}^m \right) \subset (\Omega, \partial\Omega).$$

The original Kansa's method always has $N = \mathcal{M} := M + m$ and $\Xi_N = X_M \bigcup \overline{X}_m$. Hence, it results in square resultant matrices.

For any continuous function $f \in C(\overline{\Omega})$ and any set of distinct points X, we define the $\ell^2(X)$-norm as

$$\|y\|_{\ell^2(X)}^2 := \sum_{x_i \in X} y(x_i)^2. \tag{2.3}$$

For sufficiently large $\mathcal{M}$ (that will be made precise in Section 3.3), using these interior and boundary collocation points with definition (2.3), we define the *discretized* residual norm on $\mathcal{U}_N$ as

$$\|u\|_{\Lambda_\mathcal{M}}^2 := \frac{1}{M}\|Lu\|_{\ell^2(X_M)}^2 + \frac{1}{m}\|u\|_{\ell^2(\overline{X}_m)}^2. \tag{2.4}$$

Note that (2.4) is in fact the commonly used root-mean-squares norm with different weights applying to the interior and boundary points.

With the assumption that $\mathcal{M} > N$, the approximated solution of the overdetermined least-squares based Kansa's method is defined to be the minimizer of the discrete residual over the finite trial subspace

$$u_N := \arg\min_{v \in \mathcal{U}_N} \|v - u^*\|_{\Lambda_\mathcal{M}}^2. \tag{2.5}$$

By construction, the solution to the minimization problem (2.5) is equivalent to the least-squares solution of the matrix system $A\lambda = b$ where the resultant matrix is given by

$$[A]_{ij} = \begin{cases} \sqrt{m} \cdot L\phi(x_i - \xi_j), & 1 \leq i \leq M, & 1 \leq j \leq N, \\ \sqrt{M} \cdot \phi(\overline{x}_i - \xi_j), & M+1 \leq i \leq \mathcal{M}, & 1 \leq j \leq N, \end{cases}$$

and right-hand vector is given by

$$[b]_i = \begin{cases} \sqrt{m} \cdot f(x_i), & 1 \leq i \leq M, \\ \sqrt{M} \cdot g(\overline{x}_i), & M+1 \leq i \leq \mathcal{M}. \end{cases}$$

The unknown coefficient vector λ can be found by solving the system with least-squares minimization. Once λ is obtained, the numerical solution can be evaluated anywhere over $\overline{\Omega}$. As a numerical note, the least-squares solver should be implemented by QR- or SVD-decomposition instead of the normal equation in order to avoid worsening the problem of ill-conditioning.

3 Convergence analysis

To obtain the convergence theory for the least-squares Kansa's method, we heavily depend on the RBF interpolation theories that are reviewed in Section 3.1. Here, we assume the exact solution lies in the native space of the RBF. Other necessary tools are the continuous dependency shown in Section 3.2 and the equivalence of the continuous and the discretized residual norms given in Section 3.3. For these, we further assume the solution is smooth and continuous; more specifically

$$u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

This assumption is to the fact that the native space, for the kernel theories to apply, contains rather smooth functions; these extra assumptions are not too much to ask for. When all the assumptions hold, we prove the convergence of the proposed method in Section 2. We will only give the complete proof with $d \geq 2$. After some straightforward modifications, the result also holds when $d = 1$.

3.1 RBF interpolation

For simplicity, we focus our discussion on some symmetric (conditionally) positive definite kernels $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. If we take the reproducing kernel Hilbert space associated with Φ (native space) as the trial space

$$\mathcal{U} = \mathcal{N}_\Phi := \overline{\text{span}\{\Phi(\cdot, y) : y \in \Omega\}}^{\|\cdot\|_\Phi}$$

where $\|\cdot\|_\mathcal{U} = \|\cdot\|_\Phi$ is induced from the inner product

$$\left\langle \sum_{i=1}^{k} c_i \Phi_i(\cdot, x_i), \sum_{j=1}^{k} d_j \Phi_j(\cdot, y_j) \right\rangle = \sum_{i=1}^{k} \sum_{j=1}^{k} c_i d_j \Phi_i(x_i, y_j),$$

then the standard h^β-type error bound [19, 20], where h is the fill distance of the data and β is the smoothness of Φ, can be used. For the exponential error bounds $\mathcal{O}(\lambda^{c/h})$– convergence ($c$ fixed and $0 < \lambda < 1$) for the Gaussians and (inverse) multiquadrics kernels, see [15, 16]. For functions in some Sobolev spaces but not the native space, error bounds for functions outside the native space can be found in [18]. The RBF convergence theory itself is a big topic to be included in the work. Readers are also referred to the research monographs on RBF [2, 4, 22] for recent reviews and details.

For this work, we require only that the discretized trial spaces $\mathcal{U}_N$ have certain approximation power such that for all $v \in \mathcal{U}$ there is some approximation $s_{v,\epsilon} \in \mathcal{U}_N$ with

$$\|v - s_{v,\epsilon}\|_\Lambda \le \epsilon \|v\|_\mathcal{U}, \tag{3.1}$$

with a small number $\epsilon > 0$. The exact expression for ϵ depends on our assumption on the exact solution u^* and the chosen kernel Φ, when the function $s_{u^*,\epsilon}$ will be taken as the interpolant to the PDE solution u^* on the trial centers. Since the approximation factor ϵ is a more important parameter than N, the number of RBF centers, we consider $N = N(\epsilon)$ hereafter.

3.2 Continuous dependence on data

This section is devoted to prove a special version of continuous dependence. In particular, the H^1-norm of the solution of (2.1) can be bounded by the L^2-norm of the *residual*. We begin with some necessary lemmas.

Lemma 3.1. *Let $u \in H^1(\Omega)$ be the solution of (2.1). Then, when $d \ge 2$, we have*

$$\|u\|_{H^1(\Omega)} \le C_1(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}),$$

where C_1 is a generic constant and depends only on k and Ω.

Proof. Consider the $H^1(\Omega)$-norm of u,

$$\|u\|_{H^1(\Omega)}^2 := \int_\Omega \left(|\nabla u|^2 + u^2\right) \le \max\left\{1, \frac{1}{k^2}\right\} a(u,u), \tag{3.2}$$

where the standard bilinear form is given by

$$a(u,v) := \int_\Omega \nabla u \nabla v + k^2 \int_\Omega uv, \tag{3.3}$$

with any test function $v \in H^1(\Omega)$. Taking $v = u$ in (3.3), the Green formula yields a bound for $a(u,u)$

$$0 \le a(u,u) = \int_{\partial\Omega} u\frac{\partial u}{\partial n} - \int_\Omega uf$$
$$\le \left|\int_{\partial\Omega} u\frac{\partial u}{\partial n} + \int_\Omega uf\right| \le \int_{\partial\Omega} \left|u\frac{\partial u}{\partial n}\right| + \int_\Omega |uf|. \tag{3.4}$$

By the Cauchy-Schwarz theorem, we have

$$\int_{\partial\Omega}\left|u\frac{\partial u}{\partial n}\right| \leq \|g\|_{L^2(\partial\Omega)}\left\|\frac{\partial u}{\partial n}\right\|_{L^2(\partial\Omega)}.$$

Using the trace theorem, we obtain

$$\left\|\frac{\partial u}{\partial n}\right\|_{L^2(\partial\Omega)} \leq K_1\|u\|_{H^2(\Omega)} = K_1\sqrt{\|u\|_{H^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2}$$

$$\leq K_1\sqrt{(\|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)})^2} = K_1\left(\|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right)$$

$$\leq K_1\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + k^2\|u\|_{L^2(\Omega)}\right),$$

where the constant K_1 depends only on Ω. Hence, we obtain

$$\int_{\partial\Omega}\left|u\frac{\partial u}{\partial n}\right| \leq K_1\|g\|_{L^2(\partial\Omega)}\left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + k^2\|u\|_{L^2(\Omega)}\right)$$

$$\leq K_1\left(\|g\|_{L^2(\partial\Omega)}\left(\|u\|_{H^1(\Omega)} + k^2\|u\|_{L^2(\Omega)}\right) + K_2\|u\|_{H^1(\Omega)}\|f\|_{L^2(\Omega)}\right). \quad (3.5)$$

The last inequality follows from another application of the trace theorem that

$$\|g\|_{L^2(\partial\Omega)} = \|u\|_{L^2(\partial\Omega)} \leq K_2\|u\|_{H^1(\Omega)},$$

and K_2 depends only on Ω.

A bound for the second term of (3.4) can be found similarly; by applying the Cauchy-Schwarz theorem,

$$\int_{\Omega}|uf| \leq \|u\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}\|f\|_{L^2(\Omega)}. \quad (3.6)$$

Combining (3.5) and (3.6), we can rewrite the bound (3.4) as

$$a(u,u) \leq \|u\|_{H^1(\Omega)}\left(K_1(1+k^2)\|g\|_{L^2(\partial\Omega)} + (K_1K_2+1)\|f\|_{L^2(\Omega)}\right). \quad (3.7)$$

Therefore, with the generic constant

$$C_1 := \max\left\{1, \frac{1}{k^2}\right\}\max\left\{K_1(1+k^2), (K_1K_2+1)\right\},$$

the lemma is proven.   □

The bound in Lemma 3.1 does not fit the overdetermined least-squares Kansa's method exactly. Below is a slight modification:

Theorem 3.1. *Let $u \in H^1(\Omega)$ be the solution of the modified Helmholtz Eq. (2.1). Then, we have*

$$\|u\|_{H^1(\Omega)} \leq C_2\sqrt{\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\partial\Omega)}^2}.$$

where C_2 is a generic constant and depends only on k and Ω.

Proof. It is obvious to show that

$$a + b \leq \sqrt{2}\sqrt{a^2 + b^2}.$$

Putting $a = \|f\|_{L^2(\Omega)}$ and $b = \|g\|_{L^2(\partial\Omega)}$ into the inequality, gives

$$\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \leq \sqrt{2}\sqrt{\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)}}.$$

Using Lemma 3.1, we obtain a new bound

$$\|u\|_{H^1(\Omega)} \leq C_1\sqrt{2}\sqrt{\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)}},$$

that proves the Theorem. □

3.3 The Denseness requirement for meshless collocation

In order to allow points to get dense in a controllable manor, we assume that the collocation points are getting dense uniformly: If the sequence of distinct scattered point sets $X_M := \{x_i\}_{i=1}^M$ in Ω is getting dense quasi-uniformly as $M \to \infty$, then for any partitions $P_M = \{\Omega_i\}_{i=1}^M$ of Ω using X_M (i.e. Voronoi diagram) such that

$$x_i \in \Omega_j, \quad \text{if and only if} \quad i = j,$$

there exists two positive constants K_3 and K_4 independent of M, such that for each fixed M and all i, the following inequalities hold

$$K_3 \operatorname{Vol}(\Omega) \leq M \operatorname{Vol}(\Omega_i) \leq K_4 \operatorname{Vol}(\Omega). \tag{3.8}$$

The following Lemma addresses the denseness requirement for discretizing the test space.

Lemma 3.2. *Suppose Ω is regular and bounded. The sets of collocation points $X_M := \{x_i\}_{i=1}^M$ are getting dense quasi-uniformly. Let $y \in C(\overline{\Omega})$ be a positive Riemann square integrable function in Ω. Let the $\ell^2(X_M)$-norm be defined as (2.3). Then for sufficiently large M, there exists two positive constants C_3 and C_4 independent of M such that*

$$C_3\|y\|_{L^2(\Omega)} < \frac{1}{\sqrt{M}}\|y\|_{\ell^2(X_M)} < C_4\|y\|_{L^2(\Omega)}.$$

Proof. Supposed $\{X_M\}_M$ satisfies the assumption stated. Let $\|P_M\| := \max \operatorname{Vol}(\Omega_i)$ denotes the *gap* of the partition P_M.

If $\|y\|_{L^2(\Omega)} = 0$, then $y \equiv 0$ on Ω. Therefore, $\|y\|_{\ell^2(X_M)} = 0$ for all M and the assertion holds. Consider the case when $\|y\|_{L^2(\Omega)} > 0$. Since y is Riemann integrable over Ω,

$$\|y\|^2_{L^2(\Omega)} = \lim_{M \to \infty} \sum_{i=1}^M y^2(x_i) \operatorname{Vol}(\Omega_i) = \lim_{\|P_M\| \to 0} \sum_{i=1}^M y^2(x_i) \operatorname{Vol}(\Omega_i). \tag{3.9}$$

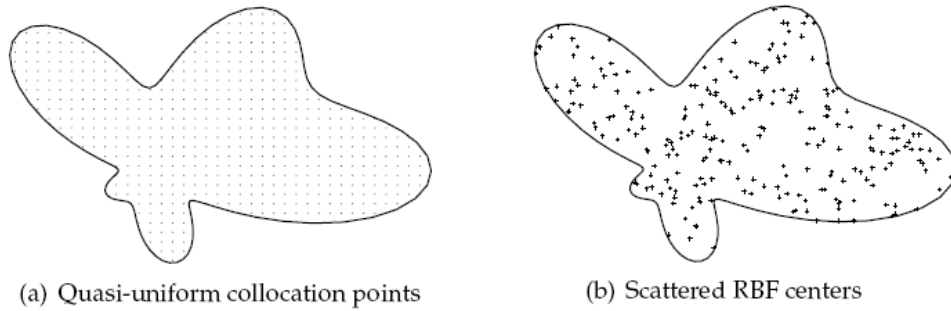(a) Quasi-uniform collocation points   (b) Scattered RBF centers

Figure 1: Schematic of collocation points and RBF centers distributions.

We take $\varepsilon = \frac{1}{2}\|y\|_{L^2(\Omega)}^2$. Knowing that the limit in (3.9) exists, there must exist a M independent constant $\delta(\varepsilon(y, \Omega)) > 0$ such that

$$\left| \sum_{i=1}^M y^2(x_i)\operatorname{Vol}(\Omega_i) - \|y\|_{L^2(\Omega)}^2 \right| < \varepsilon, \qquad \text{if } \|P_M\| < \delta.$$

Equivalently, if M is sufficiently large, we have

$$\frac{1}{2}\|y\|_{L^2(\Omega)}^2 < \sum_{i=1}^M y^2(x_i)\operatorname{Vol}(\Omega_i) < \frac{3}{2}\|y\|_{L^2(\Omega)}^2. \tag{3.10}$$

Combining (3.10) and (3.8) yields the desired inequalities with

$$C_3 = \left(2K_4 \operatorname{Vol}(\Omega)\right)^{-\frac{1}{2}} \quad \text{and} \quad C_4 = \left(\frac{2}{3}K_3 \operatorname{Vol}(\Omega)\right)^{-\frac{1}{2}}.$$

Intuitively, in Lemma 3.2, we would expect that the constants C_3 and C_4 are closer to 1 for point distribution in Fig. 1(a) comparing to those in Fig. 1(b). $\qquad\square$

Repeating arguments on $\partial\Omega$, we have the following corollary.

Corollary 3.1. Suppose $\partial\Omega$ be an oriented closed hypersurface. Let $y \in C(\partial\Omega)$ be a positive Riemann square integrable function on $\partial\Omega$. Let $\ell^2(\overline{X}_m)$-norm be defined as (2.3) where $\overline{X}_m := \{\overline{x}_i\}_{i=1}^m$ are the sets of distinct scattered points on $\partial\Omega$ getting dense quasi-uniformly as $m \to \infty$. Then, for sufficiently large m, we have

$$C_5 \|y\|_{L^2(\partial\Omega)} < \frac{1}{\sqrt{m}} \|y\|_{\ell^2(\overline{X}_m)} < C_6 \|y\|_{L^2(\partial\Omega)},$$

where C_5 and C_6 are positive constants independent of m.

Lemma 3.3. *Suppose the assumptions in Lemma 3.2 and Corollary 3.1 hold. For sufficiently large M and m, the continuous residual norm and the discretized residual norm are equivalent; i.e.,*

$$C_7 \|u\|_\Lambda < \|u\|_{\Lambda_M} < C_8 \|u\|_\Lambda,$$

where C_7 and C_8 are positive constants independent of M and m.

Proof. Using Lemma 3.2 with $y=Lu$ and Corollary 3.1 with $y=u$, we obtain

$$\|u\|_\Lambda^2 < \frac{1}{C_3^2 M}\|Lu\|_{\ell^2(X_M)}^2 + \frac{1}{C_5^2 m}\|u\|_{\ell^2(X_m)}^2.$$

After simplification, we have

$$C_7^2\|u\|_\Lambda^2 < \frac{1}{M}\|Lu\|_{\ell^2(X_M)}^2 + \frac{1}{m}\|u\|_{\ell^2(X_m)}^2 = \|u\|_{\Lambda_\mathcal{M}}^2,$$

where $C_7 = \min\{C_3, C_5\}$. Using similar arguments, the other inequity can be proved with $C_8 = \max\{C_4, C_6\}$. □

In our context, the numbers of collocation points M and m in Lemma 3.3 must depend on the number of RBF centers, N. If this is not the case, $\|u - u^*\|_{\Lambda_\mathcal{M}}$ will be zero as N increases and becomes equal or larger than $\mathcal{M}=M + m$, whereas the error with respect to continuous residual norm is non-zero. This justifies the earlier assumption made that $\mathcal{M}>N$.

3.4 Error estimates

Suppose that f and g are Riemann integrable in Ω and on $\partial\Omega$, respectively. Further assume that the collocation points are dense enough, with respect to ϵ, in the sense of Lemma 3.2 and Corollary 3.1. We are ready to derive the error bounded between the exact solution u^* and the approximate solution u_N.

From Lemma 3.3, the numerical error with respect to the continue residual norm can be bounded by that error with respect to the discretized residual norm

$$\|u_N - u^*\|_\Lambda \leq \frac{1}{C_7}\|u_N - u^*\|_{\Lambda_\mathcal{M}}.$$

By the minimization property of u_N, the (unknown) interpolation to the exact solution $s_{u^*,\epsilon} \in \mathcal{U}_N \subset \mathcal{U}$ gives an upper bound

$$\|u_N - u^*\|_{\Lambda_\mathcal{M}} \leq \|s_{u^*,\epsilon} - u^*\|_{\Lambda_\mathcal{M}}.$$

Using Lemma 3.3 again, the upper bounded can be relaxed to

$$\|s_{u^*,\epsilon} - u^*\|_{\Lambda_\mathcal{M}} \leq C_8\|s_{u^*,\epsilon} - u^*\|_\Lambda.$$

Combining the above observations, we obtain

$$\|u_N - u^*\|_\Lambda \leq \frac{C_8}{C_7}\|s_{u^*,\epsilon} - u^*\|_\Lambda. \tag{3.11}$$

Note that $\mathcal{M}$, which is important in making the continuous and discretized residual norms equivalent, does not appear in the final error bound. Eq. (3.11) also suggests that the numerical solution of the proposed method converges faster than the interpolant, with respect to the Λ-norm, to the exact solution. We conclude the results as the following Theorem.

Theorem 3.2. *Suppose Ω is regular and bounded on which the RBF interpolation theories applied. Let $u^* \in \mathcal{U}$ be the exact solution of the modified Helmholtz Eq. (2.1) with Riemann square integrable data. Let $X_{M(\epsilon)}$ and $\overline{X}_{m(\epsilon)}$ be the sets of distinct scattered collocation points in Ω and on $\partial\Omega$, respectively, getting dense quasi-uniformly. Assume that $\{\mathcal{U}_{N(\epsilon)}\}_\epsilon$ be a sequence of subspaces of $\mathcal{U}$ for $\epsilon \rightarrow 0$ such that for all $v \in \mathcal{U}$ there is an approximation $s_{v,\epsilon} \in \mathcal{U}_{N(\epsilon)}$ with approximation power as in (3.1). Define the approximate solution $u_{N(\epsilon)}$ in the discretized RBF trial space $\mathcal{U}_N \subset \mathcal{U}$,*

$$u_N := \arg\min_{v \in \mathcal{U}_N} \|v - u^*\|^2_{\Lambda_\mathcal{M}}.$$

Then, for sufficiently large M and m, the convergence and error bound are given by

$$\|u_N - u^*\|_{H^1(\Omega)} \leq C_9\,\epsilon\|u^*\|_\mathcal{U},$$

where C_9 is a positive constant independent of M and m.

Proof. Combining Theorem 3.1 and 3.11, we have

$$
\begin{aligned}
&\|u_N - u^*\|_{H^1(\Omega)} \\
&\leq C_2\sqrt{\|L(u_N - u^*)\|^2_{L^2(\Omega)} + \|u_N - u^*\|^2_{L^2(\partial\Omega)}} \\
&= C_2\|u_N - u^*\|_\Lambda \leq \frac{C_8 C_2}{C_7}\|s_{u^*,\epsilon} - u^*\|_\Lambda \leq \frac{C_8 C_2}{C_7}\epsilon\|u^*\|_\mathcal{U}.
\end{aligned}
$$

The last inequality can be refined if the employed RBF is specified. In particular, if one consider (inverse) multiquadric- or Gaussian-RBF, than the proposed least-squares Kansa's method would enjoy exponential convergence [14, 15] if the exact solution u^* lies in the associated native space. □

4 Numerical demonstrations of exponential convergence

In this section, examples in high precision (HP) and double precision (DP) computations, carried out in *Maple*© and *Matlab*© respectively, are shown to demonstrate the proven convergence theory.

Consider the modified Helmholtz Eq. (2.1) in $\Omega = [0,1]^d$ with Dirichlet boundary conditions and exact solution u^* where d is the dimension of the corresponding test problem. In all examples, for simplicity, both the modified Helmholtz parameter $k = 1$ and the shape parameter $c = 1$ are fixed throughout the section. Using different c will have no effect to all HP computations; the error profiles are similar in shape. On the other hand, large c will make the DP results departs from the HP results earlier due to the problem of ill-conditioning. The set of collocation points X and trial centers Y are regularly spaced in Ω with grid spacings, respectively, h_X and h_Y. We let h_Y be some multiples of h_X; namely, $h_Y = wh_X$. Moreover, the multiquadrics kernel

$$\Phi_c(\mathbf{x}, \mathbf{y}) = \sqrt{1 + \frac{\|\mathbf{x} - \mathbf{y}\|^2}{c^2}},$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $c > 0$ is the shape parameter, will be used in the expansion of the unknown solutions

$$u_N(\mathbf{x}) = \sum_{\mathbf{y}_i \in Y} \lambda_i \Phi(\mathbf{x}, \mathbf{y}_i), \ x \in \bar{\Omega}.$$

Accuracy of the approximation is measured by the root-mean-squares

$$RMS(u_N) = \sqrt{\frac{\sum_{\mathbf{z}_i \in Z} (u_N(\mathbf{z}_i) - u^*(\mathbf{z}_i))^2}{|Z|}},$$

where $Z \subset \bar{\Omega}$ is a set of points for evaluating error and $|Z|$ denotes the number of points in Z that is larger than $|X|$. Note that both the theories and numerical algorithms can be applied to irregular domains. Moreover, the sets of points are not necessarily uniform. These settings are imposed only for easy comparison whenever the readers found necessary.

4.1 Example I (1D-HP)

There is an implicit requirement, in Lemma 3.3, imposed to the number of collocation points $\mathcal{M} = |X|$ such that the continuous and discretized residual norms are equivalent. Our first example aims to investigate the effect of $\mathcal{M}$ on the approximation accuracy. In HP computations, the problem of ill-conditioning can be completely ignored. Moreover, the magnitude of errors is somehow irrelevant in this example; we are mainly interested in the convergence behaviors.

Let the functions f and g in (2.1) be generated by the exact solution

$$u_1^*(x) = \exp(-x^2), \quad x \in \mathbb{R}.$$

The grid spacing of trial centers $h_Y = (4i)^{-1}$ for $i = 1, 2, \ldots, 30$. The sets of collocation points are generated by $w = 1, \ldots, 4$. The resulting $\mathcal{M} \times N$ collocation system is solved by the *Maple*© built-in function LEASTSQRS.

Using the MQ kernel, exponential convergence is expected if the exact solution lies in the native space[†] and if the number of collocation points $\mathcal{M}$ is sufficiently large. Fig. 2 shows the error profiles of against different separating distances h_Y with different tested values of w. Immediately, one can see that having more collocation points (e.g. large w value in the figure) results in better accuracy and faster convergence. Firstly, accuracy improves with increasing w can be account by the constants C_7 and C_8 in Lemma 3.3. As w increases, the discretized residual norm is expected to better approximate the continuous one; hence, $C_7 \nearrow 1$ and $C_8 \searrow 1$. This suggests that the error constant $C_{10} = C_2 C_8 / C_9$ in Theorem 3.2 will decrease from above to C_2 but not to 0 when w increases. In other words, for any fixed set of trial centers, increasing the number of collocation points will not yield any convergence.

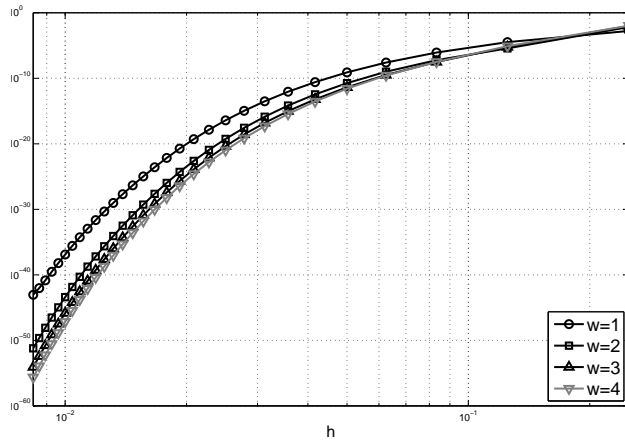[†]Note that we do not know for sure if u_1^* is in $\mathcal{N}_\Phi$ or not.

Figure 2: Example I: RMS errors obtained through high precision (HP) computation for various w as a function of h_Y.

Secondly, the tested cases show monotone decreasing errors for all h's and their errors follow a nice pattern of exponential convergence. The important observation here is that $w = 1$ results in sufficient denseness for the need of exponential convergence.

4.2 Example II (2D-HP and DP)

We now consider a two-dimensional problem in high precision (HP) computations with the exact solution

$$u_2^*(x,y) = \frac{1}{\sqrt{(x-2)^2 + (y-2)^2}}, \quad x, y \in \mathbb{R}.$$

The grid spacing of trial centers $h_Y = (2i + 2)^{-1}$ for $i = 1, 2, \ldots, 9$. The sets of collocation points are generated by $w = 1, 2$. Due to the limitation of computational power, we cannot decrease h_Y further.

Fig. 3 shows the error profiles of different separating distances h_Y against different tested values of w. The solid lines are the resulting profiles arising from HP computations in $Maple^{©}$. Similar to the 1D cases, we can see that having more collocation points results in faster convergence rate.

It is not practical, if not impossible, to solve large-scale problems with HP computations. We solve such problems with DP computations.

The same $\mathcal{M} \times N$ collocation systems are solved by the $Matlab^{©}$ built-in function `Backslash`. When $w = 1$, M is equal to N and the the collocation system is an square system. Based on the setting of `Backslash`, the system will be solved by Gaussian elimination. When $w \geq 2$, the collocation systems are overdetermined that will be solved by the QR algorithm.

Dotted lines in Fig. 3 show the error profiles of different separating distances h_Y against different tested values of w. For $h_Y < 2.5 \times 10^{-1}$, when the collocation systems
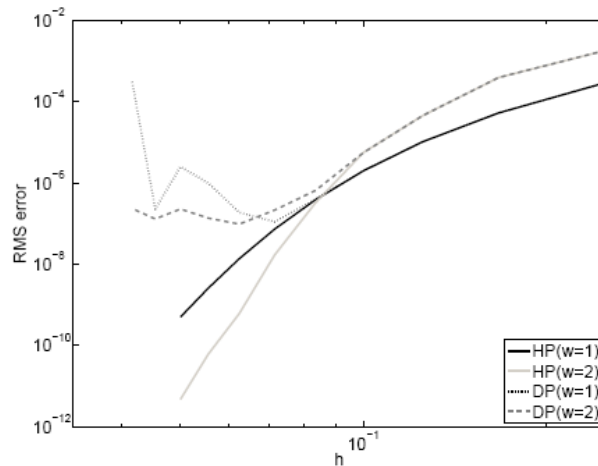
Figure 3: Example III: RMS errors obtained through high precision (HP) and double precision (DP) computations as a function of h_Y.

are relatively well-conditioned in DP, the accuracy in double precision computations coincide with that in HP computations. When h_Y decreases and the problem of ill-conditioning greatly affects the accuracy of the (proposed) Kansa's method. Therefore, the error blows up when $w=1$. For $w=2$, the error is stabilized by the regularization (QR). This observation is not new, see [3, 23], and it is the result of bad chosen basis. In the next example, we will circumvent this problem with an adaptive basis selection algorithm.

4.3   Example III (3D-DP-Greedy)

Considering three-dimensional problems, it are not easy to solve even in DP computations. (For example, `Backslash` cannot be used for large-scale problems.) Besides, when computations are carried out in DP, it is well-known that the problem of ill-conditioning greatly affects the accuracy of the (proposed) Kansa's method. There are different techniques to circumvent such problem; for example, preconditioning technique [1], domain decomposition [10], null space projection method [9], etc.

Suppose the $\mathcal{M} \times N$ overdetermined resultant matrix of the proposed method is ill-conditioned. A trial subspace selection is performed using the (improved) adaptive greedy algorithm proposed in [12]. Based on the primal- and dual-residuals, the algorithm adaptively selects the "best columns" in the original matrix for the best approximation without making the linear system solver breakdown. Note that selecting columns in the original matrix is equivalent to selecting trial centers. Moreover, the adaptive greedy algorithm can be implemented in a matrix-free way–evaluation of the original matrix is not necessary.

The computational cost for solving the original system using direct method is $\mathcal{O}(\mathcal{M}N^2 + N^3)$. Instead of solving the original system, we solve the an $k \times N$ subsystem as our numerical solution with $k \leq \mathcal{M}$ selected columns. The computational cost is then reduced to $\mathcal{O}(k^3 + k^2\mathcal{M} + k^2N)$.

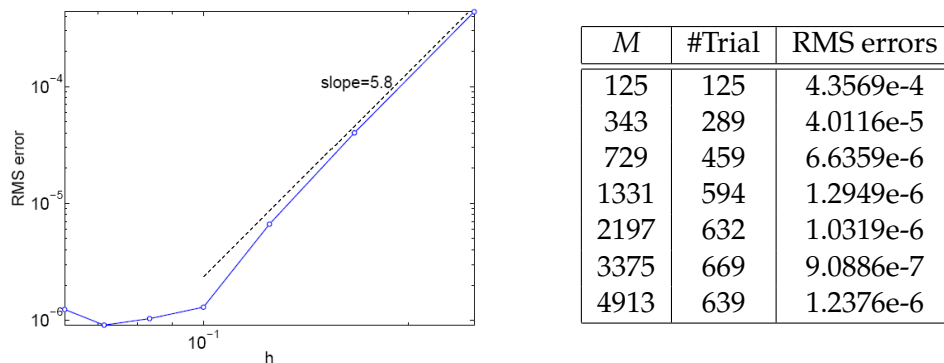| M	#Trial	RMS errors
125	125	4.3569e-4
343	289	4.0116e-5
729	459	6.6359e-6
1331	594	1.2949e-6
2197	632	1.0319e-6
3375	669	9.0886e-7
4913	639	1.2376e-6

Figure 4: Example III: RMS errors as a function of collocation points' spacings h_Y with trial subspace selection.

We now consider a three-dimensional problems with exact solution

$$u_3^*(x, y, z) = \exp(-x^2 - y^2 - z^2).$$

The grid spacing of trial centers

$$h_Y = (2i + 2)^{-1}, \quad \text{for} \quad i = 1, 2, \dots, 7.$$

The sets of collocation points are generated by $w=1$. Therefore, we begin with $h_X = h_Y$ as we are sure that the actual number of collocation points will be larger then the number of trial centers after subspace selection.
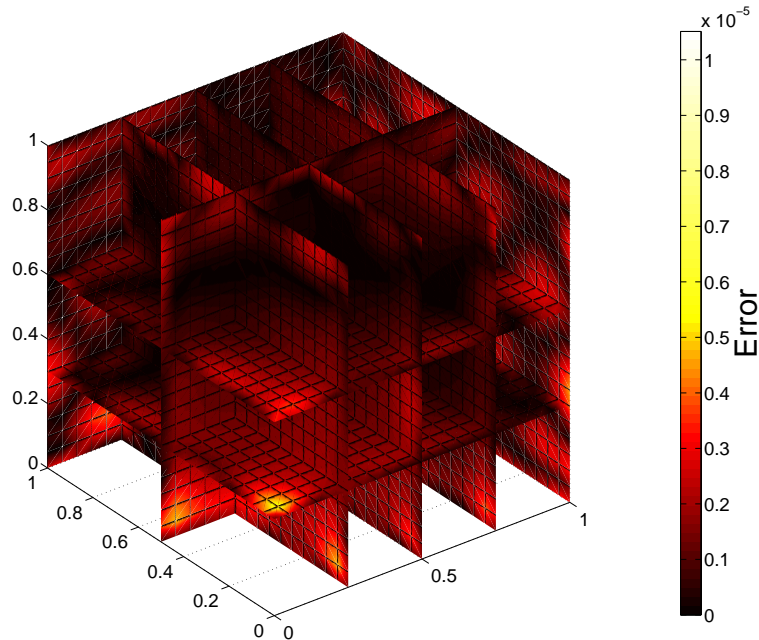

Figure 5: Solving three-dimensional Helmholtz equation with MQ kernel on a square domain: Error plot of the exact solution $u_3^*(x, y, z) = \exp(-x^2 - y^2 - z^2)$.

With the subspace selection process, instead of a clear exponential convergence behavior for small h, high order algebraic convergence is observed, see Fig. 4. Such (faster than) high order (order-5.8 in this example) of algebraic convergence all the way down to h=0.1. Since the trial centers selection process will stop when the ill-conditioning problem of the collocation system is too serious, the numbers of selected trial centers, given under #*Trial* in Fig. 4, do not increase at the same rate as $\mathcal{M}$.

Lastly, in Fig. 5, we show the error function for the case $h_X=h_Y=0.1$ in which we can see that the error are rather uniformly distributed.

5 Conclusions

In the theoretical part, if there are smooth kernel-based trial functions and sufficiently many test functionals, we proved that a modified unsymmetric meshless collocation method converges at the same rate as interpolation of the solution. In particular, trial spaces formed by multiquadrics and Gaussian basis would result in exponential convergence if the solution is analytic. In the practical part, we show some numerical examples with high precision computations, mainly in one dimensional cases and partially in two dimensional cases to demonstrate the proven results on exponential convergence. Because of the limitation of computational power and the problem of ill-conditioning, in two and three dimensional cases, some calculations in double-precision computations are shown. In the three dimensional cases, we couple the previously proposed greedy technique to perform a trial subspace (or RBF centers) selection process and solve the resultant overdetermined system by least square approximation. In these cases, high order algebraic convergence is observed.

Acknowledgements

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