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# A Note on the GMRES Method for Linear Discrete Ill-Posed Problems

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**Abstract.** In this paper, we are presenting a proposal for new modified algorithms for RRGMRES and AGMRES. It is known that RRGMRES and AGMRES are viable methods for solving linear discrete ill-posed problems. In this paper we have focused on the residual norm and have come-up with two improvements where successive updates and the stabilization of decreases for the residual norm improve performance respectively. Our numerical experiments confirm that our improved algorithms are effective for linear discrete ill-posed problems.

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**Key words**: Numerical computation, GMRES, iterative method, linear discrete ill-posed problem.

## 1 Linear dicrete ill-posed problems

Recently it is tried to use GMRES methods for *linear discrete ill-posed problems (LDIP)*. Conjugate gradient method and SVD is also applied to solve them, but we focus on the GMRES methods for LDIP in this paper. As an introduction, we will shortly describe the LDIP. The details of the GMRES methods for them are taken up in later sections.

Hansen [5], which is a good introduction to *discrete ill-posed problems (LDIP)*, says that the LDIP arise from the discretization of ill-posed problems such as the first kind of Fredholm integral equation. The first kind of Fredholm integral equation

$$\int_0^1 K(s,t)f(t)dt = g(s),\ 0 \le s \le 1, \tag{1.1}$$

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where the right-hand side $g(s)$ and the kernel K are known, but $f$ is unknown, is one of inverse problems. We obtain "input" from "output" when we deal with inverse problems. After descretizing (1.1), a linear system like

$$Ax = \boldsymbol{b}, \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}, \boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^n$, is derived. The coefficient matrix $A$ appeared from the LDIP is generally ill-conditioned, because it has clustered tiny singular values or singular values decaying to zero. The right-hand side vector in (1.2) represents the "output", so it often includes measurement errors. Then the known right-hand side vector is

$$\bar{\boldsymbol{b}} = \boldsymbol{b} + \boldsymbol{b}_{error}. \tag{1.3}$$

Since usually we don't know $\boldsymbol{b}, \boldsymbol{b}_{error}$, the approximate solution is written as

$$\bar{\boldsymbol{x}}_* = \underset{\bar{\boldsymbol{x}}_j,\, j\geq 0}{\arg\min} \|\bar{\boldsymbol{x}}_j - \boldsymbol{x}\|, \tag{1.4}$$

in which $\bar{\boldsymbol{x}}_j, j \geq 0$ is generated in $j$ steps of the GMRES methods. When the size of LDIP is small, the analogous of SVD are used for them. However, some iterative methods such as the CG method [5, 6, 8] and the GMRES meshod [1–3] are applied to the large scale LDIP for regularization.

## 2 GMRES methods for LDIP

The GMRES method by Saad and Shultz [10] is one of the popular iterative methods for the linear system like (1.2) In particular the method works well when the coefficient matrix $A$ is non-symmetric. The GMRES generates an approximate solution whose residual norm is minimum by using a Krylov subspace as follow.

$$\|\boldsymbol{b} - A\boldsymbol{x}_j\|_2 = \min_{\boldsymbol{x}_0 + \mathscr{K}_j(A, \boldsymbol{r}_0)} \|\boldsymbol{b} - A\boldsymbol{x}\|_2, \tag{2.1}$$

$$\mathscr{K}_j(A, \boldsymbol{r}_0) = \mathrm{span}\{\boldsymbol{r}_0, \mathrm{A}\boldsymbol{r}_0, \ldots, \mathrm{A}^{\mathrm{j}-1}\boldsymbol{r}_0\}, \tag{2.2}$$

where $j$ is the iteration number, $\boldsymbol{x}_0$ is the initial guess and $\boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{x}_0$ is the initial residual.

One of the GMRES methods for solving LDIP is the Range Restricted GMRES (RRGMRES) method by Calvetti et al. [2]. This method restricts the Krylov subspace to generating an approximate solution within the range of coefficient matrix $A$. The least squares problem is solved as follows:

$$\|\boldsymbol{b} - A\boldsymbol{x}_j\|_2 = \min_{\boldsymbol{x}_0 + \mathscr{K}(A, A\boldsymbol{r}_0)} \|\boldsymbol{b} - A\boldsymbol{x}\|_2, \tag{2.3}$$

$$\mathscr{K}_j(A, \boldsymbol{r}_0) = \mathrm{span}\{\mathrm{A}\boldsymbol{r}_0, \mathrm{A}^2\boldsymbol{r}_0, \ldots, \mathrm{A}^{\mathrm{j}}\boldsymbol{r}_0\}. \tag{2.4}$$