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A Note on Exact Solutions to Linear Differential Equations by the Matrix Exponential

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Abstract. It is known that the solution to a Cauchy problem of linear differential equations:

$$x'(t) = A(t)x(t), \quad \text{with} \quad x(t_0) = x_0,$$

can be presented by the matrix exponential as $\exp(\int_{t_0}^t A(s)\,ds)x_0$, if the commutativity condition for the coefficient matrix $A(t)$ holds:

$$\Big[\int_{t_0}^t A(s)\,ds, A(t)\Big] = 0.$$

A natural question is whether this is true without the commutativity condition. To give a definite answer to this question, we present two classes of illustrative examples of coefficient matrices, which satisfy the chain rule

$$\frac{d}{dt}\exp(\int_{t_0}^t A(s)\,ds) = A(t)\exp(\int_{t_0}^t A(s)\,ds),$$

but do not possess the commutativity condition. The presented matrices consist of finite-times continuously differentiable entries or smooth entries.

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1 Introduction

It is known that one of the important problems in the theory of differential equations is how to solve Cauchy problems of linear differential equations [1]. If the fundamental matrix solution is found, unique solutions to the Cauchy problems of linear differential equations can be automatically presented.

However, on one hand, it is not always possible to compute the fundamental matrix solution explicitly. On the other hand, linear differential equations are also used in solving nonlinear integrable equations, in both continuous and discrete cases [2,3]. Therefore, the explicit representation of solutions to the Cauchy problems of linear differential equations is a crucial issue in the theory of both linear and nonlinear differential equations.

Let us specify a system of linear differential equations on an interval $I=(a,b)\subseteq\mathbb{R}$ as follows:

$$x'(t) = A(t)x(t) + f(t), \tag{1.1}$$

where $f(t)\in\mathbb{R}^n$ is continuous on I and $A(t)$ is an $n \times n$ matrix of real continuous functions on I. Any higher-order scalar linear differential equation of Kovalevskaia type can be transformed into the above linear system. Of significant importance in the theory of differential equations is how to solve the Cauchy problem on I:

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0,$$

where $t_0\in I$ and $x_0\in\mathbb{R}^n$ are given. Various examples of finding solutions to Cauchy problems of differential equations, both linear and nonlinear, can be found in [1,4].

If the coefficient matrix $A(t)$ commutes with its integral $\int_{t_0}^t A(s)\,ds$:

$$[A(t), \int_{t_0}^t A(s)\,ds] = 0, \quad t \in I, \tag{1.2}$$

then the fundamental matrix solution $U(t,t_0)$ of the homogeneous system $x'(t)= A(t)x(t)$ is determined by the matrix exponential (see, say, [5]):

$$U(t,t_0) = \exp \int_{t_0}^t A(s)\,ds, \quad t \in I. \tag{1.3}$$

That is to say, if we have the commutativity condition (1.2), the following chain rule holds:

$$\frac{d}{dt} \exp \int_{t_0}^t A(s)\,ds = A(t) \exp \int_{t_0}^t A(s)\,ds, \quad t \in I, \tag{1.4}$$

and so, the unique solution to the Cauchy problem (1) is given by the variation of parameters formula:

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^t U(t,s)f(s)\,ds, \quad t \in I. \tag{1.5}$$

However, if the commutativity condition (1.2) does not hold, then we cannot expect to have the proceeding chain rule (1.4). A counterexample on the real line given by Liu [6] is

$$A(t) = \begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}. \tag{1.6}$$

It is computed in [6] that

$$\exp \int_0^t A(s)\, ds = \begin{bmatrix} e^{t^2/2} & \frac{2}{t}(e^{t^2/2} - 1) \\ 0 & 1 \end{bmatrix},$$

and thus, the chain rule (1.4) does not hold for the matrix $A(t)$ determined by (1.6).

It is natural that we wonder if the commutativity condition (1.2) is necessary to guarantee the chain rule (1.4), at a given point $t_0 \in I$. Namely, is there any matrix $A(t)$ of real continuous functions, which satisfies the chain rule (1.4) but violates the commutativity condition (1.2) at a point $t_0 \in I$? This is about the solution representation of the Cauchy problem at a point $t_0 \in I$, and thus, will help us answer the question raised by Ma and Shekhtman in [7]: Can we have the matrix exponential representation (1.3) for the fundamental matrix solution without the commutativity condition (1.2)? This is one of the most fundamental questions and should not be overlooked in the theory of differential equations.

In this paper, we would like to present two classes of illustrative examples which satisfy the chain rule (1.4) but do not satisfy the commutativity condition (1.2) at a fixed point $t_0 \in I$. The paper is organized as follows. In Section 2, two classes of concrete examples of continuous matrices are constructed, in which matrices consist of finite-times continuously differentiable entries or smooth entries. Other matrix forms are also analyzed under two transformations to present different classes of the required matrices. In Section 3, a few further remarks are finally given.

2 Presenting illustrative examples

Let $I=(a,b)\in\mathbb{R}$. We will use the following result by Ma and Shekhtman [7], in our construction of illustrative examples. Assume that

$$B(t) = \begin{bmatrix} c & F(t) \\ 0 & 0 \end{bmatrix}, \quad t \in I, \tag{2.1}$$

where $F(t)$ is a non-constant differentiable function and c is a nonzero complex number satisfying $e^c=1+c$ (see [8] for more details on such complex numbers c). Then on the interval I, we have

$$\frac{d}{dt} e^{B(t)} = B'(t)\, e^{B(t)}, \quad \text{but} \quad [B'(t), B(t)] \not\equiv 0. \tag{2.2}$$

Let us now fix $t_0 \in I$. We are going to construct two classes of continuous matrices $A(t)$ on I, which satisfy our requirements:

$$\frac{d}{dt} e^{B(t)} = A(t)\, e^{B(t)}, \quad \text{but} \quad [A(t), B(t)] \not\equiv 0 \ \text{ on } \ I, \tag{2.3}$$

where $B(t) = \int_{t_0}^{t} A(s)\, ds$. That is to say, the matrix exponential $e^{B(t)}$ produces the unique solution $e^{B(t)} x_0$ to the Cauchy problem of the corresponding homogeneous linear differential equations with $x(t_0) = x_0$, but the commutativity condition (1.2) is not satisfied.

Example 2.1. (Matrices with Finite-times Continuously Differentiable Entries)
First, let $m \in \mathbb{N}$, and we select $t_0 < t_1 < b$ and a complex number $d \neq 0$ so that

$$h := \int_{t_0}^{t_1} \sin^m (s - t_1)\, ds \neq 0,$$

and $c := dh$ satisfies $e^c = 1 + c$. Define a continuous matrix $A(t)$ on I as follows:

$$A(t) = \begin{cases} A_1, & \text{if } t < t_1, \\ A_2, & \text{if } t \geq t_1, \end{cases} \tag{2.4a}$$

$$A_1 = \begin{bmatrix} d \sin^m (t - t_1) & d \sin^m (t - t_1) \\ 0 & 0 \end{bmatrix}, \tag{2.4b}$$

$$A_2 = \begin{bmatrix} 0 & g(t) \sin^m (t - t_1) \\ 0 & 0 \end{bmatrix}, \tag{2.4c}$$

where g is a nonzero smooth function on $[t_1, b)$. Obviously, any matrix of this class is $(m-1)$-times continuously differentiable on I, but its m-th order derivative doesn't exist at $t = t_1$. Actually at $t = t_1$, its $(1,1)$th entry has no m-th order derivative, but its $(1,2)$th entry may have the m-th order derivative, which depends on what kind of functions $g(t)$ the matrix $A(t)$ involves. A direct computation yields

$$B(t) := \int_{t_0}^{t} A(s) ds = \begin{cases} B_1, & \text{if } t < t_1, \\ B_2, & \text{if } t \geq t_1, \end{cases} \tag{2.5a}$$

$$B_1 = \begin{bmatrix} d \displaystyle\int_{t_0}^{t} \sin^m (s - t_1)\, ds & d \displaystyle\int_{t_0}^{t} \sin^m (s - t_1)\, ds \\ 0 & 0 \end{bmatrix}, \tag{2.5b}$$

$$B_2 = \begin{bmatrix} c & c + \displaystyle\int_{t_1}^{t} g(s)\, \sin^m (s - t_1)\, ds \\ 0 & 0 \end{bmatrix}. \tag{2.5c}$$

Example 2.2. (Matrices with Smooth Entries)
Second, we select $t_0 < t_1 < b$ and a complex number $d \neq 0$ so that the number

$$c = d\, e^{\frac{1}{t_0 - t_1}},$$

satisfies $e^c=1+c$. Define a continuous matrix $A(t)$ on I as follows:

$$A(t)=\begin{cases} A_1, & \text{if } t<t_1, \\ A_2, & \text{if } t=t_1, \\ A_3, & \text{if } t>t_1, \end{cases} \tag{2.6a}$$

$$A_1=\begin{bmatrix} \dfrac{d}{(t-t_1)^2}\, e^{\frac{1}{t-t_1}} & \dfrac{d}{(t-t_1)^2}\, e^{\frac{1}{t-t_1}} \\ 0 & 0 \end{bmatrix}, \tag{2.6b}$$

$$A_2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3=\begin{bmatrix} 0 & g(t)\, e^{\frac{1}{t_1-t}} \\ 0 & 0 \end{bmatrix}, \tag{2.6c}$$

where $g(t)$ is a nonzero Laurent polynomial (possibly polynomial) in $t-t_1$. Obviously, any matrix of this class is infinitely differentiable on I. Actually, any order left-sided and right-sided derivatives of the $(1,1)$th and $(1,2)$th entries at $t=t_1$ are equal to zero. A direct computation gives rise to

$$B(t):=\int_{t_0}^{t} A(s)\,ds=\begin{cases} B_1, & \text{if } t<t_1, \\ B_2, & \text{if } t=t_1, \\ B_3, & \text{if } t>t_1, \end{cases} \tag{2.7a}$$

$$B_1=\begin{bmatrix} c-d\,e^{\frac{1}{t-t_1}} & c-d\,e^{\frac{1}{t-t_1}} \\ 0 & 0 \end{bmatrix}, \tag{2.7b}$$

$$B_2=\begin{bmatrix} c & c \\ 0 & 0 \end{bmatrix}, \quad B_3=\begin{bmatrix} c & c+\displaystyle\int_{t_1}^{t} g(s)\, e^{\frac{1}{t_1-s}}\,ds \\ 0 & 0 \end{bmatrix}. \tag{2.7c}$$

Proof: Now, let us show that the above two classes of matrices provide the required examples. Namely, we need to verify that the two classes of matrices defined above satisfy the requirements in (2.3).

When $t<t_1$, we have the commutativity condition:

$$[A(t),B(t)]=0, \tag{2.8}$$

and thus, the chain rule

$$\frac{d}{dt}\,e^{B(t)}=A(t)\,e^{B(t)}, \tag{2.9}$$

holds on the sub-interval (a,t_1).

When $t>t_1$, the differentiable functions

$$F(t)=c+\int_{t_1}^{t} g(s)\,\sin^m(s-t_1)\,ds, \quad F(t)=c+\int_{t_1}^{t} g(s)\,e^{\frac{1}{t_1-s}}\,ds, \quad t>t_1,$$

have nonzero derivatives on the sub-interval (t_1,b), and so, they are not constant functions on (t_1,b). It then follows from the aforementioned result in (2.1) and (2.2) that the chain rule (2.9) holds but the commutativity condition (2.8) is not satisfied on the sub-interval (t_1, b).

When $t{=}t_1$, the chain rule (2.9) follows from the continuity of

$$\frac{d}{dt}e^{B(t)}, \text{ and } A(t)e^{B(t)}.$$

To conclude, for the above two classes of matrices, the chain rule (2.9) holds for all $t{\in}I$ but the commutativity condition (2.8) is not satisfied on the whole interval I. $\square$

Note that two transformations

$$B(t) \mapsto PB(t)P^{-1}, \quad B(t) \mapsto B(t) + \alpha(t)I_2, \tag{2.10}$$

where

$$P = P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$\alpha(t)$ is an arbitrary differentiable function and I_2 is the identity matrix of size 2, do not change the properties in (2.2). Therefore, based on these two transformations, we can also begin with the following forms of basic complex matrices:

$$B(t) = \begin{bmatrix} 0 & 0 \\ F(t) & c \end{bmatrix}, \quad \begin{bmatrix} 0 & F(t) \\ 0 & -c \end{bmatrix}, \quad \begin{bmatrix} -c & 0 \\ F(t) & 0 \end{bmatrix},$$

which satisfy (2.2). Similarly, other classes of matrices satisfying our requirements in (2.3) can be presented.

However, all examples of matrices constructed above have complex entries. To construct examples of purely real matrices, we use the real matrix representation of complex numbers as in [7]:

$$\bar{a}_{ij}(t) = \begin{bmatrix} \mathrm{Re}(a_{ij}(t)) & \mathrm{Im}(a_{ij}(t)) \\ -\mathrm{Im}(a_{ij}(t)) & \mathrm{Re}(a_{ij}(t)) \end{bmatrix}, \tag{2.11}$$

and introduce real square matrices

$$\bar{A}(t) := \begin{bmatrix} \bar{a}_{11}(t) & \bar{a}_{12}(t) \\ \bar{a}_{21}(t) & \bar{a}_{22}(t) \end{bmatrix}_{4\times 4}, \tag{2.12}$$

associated with $A(t){=}(a_{ij}(t))_{2\times 2}$ defined by either (2.4) or (2.6). These matrices give us the required matrices of real continuous functions, for which the chain rule (2.9) holds without the commutativity condition (2.8).

3 Conclusions and remarks

We have analyzed a few sets of possibilities to represent solutions to Cauchy problems of linear differential equations by the matrix exponential. Two classes of illustrative examples of continuous matrices were presented, whose entries are finite-times differentiable and smooth, respectively. The corresponding systems of linear differential equations have the matrix exponential form for solutions to their Cauchy problems, but their coefficient matrices do not possess the commutativity condition.

We remark that as did in [7], we can construct much bigger size matrices to satisfy our requirements in (2.3) by inserting arbitrary sub-matrices. Two of the key points in our construction above are to use the complex number c satisfying $e^c=1+c$ and to define coefficient matrices piecewise. The presented examples are counterparts of the insightful examples by Horn and Johnson [9] about the relation between

$$[A,B]=0, \quad \text{and} \quad e^{A+B}=e^A\,e^B,$$

where A and B are two square matrices of the same size. Furthermore, it is known [6] that the chain rule (2.9) may not hold, if we do not require the commutativity condition (2.8). Therefore, our results also provide a complete supplement to the chain rule without the commutativity condition.

It is interesting to note that our illustrative examples using $e^c=1+c$ does not obey the condition [10]:

$$e^{\lambda_1(t)-\lambda_2(t)}-[\lambda_1(t)-\lambda_2(t)]-1\neq 0, \tag{3.1}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are distinct eigenvalues of $B(t)$. It is evident that every matrix $B(t)$ in Section 2, defined by either (2.1) with any $t\in I$, or (2.5) with $t\geq t_1$, or (2.7) with $t\geq t_1$, has two eigenvalues c and 0. The above condition (3.1) with $\lambda_1=c$ and $\lambda_2=0$ clearly becomes $e^c\neq 1+c$, which violates our selection criteria for the constant c. Actually, the condition (3.1) at a point $t\in I$ is sufficient to guarantee that the chain rule

$$\frac{d}{dt}e^{B(t)}=B'(t)e^{B(t)},$$

implies the commutativity

$$[B(t),B'(t)]=0,$$

at this point $t\in I$ (see, e.g., [10,11] for details), but it is not necessary since the matrices $B(t)$ defined by either (2.5) or (2.7) with $t=t_1$ generate counterexamples. Ziebur [11] gave more general examples of matrices than the class of matrices $B(t)$ in (2.1), which satisfy (2.2) but violate (3.1). Discussions on chain rules for general functions of matrices can also be found in [7,12].

We finally point out that the definition of $A(t)$ in our examples depends on the initial time $t_0\in I$. It remains an open question whether there is a continuous matrix $A(t)$ not depending on $t_0\in I$ such that the chair rule (2.9) holds without the commutativity condition (2.8) over the interval I. This is the exact question on the matrix fundamental solution of linear differential equations presented in [7].

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