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Exact and Approximate Values of the Period for a "Truly Nonlinear" Oscillator: $\ddot{x} + x + x^{1/3} = 0$

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Abstract. We investigate the mathematical properties of a "truly nonlinear" oscillator differential equation. In particular, using phase-space methods, it is shown that all solutions are periodic and the fixed-point is a nonlinear center. We calculate both exact and approximate analytical expressions for the period, where the exact solution is given in terms of elliptic functions and the method of harmonic balance is used to calculate the approximate formula.

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1 Introduction

For the past several centuries the study of nonlinear oscillations has centered on systems for which the elastic force terms have a harmonic oscillator limiting form [1,3,8]. Thus, the corresponding mathematical models have the structure

$$\ddot{x} + \omega^2 x = \epsilon f(x, \dot{x}), \tag{1.1}$$

where ϵ is a parameter which can be taken to be non-negative; the dot over x indicates a time derivative, i.e., $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$; and $f(x, \dot{x})$ is, generally, a nonlinear function of its arguments. In the limit $\epsilon \to 0$, Eq. (1.1) becomes

$$\ddot{x} + \omega^2 x = 0, \tag{1.2}$$

which is the equation of motion for the linear harmonic oscillator. If the parameter ϵ is small, i.e., $0 < \epsilon \ll 1$, then a number of techniques [1, 3] can be applied to construct/calculate analytic approximations to the oscillatory solutions of Eq. (1.1). However, it should be noted that all such standard procedures are based on the assumption

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that the limiting equation Eq. (1.2) can be obtained from Eq. (1.1). From this it follows that the required oscillatory solutions can be represented by means of generalized asymptotic expansions in ϵ [1].

In recent years, a new class of nonlinear oscillatory differential equations have been studied [6,7]. These equations do not have a harmonic oscillator limiting form when the parameter ϵ goes to zero. Specific examples include the following:

$$\ddot{x} + x^3 = \epsilon(1 - x^2)\dot{x}, \tag{1.3a}$$
$$\ddot{x} + |x|x = -\epsilon\dot{x}, \tag{1.3b}$$
$$\ddot{x} + x + x^{1/3} = -\epsilon x^2\dot{x}. \tag{1.3c}$$

Observe that Eqs. (1.3a) and (1.3b) have elastic force terms given respectively, by $-x^3$ and $-|x|x$, and therefore they have no linear terms. However, the elastic force in Eq. (1.3c) is $-x - x^{1/3}$, and for small x is dominated by $x^{1/3}$ rather than x; consequently, for sufficiently small x, its equation of motion can be approximated by

$$\ddot{x} + x^{1/3} \approx 0. \tag{1.4}$$

Thus, none of these three equations have the proper harmonic oscillator limiting form required for the application of the standard perturbation methods [1,3].

All of these examples are specific cases of the general equation

$$\ddot{x} + g(x) = \epsilon f(x, \dot{x}), \tag{1.5}$$

where $f(x, \dot{x})$ is a polynomial-type function of x and $\dot{x}$, and $g(x)$ the elastic force term, does not have a linear approximation for $x \to 0$, i.e.,

$$g(x) = \mathcal{O}(x^\alpha), \qquad \alpha \neq 1. \tag{1.6}$$

We denote such equations "truly nonlinear oscillatory" (TNL) differential equations, and in several publications have presented techniques to calculate analytic approximations to any periodic solutions they may possess [6,7].

The main purpose of this paper is to investigate the general properties of the following TNL differential equation

$$\ddot{x} + x + x^{1/3} = 0, \tag{1.7}$$

with the main effort placed on calculating both exact and approximate expressions for the period of oscillations. In the next section, we give some preliminaries relating to the fundamental properties of the solutions to Eq. (1.7). Section 3 gives our calculation for the exact period, while in Section 4, an approximate formula for the period is derived. Finally, in the last section, we summarize our results and indicate the next steps in the analysis of Eq. (1.7).

2 Preliminaries

Eq. (1.7) corresponds to a nonlinear, conservative oscillator [1,3,5]. This follows from the fact that it can be rewritten to the form

$$\ddot{x} = -\frac{dU(x)}{dx}, \tag{2.1}$$

where the potential energy function, $U(x)$, is

$$U(x) = \left(\frac{1}{2}\right) x^2 + \left(\frac{3}{4}\right) x^{4/3}. \tag{2.2}$$

Eq. (1.7) may also be expressed as a system of two first-order differential equations

$$\frac{dx}{dt} = y, \tag{2.3}$$

$$\frac{dy}{dt} = -x - x^{1/3}, \tag{2.4}$$

where (x, y) are the two-dimensional phase space variables [3]. Inspection of Eqs. (2.3) and (2.4) shows that the unique fixed-point (or equilibrium solution) is

$$(\bar{x}, \bar{y}) = (0, 0), \tag{2.5}$$

and corresponds to a nonlinear center; see Mickens [5], Section 4.5.8.

The first-order differential equation, from which the trajectories in phase-space, i.e., $y = y(x)$, can be determined, is

$$\frac{dy}{dx} = -\frac{x + x^{1/3}}{y}. \tag{2.6}$$

Since this equation is separable, a first integral can be readily found

$$\left(\frac{1}{2}\right) y^2 + \left(\frac{1}{2}\right) x^2 + \left(\frac{3}{4}\right) x^{4/3} = \left(\frac{1}{2}\right) A^2 + \left(\frac{3}{4}\right) A^{4/3}. \tag{2.7}$$

To obtain this result, we applied the initial conditions

$$x(0) = A, \qquad y(0) = \dot{x}(0) = 0. \tag{2.8}$$

Using standard arguments [3], it can be easily demonstrated that Eq. (2.7) is a closed curve in the x-y phase space. From this result, it follows that all solutions to our original differential equation Eq. (1.7) are periodic [3,5].

Inspection of Eq. (2.6) also shows that it is invariant under the following three transformations:

$$\begin{array}{llll} T_1: & x \to -x, & y \to y, & \text{(reflection in the } y\text{-axis)}, \\ T_2: & x \to x, & y \to -y, & \text{(reflection in the } x\text{-axis)}, \\ T_3: & x \to -x, & y \to -y, & \text{(inversion through the origin)}. \end{array}$$

It should be indicated that independently of the direct use of the first-integral Eq. (2.7), these symmetry operations can be used to prove "geometrically" that all solutions to Eq. (1.7) are periodic [3, 5]. Further, Eq. (1.7) is an odd-parity system [4] with the consequence that the Fourier representation of its periodic solutions contain only odd multiples of the (angular) frequency Ω, i.e.,

$$x(t) = \sum_{k=0}^{\infty} a_k \cos(2k+1)\Omega t, \tag{2.9}$$

where a_k are the Fourier coefficients and where if T is the period, we have

$$\Omega = \frac{2\pi}{T}. \tag{2.10}$$

In general, since Eq. (1.7) is a nonlinear oscillator, it is expected that the period and hence the frequency Ω are both functions of the initial conditions, i.e.,

$$T = T(A), \qquad \Omega = \Omega(A).$$

3 Exact solution for the period

We now present the calculations for the period of the TNL oscillator modeled by Eq. (1.7). Since few nonlinear oscillators have known, closed form expressions for their periods, the results that we obtain will be of broader interest than just to the specific equation that is being investigated. It clearly demonstrates the complexity of the various exact functions calculated for general nonlinear oscillators and further shows the importance of determining valid approximations for both the period and Fourier representations of the solution [6,7].

From the first-integral, the following expression can be derived for the period $T(A)$ (see, e.g., [3])

$$T(A) = 4\int_0^A \frac{dx}{\sqrt{(A^2-x^2)+\left(\frac{3}{2}\right)\left(A^{4/3}-x^{4/3}\right)}}. \tag{3.1}$$

The linear transformation $x{=}Ay$ gives

$$T(A) = 4\int_0^1 \frac{dy}{\sqrt{(1-y^2)+\beta(1-y^{4/3})}}, \tag{3.2}$$

where $\beta{=}3/2A^{2/3}$. The additional transformation $w{=}y^{2/3}$ further yields the result

$$\frac{T(A)}{6} \equiv \psi(\beta) = \int_0^1 \sqrt{\frac{w}{(1+\beta)-\beta w^2-w^3}}\,dw. \tag{3.3}$$

Note that the cubic expression in the denominator can be written as

$$w^3-\beta w^2-(1+\beta) = (w-1)(w-w_1)(w-w_2), \tag{3.4}$$

where

$$w_{1,2} = \left(\frac{1}{2}\right)\left[-(1+\beta) \pm \sqrt{(\beta+1)(\beta-3)}\right]. \tag{3.5}$$

Note also that $w_{1,2}$ are either negative or are complex conjugate with negative real parts.

The integral given in Eq. (3.3) is a special form of formula 3.16.23 given in Gradshteyn and Ryzhik [2]:

$$\int_u^a \sqrt{\frac{w-b}{(a-w)(w-c)(w-d)}}\,dx$$

where for our case

$$a=1, \quad u=0, \quad b=0, \quad d<c<0, \tag{3.6}$$

with c=w_1 and d=w_2. (More generally, the last inequalities can be taken to be real(c)= real(d)<0.) Thus, we obtain for $\psi(\beta)$

$$\begin{aligned}\psi(\beta) = \frac{2}{\sqrt{(w_1-1)w_2}}\Big\{&(w_2-1)\Pi\left(\frac{\pi}{2}, \frac{1}{w_2}, \frac{w_1-w_2}{w_1 w_2}\right) \\ &+w_2 F\left(\frac{\pi}{2}, \frac{w_1-w_2}{w_1 w_2}\right)\Big\},\end{aligned} \tag{3.7}$$

where $\Pi(\phi, n, k)$ is the elliptic integral of the third-kind and $F(\phi, k)$ is the elliptic integral of the first kind [2].

While these elliptic integrals are well known and their many interesting properties are provided in handbooks, such as Gradshteyn and Ryzhik [2], the absence of elementary representations essentially eliminates them for routine calculations. Thus, while we have an exact analytic expression for the period of the TNL oscillator given by Eq. (1.7), we need something more usable for practicable and/or routine applications. In the next section, we show that such an approximate formula for the period as a function of the amplitude A can be derived.

4 Approximate expression for the period

In the previous section, a closed form but very complex formula was derived for the period of the TNL oscillator of Eq. (1.7). Here, we demonstrate that a much more useful approximation can be derived using the method of harmonic balance. We will merely outline the basic methodology of this technique and refer to Mickens [3,5] for the full details.

The method of simple harmonic balance (see Mickens [3], Section 4.3), in general works extremely well for conservative nonlinear oscillator differential equations. It is a well tested, reliable technique that gives elementary mathematical expressions which provide analytical approximations for the period as a function of the amplitude [3,5]. The basic procedure is as follows:

i) Assume that a first approximation to the periodic solution is $x_a(t){=}A\cos(\Omega_a t)$, where the initial conditions $x_a(0){=}A$ and $\dot{x}_a(0){=}0$ are selected; see Eq. (2.8).

ii) Assume that the needed derivatives are given by $\dot{x}_a(t){=}-\Omega_a A\sin(\Omega_a t)$ and $\ddot{x}_a(t){=}-\Omega_a^2 A\cos(\Omega_a t)$.

iii) Let the differential equation of interest be denoted by

$$H(x,\dot{x},\ddot{x}) = 0, \tag{4.1}$$

and substitute the results of i) and ii) into it and carry out the necessary trigonometric expansions to obtain

$$H_1(A,\Omega_a)\cos(\Omega_a t) + H_2(A,\Omega_a)\sin(\Omega_a t) + \text{(higher-order harmonics)} \simeq 0. \tag{4.2}$$

(Note that for a conservative oscillator $H_2(A,\Omega_a){\equiv}0$.) For the general case, harmonic balancing gives

$$H_1(A,\Omega_a) = 0, \qquad H_2(A,\Omega_a) = 0. \tag{4.3}$$

These give two relations which can be solved for A and Ω_a. As noted, for a conservative oscillator, we only have

$$H_1(A,\Omega_a) = 0, \tag{4.4}$$

and the solution of this relation will provide $\Omega_a(A)$, i.e., the frequency will be expressed as a function of the amplitude [3].

To apply simple harmonic balance to our equation

$$\ddot{x} + x + x^{1/3} = 0, \tag{4.5}$$

we will have to have a knowledge of

$$x^{1/3} \to [A\cos(\Omega_a t)]^{1/3}.$$

The required result is

$$(\cos\theta)^{1/3} = a_1\left[\cos\theta - \frac{\cos 5\theta}{5} + \frac{\cos 5\theta}{10} - \frac{7\cos 7\theta}{110} + \cdots\right], \tag{4.6}$$

where

$$a_1 = \frac{\Gamma\left(\frac{1}{3}\right)}{2^{1/3}\left[\Gamma\left(\frac{2}{3}\right)\right]^2} = 1.159595\ldots. \tag{4.7}$$

(See Mickens [5], Section 2.7.)

Substituting these results into Eq. (4.5) gives

$$(-A\Omega_a^2 + A + a_1 A^{1/3})\cos(\Omega_a t) + \text{(higher-order harmonics)} \simeq 0, \tag{4.8}$$

and applying harmonic balance, the expression is

$$-A\Omega_a^2 + A + a_1 A^{1/3} = 0, \tag{4.9}$$

or

$$\Omega_a^2(A) = 1 + \frac{a_1}{A^{2/3}}. \tag{4.10}$$

Using the period-frequency relation $\Omega T = 2\pi$, we finally obtain the following formula for an approximation to the period

$$T_a(A) = 2\pi \left[\frac{A^{2/3}}{a_1 + A^{2/3}} \right]^{1/2}. \tag{4.11}$$

Inspection of this relation shows that

$$T_a(A) = \begin{cases} 2\pi, & \text{for large } A; \\ \left(\frac{2\pi}{\sqrt{a_1}} \right) A^{1/3}, & \text{for small } A; \end{cases}$$

where

$$\begin{cases} \text{large } A \Rightarrow A \gg a_1^{3/2}, \\ \text{small } A \Rightarrow A \ll a_1^{3/2}. \end{cases}$$

Note that in these limits, the "dominant" differential equations governing these behaviors are

$$\begin{cases} \ddot{x} + x \simeq 0, & A \text{ large}; \\ \ddot{x} + x^{1/3} \simeq 0, & A \text{ small}. \end{cases}$$

Clearly, Eq. (4.11) is exact for both large and small values of the amplitude, and provides th proper extrapolation for all intermediate values of A.

5 Summary

We have achieved the purpose of this paper, namely, the determination of both exact and approximate expressions for the period of a TNL oscillator modeled by Eqs. (1.7) or (4.5). The exact expression for $T(A)$ is clearly complex in mathematical structure, involving both first- and third-kind elliptic integrals, and would not in general be very useful for application in practical computations. However, the approximate expression is rather elementary in form and can be easily interpreted in terms of small and large amplitude oscillations for the original nonlinear differential equation. This formula for $T(A)$ is a further demonstration of the power of the harmonic balance method when used to calculate approximations to the solutions of very nonlinear oscillators.

A future problem is to consider the generalization of Eq. (1.7) to

$$\ddot{x} + x + x^{1/3} = \epsilon(1 - x^2)\dot{x}, \tag{5.1}$$

which is an extension of the standard van der Pol oscillator [3,4]

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}. \tag{5.2}$$

Preliminary work gives reasons to believe that Eq. (5.1) has a unique, stable limit-cycle [7].

Finally, an additional research problem might be the way the analytic findings of this paper relate to the discrete properties of the nonstandard finite difference schemes considered in the recent paper of Ehrhardt and Mickens [9].

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