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Collocation Methods for Hyperbolic Partial Differential Equations with Singular Sources

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Abstract. A numerical study is given on the spectral methods and the high order WENO finite difference scheme for the solution of linear and nonlinear hyperbolic partial differential equations with stationary and non-stationary singular sources. The singular source term is represented by the δ-function. For the approximation of the δ-function, the direct projection method is used that was proposed in [6]. The δ-function is constructed in a consistent way to the derivative operator. Nonlinear sine-Gordon equation with a stationary singular source was solved with the Chebyshev collocation method. The δ-function with the spectral method is highly oscillatory but yields good results with small number of collocation points. The results are compared with those computed by the second order finite difference method. In modeling general hyperbolic equations with a non-stationary singular source, however, the solution of the linear scalar wave equation with the non-stationary singular source using the direct projection method yields non-physical oscillations for both the spectral method and the WENO scheme. The numerical artifacts arising when the non-stationary singular source term is considered on the discrete grids are explained.

AMS subject classifications: 65M06, 65M30, 65M70

Key words: Singular sources, Dirac-δ-function, Direct projection method, Chebyshev collocation method, WENO scheme.

1 Introduction

In various mathematical modeling of physical processes, such as the shock-particle laden flows [5] and the collisions of black-holes [7], the system of partial differential

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equations (PDEs) often involves point sources as a forcing term on the right hand side of PDEs. For example, consider the following 1D scalar hyperbolic equation

$$U_t + F_x(U) = S(x,t), \quad U : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \quad t > 0, \tag{1.1}$$

where $F(U)$ is the flux whose Jacobian $\partial F/\partial U$ has real eigenvalues and $S(x,t)$ is the source term consisting of a singular function such as the δ-function and its derivative(s).

In this work, we consider some numerical issues related to the solution of PDEs with a stationary source ($S(x)$ is a function of space only) and a non-stationary singular source ($S(x,t)$ is a function of both space and time) with the collocation methods such as the spectral methods and high order weighted essentially non-oscillatory (WENO) finite difference methods (see [1,3,4,12] and references therein for details of these methods). It should be noted that the WENO methods are particularly suitable for solving problem with solution containing discontinuities. Following the convention, the singular sources are represented by the Dirac-δ-function and/or its derivative(s).

Due to the singular nature of the point sources, a smooth initial condition may yield a solution containing discontinuities even for a linear scalar equation. High order numerical approximations of such nonsmooth solutions would suffer from the well-known Gibbs phenomenon [3,4]. Consequently, the numerical solutions become oscillatory near the singularity and high order convergence will be lost near the singularity. For the time-dependent problem, the scheme would even become unstable. To handle this situation, the point sources are often regularized to obtain a smoother representation of the δ-function so that the approximation converges to the δ-function in the usual limit sense. Several related regularization methods have been developed [2,10,11,13]. The Gaussian function approximation (GA) method is the simplest approximation popularly used. One can also make use of an alternative definition of the δ-function that the derivative of the Heavside function $H(x)$ is the δ-function in the distribution sense, namely, $dH(x)/dx = \delta(x)$. The derivative operator will then be incorporated into the derivative operator of the flux function F. This is what we call the Direct Projection (DP) method in this study which was proposed in [6].

In this paper, we focus our discussion on the numerical approximations of PDEs on a set of discrete grid points and we shall limit our discussion to the one-dimensional scalar linear and nonlinear problems. We will demonstrate the application of the DP method for solving the nonlinear sine-Gordon equation with a stationary singular source term using the Chebyshev collocation method [3,4,6] and compare the results obtained by the second order finite difference scheme using the GA method. The solution with the spectral method converges quickly when the number of grid points is increased.

For a non-stationary singular source that is moving in time, one has to consider the effect of the movement of the singular source in time and space. The δ-function can be located between two discrete grid points causing errors in modeling of the δ-function. We demonstrate, through a simple linear scalar wave equation with a non-stationary