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Efficient Reconstruction Methods for Nonlinear Elliptic Cauchy Problems with Piecewise Constant Solutions

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Abstract. In this article, a level-set approach for solving nonlinear elliptic Cauchy problems with piecewise constant solutions is proposed, which allows the definition of a Tikhonov functional on a space of level-set functions. We provide convergence analysis for the Tikhonov approach, including stability and convergence results. Moreover, a numerical investigation of the proposed Tikhonov regularization method is presented. Newton-type methods are used for the solution of the optimality systems, which can be interpreted as stabilized versions of algorithms in a previous work and yield a substantial improvement in performance. The whole approach is focused on three dimensional models, better suited for real life applications.

AMS subject classifications: 65J20, 35J60
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1 Introduction

We start by introducing the inverse problem under consideration. Let $\Omega\subset\mathbb{R}^3$, be an open bounded set with piecewise Lipschitz boundary $\partial\Omega$. Further, we assume that

$$\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2,$$

where Γ_i are two open disjoint parts of $\partial\Omega$. Given the function $q : \mathbb{R} \to \mathbb{R}^+$, we define the second order elliptic operator

$$\mathcal{P}(u) := -\nabla \cdot (q(u)\nabla u)\,. \tag{1.1}$$

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We denote by *nonlinear elliptic Cauchy problem* the following problem

$$\begin{cases} \mathcal{P}(u) = f, & \text{in } \Omega, \\ u = g_1, & \text{on } \Gamma_1, \\ q(u)u_\nu = g_2, & \text{on } \Gamma_1, \end{cases} \tag{CP_{nl}}$$

where the pair of functions $(g_1, g_2) \in H^{1/2}(\Gamma_1) \times H^{1/2}_{00}(\Gamma_1)'$ are given *Cauchy data* and $f \in L^2(\Omega)$ is a known source term in the model (see [32, p. 66] or [14] for a definition of the Sobolev spaces).

A solution of (CP_{nl}) is a distribution in $H^1(\Omega)$, which solves the weak formulation of the nonlinear elliptic equation $\mathcal{P}(u)=f$ in Ω and further satisfies the Cauchy data on Γ_1 in the sense of the trace operators. Notice that, if we know the Neumann (or Dirichlet) trace of u on Γ_2, say $q(u)\, u_\nu|_{\Gamma_2}=\varphi$, then u can be computed as the solution of a nonlinear mixed boundary value problem (BVP) in a stable way, namely

$$\begin{cases} \mathcal{P}(u) = f, & \text{in } \Omega, \\ u = g_1, & \text{on } \Gamma_1, \\ q(u)u_\nu = \varphi, & \text{on } \Gamma_2, \end{cases} \tag{BVP}$$

Therefore, in order to solve (CP_{nl}), it is enough to consider the task of determining the Neumann trace of u on Γ_2 (a distribution in $H^{1/2}_{00}(\Gamma_2)'$).

Remark 1.1. For simplicity of the presentation the boundary parts Γ_i are assumed to be connected. Using standard elliptic theory one can prove that the results in this article also hold without this assumption. Moreover, the theory derived here extends naturally to Cauchy problems defined on domains with $\partial\Omega=\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, where Γ_i are disjoint and some extra boundary condition (Dirichlet, Neumann, Robin, ...) is prescribed on Γ_3.

Remark 1.2. Let P be the linear elliptic operator defined in Ω by

$$P\, u := -\sum_{i,j=1}^{3} D_i(a_{i,j} D_j u),$$

where the real functions $a_{i,j} \in L^\infty(\Omega)$ are such that the matrix $A(x) := (a_{i,j})_{i,j=1}^d$ satisfies $\xi^t A(x)\xi > \alpha ||\xi||^2$, for all $\xi \in \mathbb{R}^3$ and for a.e. $x \in \Omega$. Here α is some positive constant. The *linear elliptic Cauchy problem* corresponds to the problem (CP_{nl}) obtained when the operator $\mathcal{P}$ is substituted by P and the Neumann boundary condition is substituted by $u_{\nu_A}|_{\Gamma_1}=g_2$. The linear version of (CP_{nl}) has been intensively investigated over the last years [5–8, 11, 13, 17, 19, 23, 25, 28, 30, 31].

Linear elliptic Cauchy problems were used by Hadamard in the 1920's as an example of (exponentially) ill-posed problem [22]. For linear elliptic operators with analytical coefficients, uniqueness of solutions is known for over half a century [10, 12]. Moreover, as a straightforward argumentation with the Schwarz reflection principle