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Iterative Method for Solving a Problem with Mixed Boundary Conditions for Biharmonic Equation

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Abstract. The solution of boundary value problems (BVP) for fourth order differential equations by their reduction to BVP for second order equations, with the aim to use the available efficient algorithms for the latter ones, attracts attention from many researchers. In this paper, using the technique developed by the authors in recent works we construct iterative method for a problem with complicated mixed boundary conditions for biharmonic equation which is originated from nanofluidic physics. The convergence rate of the method is proved and some numerical experiments are performed for testing its dependence on a parameter appearing in boundary conditions and on the position of the point where a transmission of boundary conditions occurs.

AMS subject classifications: 65N99, 65Z05, 76M25

Key words: Iterative method; biharmonic equation; mixed boundary conditions.

1 Introduction

The solution of fourth order differential equations by their reduction to BVP for the second order equations, with the aim of using available efficient algorithms for the latter ones, attracts attention from many researchers. Namely, for the biharmonic equation with the Dirichlet boundary condition, there has been intensive investigation on the iterative method, which leads the problem to two problems for the Poisson equation at each iteration (see, e.g., [8,9,11]). In 1992, Abramov and Ulijanova [1] proposed an iterative method for the Dirichlet problem for the biharmonic type equation, but the convergence of the method is not proved. In our previous works [3, 4, 6, 7] with the help of boundary or mixed boundary-domain operators introduced appropriately,

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we constructed iterative methods for biharmonic and biharmonic type equations associated with the Dirichlet, Neumann or simple type of mixed boundary conditions. These iterative methods are originated from our earlier works [2,5].

In this work we develop our technique for a problem with rather complicated mixed conditions for biharmonic equation, namely, we consider the following problem

$$\Delta^2 u = f, \qquad \text{in } \Omega, \tag{1.1a}$$
$$\frac{\partial u}{\partial x} = g_1, \quad \frac{\partial \Delta u}{\partial x} = g_2, \qquad \text{on } \Gamma_1, \tag{1.1b}$$
$$u = g_3, \quad \frac{\partial u}{\partial y} + b\Delta u = g_4, \qquad \text{on } \Gamma_2, \tag{1.1c}$$
$$\frac{\partial u}{\partial x} = g_5, \quad \frac{\partial \Delta u}{\partial x} = g_6, \qquad \text{on } \Gamma_3, \tag{1.1d}$$
$$u = g_7, \qquad \text{on } \Gamma_4 \cup \Gamma_5, \tag{1.1e}$$
$$\frac{\partial u}{\partial y} - b\Delta u = g_8, \qquad \text{on } \Gamma_4, \tag{1.1f}$$
$$\Delta u = g_9, \qquad \text{on } \Gamma_5, \tag{1.1g}$$

where Ω is the rectangle $(0,l_1)\times(0,l_2)$, and $\Gamma_1,\cdots,\Gamma_5$ are parts of the boundary $\Gamma=\partial\Omega$ as shown in Fig. 1, Δ is the Laplace operator, f and g_i (i=1, $\cdots$, 9) are functions given in Ω and on parts of the boundary Γ, respectively, b =const$\geq$ 0.

This problem with special right hand sides in equation and boundary conditions describes the slip behaviour in liquid films on surfaces of partterned wettability (see [12]). For the problem in general setting (1.1), we propose an iterative method which reduces it to a sequence of problems for the Poisson equation. The convergence of the method will be established and the numerical experiments will confirm the efficiency of the method under investigation.

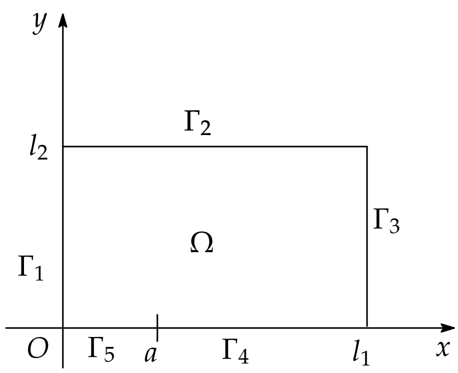

Figure 1: Domain Ω and parts of its boundary.

2 Reduction of the problem to an operator equation

As usual, we set

$$\Delta u = v, \qquad v|_{\Gamma_2} = g, \qquad v|_{\Gamma_4} = h. \tag{2.1}$$

Then the problem (1.1) is decomposed into two consecutive problems

$$\Delta v = f, \qquad \text{in } \Omega, \tag{2.2a}$$

$$\frac{\partial v}{\partial x} = \begin{cases} g_2, & \text{on } \Gamma_1, \\ g_6, & \text{on } \Gamma_3, \end{cases} \tag{2.2b}$$

$$v = \begin{cases} g, & \text{on } \Gamma_2, \\ h, & \text{on } \Gamma_4, \\ g_9, & \text{on } \Gamma_5, \end{cases} \tag{2.2c}$$

and

$$\Delta u = v, \qquad \text{in } \Omega, \tag{2.3a}$$

$$\frac{\partial u}{\partial x} = \begin{cases} g_1, & \text{on } \Gamma_1, \\ g_5, & \text{on } \Gamma_3, \end{cases} \tag{2.3b}$$

$$u = \begin{cases} g_3, & \text{on } \Gamma_2, \\ g_7, & \text{on } \Gamma_4 \cup \Gamma_5. \end{cases} \tag{2.3c}$$

It should be noticed that the solution v of Problem (2.2) and consequently the solution u of Problem (2.3) depends on the temporarily unknown boundary functions g and h. For determining these functions we shall use two remaining conditions in (1.1c) and (1.1f), which may be rewritten in the form

$$\frac{\partial u}{\partial \nu} + b\Delta u = \varphi, \quad \text{on } \Gamma_2 \cup \Gamma_4. \tag{2.4}$$

Here, ν denotes the unit outward normal to the boundary Γ and

$$\varphi = \begin{cases} g_4, & \text{on } \Gamma_2, \\ -g_8, & \text{on } \Gamma_4. \end{cases} \tag{2.5}$$

Taking into account (2.1) and denoting the trace of the function v on $\Gamma_2 \cup \Gamma_4$ by v_0, that is,

$$v_0 = \begin{cases} g, & \text{on } \Gamma_2, \\ h, & \text{on } \Gamma_4, \end{cases}$$

we rewrite (2.4) in the form

$$\frac{\partial u(v_0)}{\partial \nu} + bv_0 = \varphi, \quad \text{on } \Gamma_2 \cup \Gamma_4. \tag{2.6}$$

Now we shall represent the above equality in the form of an operator equation. For this purpose let us introduce an operator S acting on the functions v_0 defined on $\Gamma_2 \cup \Gamma_4$ by the formula

$$Sv_0 = \frac{\partial u}{\partial \nu}\Big|_{\Gamma_2 \cup \Gamma_4}, \tag{2.7}$$

where u is the function found from the problems

$$\Delta v = 0, \qquad \text{in } \Omega, \tag{2.8a}$$

$$\frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \Gamma_1 \cup \Gamma_3, \tag{2.8b}$$

$$v = \begin{cases} v_0, & \text{on } \Gamma_2 \cup \Gamma_4, \\ 0, & \text{on } \Gamma_5, \end{cases} \tag{2.8c}$$

and

$$\Delta u = v, \qquad \text{in } \Omega, \tag{2.9a}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \Gamma_1 \cup \Gamma_3, \tag{2.9b}$$

$$u = 0, \qquad \text{on } \Gamma_2 \cup \Gamma_4 \cup \Gamma_5. \tag{2.9c}$$

In order to investigate properties of the operator S we introduce the space $H=L^2(\Gamma_2 \cup \Gamma_4)$ with the scalar product

$$(v_0, \widetilde{v}_0) = \int_{\Gamma_2 \cup \Gamma_4} v_0.\widetilde{v}_0 d\Gamma, \qquad v_0, \widetilde{v}_0 \in H,$$

and the norm $||v_0||=\sqrt{(v_0, v_0)}$.

Let $\widetilde{u}$ and $\widetilde{v}$ be the solutions of Problem (2.8) and Problem (2.9) corresponding to $\widetilde{v}_0$, respectively.

Property 1. *S is a symmetric and positive operator in H.*

Proof. Let v_0 and $\widetilde{v}_0$ be two functions in H such that Sv_0 and $S\widetilde{v}_0$ belong H, too. Consider

$$J = (Sv_0, \widetilde{v}_0) = \int_{\Gamma_2 \cup \Gamma_4} \frac{\partial u}{\partial \nu}.\widetilde{v} d\Gamma. \tag{2.10}$$

Since

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \Gamma_1 \cup \Gamma_3, \quad \text{(see (2.9))},$$

and

$$\widetilde{v} = 0, \quad \text{on } \Gamma_5, \quad \text{(see (2.8) corresponding to } \widetilde{v}_0),$$

from (2.10) we obtain

$$J = \int_\Gamma \frac{\partial u}{\partial \nu}.\widetilde{v} d\Gamma.$$

Due to

$$u = 0, \quad \text{on } \Gamma_2 \cup \Gamma_4 \cup \Gamma_5, \quad \text{(see (2.9))},$$

and

$$\frac{\partial \widetilde{v}}{\partial \nu} = 0, \quad \text{on } \Gamma_1 \cup \Gamma_3, \quad (\text{see (2.8) corresponding to } \widetilde{v}_0),$$

we have

$$\int_\Gamma u \frac{\partial \widetilde{v}}{\partial \nu} d\Gamma = 0.$$

Therefore, we can write

$$J = \int_\Gamma \left(\frac{\partial u}{\partial \nu} \widetilde{v} - u \frac{\partial \widetilde{v}}{\partial \nu} \right) d\Gamma.$$

Using the Green formula we get

$$J = \int_\Omega (\widetilde{v} \Delta u - u \Delta \widetilde{v}) d\Omega.$$

Now, taking into account that $\Delta \widetilde{v} = 0$ and $\Delta u = v$, we have

$$J = \int_\Omega \widetilde{v}.v d\Omega.$$

Thus,

$$(S v_0, \widetilde{v}_0) = \int_\Omega v.\widetilde{v} d\Omega, \tag{2.11}$$

and the symmetry of S is proved. Next, from (2.11) we have

$$(S v_0, v_0) = \int_\Omega v^2.d\Omega \geqslant 0. \tag{2.12}$$

Let (Sv_0, v_0)=0. Then from (2.12), it follows $v \equiv 0$ in Ω. Consequently, $v|_\Gamma$=0 and in particular $v \,|_{\Gamma_2 \cup \Gamma_4}$=0, i.e, v_0=0. This proves the positivity of S. □

Property 2. *S is compact operator in H.*

Proof. This property can follow from the theory of elliptic problems and embedding theorems (see [10]). Namely, if $v_0 \in H^s(\Gamma_2 \cup \Gamma_4)$ $(s \geq 0)$ then the solution of Problem (2.8) $v \in H^{s+1/2}(\Omega)$, and hence, Problem (2.9) has a solution $u \in H^{s+5/2}(\Omega)$. Therefore, by the embedding theorem we have

$$\frac{\partial u}{\partial \nu}\Big|_{\Gamma_2 \cup \Gamma_4} \in H^{s+1}(\Gamma_2 \cup \Gamma_4).$$

So, the operator S maps $H^s(\Gamma_2 \cup \Gamma_4)$ into $H^{s+1}(\Gamma_2 \cup \Gamma_4)$ and consequently, is compact due to the compactness of the embedding of $H^s(\Gamma_2 \cup \Gamma_4)$ into $H^{s+1}(\Gamma_2 \cup \Gamma_4)$. □

Now, we return to Problems (2.2) and (2.3). Represent their solutions in the form

$$v = \overline{v} + \widehat{v}, \qquad u = \overline{u} + \widehat{u}\,,$$

where $\overline{v}$, $\overline{u}$ are the solutions of Problems (2.8), (2.9), and $\widehat{v}$, $\widehat{u}$ are the solutions of Problems

$$\Delta\widehat{v} = f, \quad \text{in } \Omega, \tag{2.13a}$$

$$\frac{\partial\widehat{v}}{\partial x} = \begin{cases} g_2, & \text{on } \Gamma_1, \\ g_6, & \text{on } \Gamma_3, \end{cases} \tag{2.13b}$$

$$\widehat{v} = \begin{cases} 0, & \text{on } \Gamma_2 \cup \Gamma_4, \\ g_9, & \text{on } \Gamma_5, \end{cases} \tag{2.13c}$$

and

$$\Delta\widehat{u} = \widehat{v}, \quad \text{in } \Omega, \tag{2.14a}$$

$$\frac{\partial\widehat{u}}{\partial x} = \begin{cases} g_1, & \text{on } \Gamma_1, \\ g_5, & \text{on } \Gamma_3, \end{cases} \tag{2.14b}$$

$$\widehat{u} = \begin{cases} g_3, & \text{on } \Gamma_2, \\ g_7, & \text{on } \Gamma_4 \cup \Gamma_5. \end{cases} \tag{2.14c}$$

Then from the definition of S we have

$$Sv_0 = \frac{\partial\overline{u}}{\partial\nu}\Big|_{\Gamma_2\cup\Gamma_4}.$$

Since

$$\frac{\partial u}{\partial\nu} = \frac{\partial\overline{u}}{\partial\nu} + \frac{\partial\widehat{u}}{\partial\nu} = Sv_0 + \frac{\partial\widehat{u}}{\partial\nu},$$

from (2.6) we obtain the equation

$$Sv_0 + bv_0 = \psi, \tag{2.15}$$

where

$$\psi = \varphi - \frac{\partial\widehat{u}}{\partial\nu}, \quad \text{on } \Gamma_2 \cup \Gamma_4. \tag{2.16}$$

Thus, we have obtained Eq. (2.15) for determining the unknown boundary function v_0 on $\Gamma_2 \cup \Gamma_4$.

3 Iterative method

Set

$$B = S + bI, \tag{3.1}$$

where I is the identity operator. Then Eq. (2.15) can be rewritten in the form

$$Bv_0 = \psi. \tag{3.2}$$

Since the operator S is proved to be symmetric, positive and compact in the space H, operator B is also symmetric, positive and

$$bI < B \leq (||S|| + b)I. \tag{3.3}$$

Consider the following iterative scheme

$$\begin{aligned} &\frac{v_0^{(k+1)} - v_0^{(k)}}{\tau} + Bv_0^{(k)} = \psi, \qquad k = 0, 1, ... \\ &v_0 \text{ is given.} \end{aligned} \tag{3.4}$$

Theorem 3.1. *The iterative scheme* (3.4) *is realized by the following process:*

i) Given $v_0^{(0)}$, for example, $v_0^{(0)}$=0 on $\Gamma_2 \cup \Gamma_4$.

ii) Knowing $v_0^{(k)}$ (k=0, 1, ...) solve consecutively two problems

$$\Delta v^{(k)} = f, \qquad in\ \Omega, \tag{3.5a}$$

$$\frac{\partial v^{(k)}}{\partial x} = \begin{cases} g_2, & on\ \Gamma_1, \\ g_6, & on\ \Gamma_3, \end{cases} \tag{3.5b}$$

$$v^{(k)} = \begin{cases} v_0^{(k)}, & on\ \Gamma_2 \cup \Gamma_4, \\ g_9, & on\ \Gamma_5. \end{cases} \tag{3.5c}$$

and

$$\Delta u^{(k)} = v^{(k)}, \qquad in\ \Omega, \tag{3.6a}$$

$$\frac{\partial u^{(k)}}{\partial x} = \begin{cases} g_1, & on\ \Gamma_1, \\ g_5, & on\ \Gamma_3, \end{cases} \tag{3.6b}$$

$$u^{(k)} = \begin{cases} g_3, & on\ \Gamma_2, \\ g_7, & on\ \Gamma_4 \cup \Gamma_5. \end{cases} \tag{3.6c}$$

iii) Update the new approximation by

$$v_0^{(k+1)} = v_0^{(k)} + \tau\left(\varphi - \frac{\partial u^{(k)}}{\partial \nu} - bv_0^{(k)}\right), \quad on\ \Gamma_2 \cup \Gamma_4. \tag{3.7}$$

Proof. Putting $v^{(k)} = \widehat{v} + \overline{v}^{(k)}$, $u^{(k)} = \widehat{u} + \overline{u}^{(k)}$, where $\widehat{v}$, and $\widehat{u}$ are the solution of (2.13) and (2.14), respectively, we obtain that $\overline{v}^{(k)}$, $\overline{u}^{(k)}$ satisfy the following problems

$$\Delta \overline{v}^{(k)} = 0, \qquad \text{in } \Omega, \tag{3.8a}$$

$$\frac{\partial \overline{v}^{(k)}}{\partial \nu} = 0, \qquad \text{on } \Gamma_1 \cup \Gamma_3, \tag{3.8b}$$

$$\overline{v}^{(k)} = \begin{cases} v_0^{(k)}, & \text{on } \Gamma_2 \cup \Gamma_4, \\ 0, & \text{on } \Gamma_5. \end{cases}$$

$$\Delta \overline{u}^{(k)} = \overline{v}^{(k)}, \quad \text{in } \Omega, \tag{3.9a}$$
$$\frac{\partial \overline{u}^{(k)}}{\partial \nu} = 0, \quad \text{on } \Gamma_1 \cup \Gamma_3, \tag{3.9b}$$
$$\overline{u}^{(k)} = 0, \quad \text{on } \Gamma_2 \cup \Gamma_4 \cup \Gamma_5.$$

Consequently, it is easy to see that

$$S v_0^{(k)} = \frac{\partial \overline{u}^{(k)}}{\partial \nu}\Big|_{\Gamma_2 \cup \Gamma_4}.$$

Since

$$\frac{\partial u^{(k)}}{\partial \nu} = \frac{\partial \widehat{u}}{\partial \nu} + \frac{\partial \overline{u}^{(k)}}{\partial \nu},$$

we have

$$\frac{\partial u^{(k)}}{\partial \nu} = S v_0^{(k)} + \frac{\partial \widehat{u}}{\partial \nu}, \quad \text{on } \Gamma_2 \cup \Gamma_4.$$

Substituting this formula into (3.7) we obtain

$$v_0^{(k+1)} = v_0^{(k)} + \tau\Big(\varphi - \frac{\partial \widehat{u}}{\partial \nu} - S v_0^{(k)} - b v_0^{(k)}\Big).$$

Taking into account (2.16) and (3.1) we have

$$v_0^{(k+1)} = v_0^{(k)} + \tau\big(\psi - B v_0^{(k)}\big).$$

This is an another form of iterative scheme (3.4). Thus, the theorem is proved. □

Before stating the result on convergence of the iterative process (3.5)-(3.7) we assume that the data functions of the original problem (1.1a)-(1.1g) are sufficiently smooth and this problem has a unique solution which also is sufficiently smooth.

In the case if $b>0$ we have the following theorem of the convergence of the iterative scheme (3.4).

Theorem 3.2. *Let $b>0$ and let u be the solution of the original problem* (1.1a)-(1.1g). *Then the iterative scheme* (3.4) *(or equivalently, the iterative process* (3.5)-(3.7) *converges if*

$$0 < \tau < \frac{2}{\|S\| + b}. \tag{3.10}$$

In the case if

$$\tau = \frac{1}{\|S\|/2 + b}, \tag{3.11}$$

there holds the estimate

$$\|u^{(k)} - u\|_{H^{5/2}(\Omega)} \leqslant C\rho^k, \quad k = 1, 2, ..., \tag{3.12}$$

where C is a positive constant, and

$$\rho = \frac{1-\xi}{1+\xi}, \qquad \xi = \frac{b}{b+\|S\|}. \tag{3.13}$$

Proof. The convergence of the iterative scheme (3.4) under the condition (3.10) follows directly from the theory of two-layer iterative scheme [13] applied to Eq. (3.2). Besides, if τ is chosen by the formula (3.11) then according to this theory we have

$$\|v_0^{(k)} - v_0\| \leqslant \rho^k \|v_0^{(0)} - v_0\|.$$

In view of this estimate in combination with the estimate

$$||v^{(k)} - v||_{H^{1/2}(\Omega)} \leq C_1 ||v_0^{(k)} - v_0||$$

for the solution of the problem

$$\Delta(v^{(k)} - v) = 0, \qquad \text{in } \Omega, \tag{3.14a}$$

$$\frac{\partial}{\partial x}(v^{(k)} - v) = 0, \qquad \text{in } \Gamma_1 \cup \Gamma_3, \tag{3.14b}$$

$$v^{(k)} - v = \begin{cases} v_0^{(k)} - v_0, & \text{on } \Gamma_2 \cup \Gamma_4, \\ 0, & \text{on } \Gamma_5, \end{cases} \tag{3.14c}$$

and the estimate

$$||u^{(k)} - u||_{H^{5/2}(\Omega)} \leq C_2 ||v^{(k)} - v||_{H^{1/2}(\Omega)}$$

for the solution of

$$\Delta(u^{(k)} - u) = v^{(k)} - v, \quad \text{in } \Omega, \tag{3.15a}$$

$$\frac{\partial}{\partial x}(u^{(k)} - u) = 0, \qquad \text{in } \Gamma_1 \cup \Gamma_3, \tag{3.15b}$$

$$u^{(k)} - u = 0, \qquad \text{on } \Gamma_2 \cup \Gamma_4 \cup \Gamma_5, \tag{3.15c}$$

which follows from the general theory of elliptic problems [10], we obtain the estimate (3.12) with $C = C_1 C_2 \|v_0^{(0)} - v_0\|$. Here C_1 and C_2 are positive constants. □

Now consider the case b=0. In this case the convergence of the iterative scheme (3.4) is guaranteed by Lemma A. 1 in Appendix A. due to the compactness and positivity of the operator S but we can not get any estimate for the error of approximate solution.

4 Numerical examples and discussion

We performed some experiments in MATLAB for testing the convergence of the iterative process (3.5)-(3.7) in dependence on the parameter b appearing in the boundary conditions (1.1c) and (1.1f), which later appears in Eq. (2.15) and in dependence on the position of the point a, where transmission of boundary conditions on Γ_4 and Γ_5 occurs. In the examples considered below the computational domain is the unit square, i.e, l_1=l_2=1, with uniform grids including 65×65 and 129×129 nodes. The mixed BVP for the Poisson equation (3.5), (3.6) is discretized by difference scheme of second order approximation (see Appendix B). After that the obtained system of difference equations is solved by the method of complete reduction [14]. For computing the normal derivative in (3.7) we also use a formula of second order error. Since the calculation or the estimation of $\|S\|$ is difficult, we can not choose optimal iterative parameter τ by the formula (3.10). Instead of this, by experiments we observe that the value

$$\tau^* = \frac{1}{b+0.4},$$

is best in comparison with some other values of τ.

Below we report the results of computation for several examples, where we first take exact solution $u(x,y)$, calculate its corresponding boundary conditions, and afterwards perform iterative process (3.5)-(3.7) until

$$\|u^{(k+1)} - u^{(k)}\|_\infty < h^2,$$

h being the stepsize of the grid.

Example 1 We take

$$u = xe^y + ye^x, \quad h = \frac{1}{64}.$$

We have $h^2 \approx 2.44 \times 10^{-4}$. The results of computation for the cases a=1/2 and a=3/4 are given in Table 1, where K is the number of iterations, error=$\|u^{(k)} - u\|_\infty$.

For a=1/4 and some other values of a the result of convergence of the iterative process (3.5)-(3.7) is only slightly different from the above table.

Table 1: Convergence of the iterative scheme in Example 1 on grid 65×65.

b	$a=1/2$		$a=3/4$	
	K	error	K	error
2	4	1.12e-5	4	1.18e-5
1	4	2.89e-5	4	3.21e-5
0.5	5	5.58e-5	5	5.60e-5
0.2	7	9.10e-5	6	1.93e-4
0.1	8	1.83e-4	7	3.29e-4
0.05	9	2.63e-4	9	2.63e-4
0.02	10	3.27e-4	9	5.11e-4
0.01	10	4.17e-4	10	4.41e-4
0.005	10	4.27e-4	10	4.98e-4
0.002	10	5.08e-4	10	5.37e-4
0.001	10	5.21e-4	10	5.50e-4
0	10	5.34e-4	10	5.64e-4

Table 2: Convergence of the iterative scheme in Example 1 on grid 129×129.

b	K	error
2	4	3.99e-6
1	5	5.21e-6
0.5	6	1.68e-5
0.2	8	4.20e-5
0.1	10	5.65e-5
0.05	12	6.67e-5
0.02	13	1.13e-4
0.01	13	1.56e-4
0.005	14	1.63e-4
0.002	14	1.47e-4
0.001	14	1.69e-4
0	14	1.75e-4

The convergence of the iterative scheme computed on a more dense grid, namely, with h=1/128 is also rather fast as is shown in Table 2 (for a=1/2).

Example 2 We take another function

$$u = \sin x \sin y,$$

and perform computation by the iterative scheme (3.5)-(3.7) on two different grids . The result of convergence of the scheme is given in Table 3, where K_1 and K_2 are the numbers of iterations for h=1/64 and h=1/128, respectively, a=1/2.

Example 3 We take

$$u = (x-1)^2(y-1), \qquad a = \frac{1}{2}.$$

The result of convergence rate of the iterative scheme is presented in Table 4, where as

Table 3: Convergence of the iterative scheme in Example 2.

b	K_1	K_2
2	3	4
1	4	4
0.5	4	5
0.2	5	7
0.1	6	8
0.05	7	9
0.02	7	9
0.01	7	10
0.005	7	11
0.002	8	11
0.001	8	11
0	8	11

Table 4: Convergence of the iterative scheme in Example 3.

b	K_1	K_2
2	3	4
1	4	5
0.5	5	6
0.2	6	8
0.1	7	9
0.05	8	11
0.01	9	12
0.005	9	13
0.001	9	13
0	9	13

in Table 3, K_1 and K_2 are the numbers of iterations for h=1/64 and h=1/128, respectively.

From the results of computation on different examples, as shown in Tables 1-4, we see that the convergence rate of the iterative process (3.5)-(3.7) increases with the growth of b and slightly depends on the position of the point a, where the transmission of boundary conditions occurs. This agrees with Theorem 3.2 because in the performed experiments we use the iterative parameter

$$\tau^* = \frac{1}{b+0.4},$$

which appears close to the optimal value. In this case the convergence rate of the iterative method is determined by ρ given in (3.13), namely,

$$\rho = \frac{\|S\|}{2b+\|S\|}.$$

Of course, the convergence of the discretized version of the iterative process (3.5)-(3.7) on a grid with stepsize h is determined by some ρ_h close to ρ. More precisely, S_h, as discrete analog of S , is always positive definite

$$\delta_h I \leqslant S_h \leqslant \Delta_h I, \quad \delta_h > 0,$$

then with optimal τ_h, the convergence rate is determined by

$$\rho_h = \frac{\Delta_h - \delta_h}{\Delta_h + \delta_h + 2b}.$$

Therefore, the iterative process (3.5)-(3.7) in discretized version, always converges for any $b \geqslant 0$, as was seen from Tables 1-4.

In the above examples for testing the convergence of the proposed iterative method we take the parameter b in the range $[0,2]$, but in the problem of nanofluidic physics [12] it may be any nonnegative number because it is the Navier slip length.

Appendix A

Lemma A.1 *Suppose that A is a linear, symmetric, positive and compact operator in a Hillbert space H and u is the solution of the equation*

$$Au = f, \qquad f \in R(A). \tag{A. 1}$$

Then the iterative method

$$\begin{cases} \dfrac{u_{k+1} - u_k}{\tau} + Au_k = f, & k = 0, 1, ... \\ u_0 \text{ is given,} \end{cases} \tag{A. 2}$$

converges if

$$0 < \tau < \frac{2}{\|A\|} . \tag{A. 3}$$

Proof. Since the operator A is linear, symmetric and compact in H there exists a orthonormal basis of H consisting of the eigenvectors of A. Denote this basis by $e_1, e_2, \cdots, e_n, \cdots$ and the corresponding eigenvalues which are positive by $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$. So, we have $Ae_i = \lambda_i e_i$ $(i=1, 2, \cdots)$. Suppose these eigenvalues are sorted in descending order, i.e.,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots > 0.$$

Set $z_k = u_k - u$. Then, from (A. 2) it is easy to verify that

$$z_{k+1} = (I - \tau A) z_k = (I - \tau A)^k z_0.$$

The convergence of the iterative method (A. 2) for Eq. (A. 1) will be proved if we can show that

$$\forall g \in H, \quad \|(I - \tau A)^k g\| \to 0, \qquad \text{as } k \to \infty. \tag{A. 4}$$

For this purpose we suppose that

$$g = \sum_{n=1}^{\infty} a_n e_n.$$

Then

$$(I - \tau A)^k g = \sum_{n=1}^{\infty} \xi_n^k a_n e_n ,$$

where $\xi_n = 1 - \tau \lambda_n$. We have

$$\|g\|^2 = \sum_{n=1}^{\infty} |a_n|^2, \qquad \|(I - \tau A)^k g\|^2 = \sum_{n=1}^{\infty} |\xi_n|^{2k} |a_n|^2.$$

Let ε be an arbitrary small number, $\varepsilon > 0$. Then, there is a number $N = N(\varepsilon)$ sufficiently large so that

$$\sum_{n=N+1}^{\infty} |a_n|^2 < \frac{\varepsilon^2}{2}. \tag{A. 5}$$

Since τ satisfies the condition (A. 3) and $\|A\|=\lambda_1$ we have $|\xi_n|<1$ $(n=1,2,\cdots,N)$. Therefore $|\xi_n|^{2k}\to 0$ as $k\to\infty$. Hence we can choose $K=K(\varepsilon)$ sufficiently large so that

$$\sum_{n=1}^{N}|\xi_n|^{2k}|a_n|^2<\frac{\varepsilon^2}{2},\qquad \forall\, k>K. \tag{A. 6}$$

Thus, $\forall\, k>K$ we have

$$\begin{aligned}\|(I-\tau A)^k g\|^2&=\sum_{n=1}^{N}|\xi_n|^{2k}|a_n|^2+\sum_{n=N+1}^{\infty}|\xi_n|^{2k}|a_n|^2\\&<\frac{\varepsilon^2}{2}+\sum_{n=N+1}^{\infty}|a_n|^2<\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}=\varepsilon^2,\end{aligned}$$

due to (A. 5) and (A. 6).

In summary, $\forall\varepsilon>0$, $\exists K=K(\varepsilon)$ such that

$$\|(I-\tau A)^k g\|<\varepsilon,\qquad \forall\, k>K,$$

which is the desired result (A. 4). The lemma is proved. □

Appendix B

Discretization of mixed boundary value problem for Poisson equation.
The mixed boundary value problem

$$\begin{aligned}&-\Delta u=f(x,y),\\&u(x,0)=g_1(x),\qquad u(x,l_2)=g_2(x),\\&\frac{\partial u}{\partial x}\Big|_{x=0}=\mu_1(y),\qquad \frac{\partial u}{\partial x}\Big|_{x=l_1}=\mu_2(y),\end{aligned}$$

in$(0,l_1)\times(0,l_2)$, on the uniform grid

$$\bar{\omega}_{h_1h_2}=\big\{(x_i,y_j)=(ih_1,jh_2),\ 0\le i\le M,\ 0\le j\le N\big\},$$

with $h_1=l_1/M$, $h_2=l_2/N$, is approximated by the following difference scheme

$$\begin{aligned}&-\frac{1}{h_1^2}\big(u_{i-1,j}-2u_{ij}+u_{i+1,j}\big)-\frac{1}{h_2^2}\big(u_{i,j-1}-2u_{ij}+u_{i,j+1}\big)=f(x_i,y_j),\\&\qquad\qquad\qquad\qquad 1\le i\le M-1,\quad 1\le j\le N-1,\\&-\frac{2}{h_1}\frac{1}{h_1}(u_{1j}-u_{0j})-\frac{1}{h_2^2}(u_{0,j-1}-2u_{0j}+u_{0,j+1})=-\frac{2}{h_1}\mu_1(y_j)+f(x_0,y_j),\\&\qquad\qquad\qquad\qquad 1\le j\le N-1,\\&\frac{2}{h_1}\frac{1}{h_1}(u_{Mj}-u_{M-1,j})-\frac{1}{h_2^2}(u_{M,j-1}-2u_{Mj}+u_{M,j+1})=\frac{2}{h_1}\mu_2(y_j)+f(l_1,y_j),\\&\qquad\qquad\qquad\qquad 1\le j\le N-1,\\&u_{i0}=g_1(x_i),\quad u_{iN}=g_2(x_i),\quad 0\le i\le M,\end{aligned}$$

where $u_{ij} \approx u(x_i, y_j)$. The truncation error of this difference scheme is of second-order. Introduce the notations

$$U_j = (u_{0j}, u_{1j}, ..., u_{Mj})^T, \qquad r = h_2^2/h_1^2, \qquad d = 2(1+r),$$

$$C = \begin{pmatrix} d & -2r & 0 & \dots & 0 & 0 \\ -r & d & -r & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -2r & d \end{pmatrix} \in R^{(M+1)\times(M+1)},$$

$$F_0 = \big(g_1(x_0), g_1(x_1), \dots, g_1(x_M)\big)^T, \qquad F_N = \big(g_2(x_0), g_2(x_1), \dots, g_2(x_M)\big)^T,$$

$$F_j = h_2^2\Big(f(x_0, y_j) - \frac{2}{h_1}\mu_1(y_j), f(x_1, y_j), \dots, f(x_{M-1}, y_j), f(x_M, y_j) + \frac{2}{h_1}\mu_2(y_j)\Big)^T.$$

Then the difference scheme can be written in the standard form of three- point vector equations

$$\begin{aligned} & -U_{j-1} + CU_j - U_{j+1} = F_j, \qquad 1 \le j \le N-1, \\ & U_0 = F_0, \qquad U_N = F_N, \end{aligned}$$

for which the method of complete reduction [14] is applicable.

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