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# Analysis of Two-Grid Methods for Nonlinear Parabolic Equations by Expanded Mixed Finite Element Methods

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**Abstract.** In this paper, we present an efficient method of two-grid scheme for the approximation of two-dimensional nonlinear parabolic equations using an expanded mixed finite element method. We use two Newton iterations on the fine grid in our methods. Firstly, we solve an original nonlinear problem on the coarse nonlinear grid, then we use Newton iterations on the fine grid twice. The two-grid idea is from Xu′s work [SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777] on standard finite method. We also obtain the error estimates for the algorithms of the two-grid method. It is shown that the algorithm achieve asymptotically optimal approximation rate with the two-grid methods as long as the mesh sizes satisfy $h = \mathcal{O}(H^{(4k+1)/(k+1)})$.

**AMS subject classifications**: 65N30, 65N15, 65M12

**Key words**: Nonlinear parabolic equations, two-grid scheme, expanded mixed finite element methods, Gronwall's Lemma.

## 1 Introduction

In this paper, we consider the following nonlinear parabolic equations

$$\frac{\partial p}{\partial t} - \nabla \cdot (K(p)\nabla p) = f(p, \nabla p), \qquad (x,t) \in \Omega \times J, \tag{1.1}$$

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with initial condition

$$p(x,0) = p^0(x), \qquad x \in \Omega, \tag{1.2}$$

and boundary condition

$$(K(p)\nabla p) \cdot \nu = 0, \qquad (x,t) \in \partial\Omega \times J, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and convex domain with $C^1$ boundary $\partial\Omega$, $\nu$ is the unit exterior normal to $\partial\Omega$, $J = (0,T]$, $K$ is a symmetric positive definite tensor and $K : \Omega \times \mathbb{R} \to \mathbb{R}^{2\times 2}$. Eq. (1.1) can be rewritten as following

$$\frac{\partial p}{\partial t} + \nabla \cdot \Psi = f(p, \nabla p), \tag{1.4}$$

$$K(p)^{-1}\Psi + \nabla p = 0. \tag{1.5}$$

Eqs. (1.1)-(1.3) are the simplification of the modeling of groundwater through porous media [9]. In this particular case, $p$ denotes the fluid pressure; and $K$ is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components representing the permeability divided by the viscosity; $\Psi$ represents the Darcy velocity of the flow; and $f(p, \nabla p)$ models the external flow rate.

The two-grid methods was first introduced by Xu [16, 17] as a discretization technique for nonlinear and nonsymmetric indefinite partial differential equations. It based on the fact the nonlinearity, nonsymmetry and indefiniteness behaving like low frequencies are governed by coarse grid and the related high frequencies are governed by some linear or symmetric positive definite operators. The basic idea of the two-grid method is to solve a complicated problem (nonlinear, nonsymmetric indefinite) on a coarse grid (mesh size $H$) and then solve an easier problem (linear, symmetric positive) on the fine grid (mesh size $h$ and $h \ll H$) as correction.

In many partial differential equations, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. Some specialists have made many important works on some topic of mixed finite element method for linear elliptic or reaction-diffusion equations. In [5, 6], Chen has studied the expanded mixed element methods for some quasilinear second order elliptic equations. However, there doesn't seem to exist much work on theoretical analysis for two-grid methods for mixed finite element approximation of quasilinear or nonlinear parabolic equations in the literature.

Many contributions have been done to the multi-grid schemes for finite element methods, see, for example [11,12]. In [8], the authors have studied a two-grid finite difference scheme for nonlinear parabolic equations. Xu and Zhou have considered some multi-scale schemes for finite element method of elliptic partial differential equations in [18]. Recently, we constructed a new two-grid method of expanded mixed finite