Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 6, pp. 830-844

DOI: 10.4208/aamm.09-m09S09
December 2009

Analysis of Two-Grid Methods for Nonlinear Parabolic Equations by Expanded Mixed Finite Element Methods

Yanping Chen[1,*], Peng Luan[2] and Zuliang Lu[3]

[1] *School of Mathematical Science, South China Normal University, Guangzhou 510631, Guangdong, China*

[2] *Mathematics Teaching and Research Group, Sanshui Experimental High School of Zhongshan University, Foshan 528145, Guangdong, China*

[3] *Hunan Key Lab for Computation and Simulation in Science and Engineering, Department of Mathematics, Xiangtan University, Xiangtan 411105, Hunan, China*

Received 15 April 2009; Accepted (in revised version) 08 October 2009
Available online 18 November 2009

Abstract. In this paper, we present an efficient method of two-grid scheme for the approximation of two-dimensional nonlinear parabolic equations using an expanded mixed finite element method. We use two Newton iterations on the fine grid in our methods. Firstly, we solve an original nonlinear problem on the coarse nonlinear grid, then we use Newton iterations on the fine grid twice. The two-grid idea is from Xu′s work [SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777] on standard finite method. We also obtain the error estimates for the algorithms of the two-grid method. It is shown that the algorithm achieve asymptotically optimal approximation rate with the two-grid methods as long as the mesh sizes satisfy $h = \mathcal{O}(H^{(4k+1)/(k+1)})$.

AMS subject classifications: 65N30, 65N15, 65M12

Key words: Nonlinear parabolic equations, two-grid scheme, expanded mixed finite element methods, Gronwall's Lemma.

1 Introduction

In this paper, we consider the following nonlinear parabolic equations

$$\frac{\partial p}{\partial t} - \nabla \cdot (K(p)\nabla p) = f(p, \nabla p), \qquad (x,t) \in \Omega \times J, \tag{1.1}$$

*Corresponding author.
URL: `http://202.116.32.252/user_info.asp?usernamesp=%B3%C2%D1%DE%C6%BC`
Email: yanpingchen@scnu.edu.cn (Y. Chen), 34122lp@163.com (P. Luan), zulianglux@126.com (Z. Lu)

with initial condition

$$p(x,0) = p^0(x), \qquad x \in \Omega, \tag{1.2}$$

and boundary condition

$$(K(p)\nabla p) \cdot \nu = 0, \qquad (x,t) \in \partial\Omega \times J, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and convex domain with C^1 boundary $\partial\Omega$, ν is the unit exterior normal to $\partial\Omega$, $J = (0,T]$, K is a symmetric positive definite tensor and $K : \Omega \times \mathbb{R} \to \mathbb{R}^{2\times 2}$. Eq. (1.1) can be rewritten as following

$$\frac{\partial p}{\partial t} + \nabla \cdot \Psi = f(p, \nabla p), \tag{1.4}$$

$$K(p)^{-1}\Psi + \nabla p = 0. \tag{1.5}$$

Eqs. (1.1)-(1.3) are the simplification of the modeling of groundwater through porous media [9]. In this particular case, p denotes the fluid pressure; and K is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components representing the permeability divided by the viscosity; Ψ represents the Darcy velocity of the flow; and $f(p,\nabla p)$ models the external flow rate.

The two-grid methods was first introduced by Xu [16, 17] as a discretization technique for nonlinear and nonsymmetric indefinite partial differential equations. It based on the fact the nonlinearity, nonsymmetry and indefiniteness behaving like low frequencies are governed by coarse grid and the related high frequencies are governed by some linear or symmetric positive definite operators. The basic idea of the two-grid method is to solve a complicated problem (nonlinear, nonsymmetric indefinite) on a coarse grid (mesh size H) and then solve an easier problem (linear, symmetric positive) on the fine grid (mesh size h and $h \ll H$) as correction.

In many partial differential equations, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. Some specialists have made many important works on some topic of mixed finite element method for linear elliptic or reaction-diffusion equations. In [5, 6], Chen has studied the expanded mixed element methods for some quasilinear second order elliptic equations. However, there doesn't seem to exist much work on theoretical analysis for two-grid methods for mixed finite element approximation of quasilinear or nonlinear parabolic equations in the literature.

Many contributions have been done to the multi-grid schemes for finite element methods, see, for example [11,12]. In [8], the authors have studied a two-grid finite difference scheme for nonlinear parabolic equations. Xu and Zhou have considered some multi-scale schemes for finite element method of elliptic partial differential equations in [18]. Recently, we constructed a new two-grid method of expanded mixed finite element method for semi-linear reaction-diffusion equation [4]. As a continued work of Chen [3], in this paper, we shall discuss the case that the coefficient matrix function K and the reaction term f are nonlinear, namely, $K = K(p)$ and $f = f(p, \nabla p)$.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p,$$

and a semi-norm $|\cdot|_{m,p}$ given by

$$|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p.$$

We set

$$W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}.$$

For p=2, we denote

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega), \quad \|\cdot\|_m = \|\cdot\|_{m,2}, \quad \|\cdot\| = \|\cdot\|_{0,2}\ .$$

We denote by $L^s(J; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm

$$\|v\|_{L^s(J;W^{m,p}(\Omega))} = \Big(\int_0^T ||v||_{W^{m,p}(\Omega)}^s dt\Big)^{\frac{1}{s}}, \quad \text{for} \quad s \in [1, \infty),$$

and the standard modification for $s = \infty$. The details can be found in [10]. In addition C or c denotes a general positive constant independent of h.

At first, we make the following assumptions for the nonlinear parabolic equations.

(**A1**) There exist positive constants K_* and K^*, such that for $z \in \mathbb{R}^2$,

$$K_* \|z\|^2 \leq z^t K(x,s) z \leq K^* \|z\|^2, \qquad \text{for} \quad x \in \Omega,$$

and that each element of K is twice continuously differential with derivatives up to second order bounded above by K^*.

(**A2**) For some integer $k \geq 1$, we assume that the solution function (1.4)-(1.5) has the following regularity:

$$p \in L^2\left(J; W^{k+2,4}(\Omega)\right), \qquad \Psi \in \left(L^2(J; H^{k+1}(\Omega))\right)^2.$$

To analyze the discretization on a time interval $(0, T)$, let $N > 0$, $\triangle t = T/N$, and

$t^n = n\triangle t$, and set

$$
\begin{aligned}
&\phi^n = \phi(\cdot, t^n), \qquad \partial_t \phi^n = \frac{\phi^n - \phi^{n-1}}{\triangle t},\\
&\|\phi\|_{l^2((0,T);X)} = \Big(\sum_{n=1}^{N} \triangle t \|\phi^n\|_X^2\Big)^{\frac{1}{2}},\\
&\|\phi\|_{l^\infty((0,T);X)} = \max_{1\le n\le N} \|\phi^n\|_X,\\
&\|\varphi\|_{L^2((0,T);X)} = \Big(\int_0^T \|\varphi^n(\cdot,t)\|_X^2 dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

The plan of this paper is as follows. In next section, we introduce some notations and projections. The approximation properties of these projections will be recalled. In section 3, we construct a expanded mixed finite element discretization for the nonlinear parabolic equations (1.1)-(1.3). A two-grid scheme and error estimates for two-dimensional nonlinear parabolic equations using an expanded mixed finite element method are discussed in section 4. Finally, we give the conclusion and future work in section 5.

2 Projection operators and approximation properties

Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product. Let

$$
\begin{aligned}
&\boldsymbol{V} = H(\mathrm{div};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\},\\
&\tilde{\boldsymbol{V}} = \boldsymbol{V} \cap \{\boldsymbol{v}\cdot\nu = 0\}, \qquad W = L^2(\Omega).
\end{aligned}
$$

The Hilbert space $\boldsymbol{V}$ is equipped with the following norm:

$$\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \big(\|\boldsymbol{v}\|^2 + \|\mathrm{div}\boldsymbol{v}\|^2\big)^{\frac{1}{2}}.$$

Let Γ_h denote a quasi-uniform partition of Ω into rectangles or triangles with the partition step h. We form $\boldsymbol{V}_h$ and W_h, discrete subspaces of $\boldsymbol{V}$ and W, using standard mixed finite element space such as the RT [13] spaces of order k, RT_k, or Brezzi-Douglas-Marini [1] spaces of order k, BDM_k. In addition, we assume that
(**A3**) The following inclusion hold for the RT spaces or BDM spaces

$$\nabla\cdot\boldsymbol{v}_h \in W_h, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

We shall employ some projection operators. Let Q_h denote the L^2 projection defined by

$$(\phi, w_h) = (Q_h\phi, w_h), \qquad \forall w_h \in W_h, \tag{2.1}$$

for any $\phi \in L^2(\Omega)$, and

$$(\phi, \boldsymbol{v}_h) = (Q_h\phi, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.2}$$

for any vector valued function $\phi \in (L^2(\Omega))^2$.

We also assume that the following approximation properties hold. For $1 < q \leq \infty$, there exists a positive constant C independent of h such that [1]: for all $\phi \in W^{k+1,q}(\Omega)$ (or $\phi \in (W^{k+1,q}(\Omega))^2$)

$$\|Q_h\phi\|_{0,q} \leq C\|\phi\|_{0,q}, \qquad 2 \leq q < \infty, \tag{2.3}$$

$$\|\phi - Q_h\phi\|_{0,q} \leq C\|\phi\|_{r,q}h^r, \qquad 0 \leq r \leq k+1. \tag{2.4}$$

We use the well known Π_h projection for mixed finite element approximation spaces. We shall assume that there exists a projection operator $\Pi_h : (H^1(\Omega))^2 \to \boldsymbol{V}_h$ such that for $q \in H(\text{div};\Omega)$,

$$(\nabla \cdot \Pi_h q, w_h) = (\nabla \cdot q, w_h), \qquad \forall w_h \in W_h. \tag{2.5}$$

The following approximation properties [1] hold for the projection Π_h:

$$\|q - \Pi_h q\|_{0,q} \leq C\|q\|_{r,q}h^r, \qquad \frac{1}{q} < r \leq k+1, \tag{2.6}$$

$$\|\nabla \cdot (q - \Pi_h q)\|_{0,q} \leq C\|\nabla \cdot q\|_{r,q}h^r, \qquad 0 \leq r \leq k+1. \tag{2.7}$$

We also assume that

$$\|\phi - Q_h\phi\|_\infty \leq C\|\phi\|_\infty h^r, \qquad 0 \leq r \leq k+1, \tag{2.8}$$

$$\|\boldsymbol{v}_h\|_\infty \leq C\|\boldsymbol{v}_h\|_\infty h^{-1}, \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.9}$$

Now, we recall the discrete Gronwall's Lemma (see, e.g., [14]):

Lemma 2.1. *Assume that k_n is a non-negative sequence, and that the sequence φ_n satisfies*

$$\begin{cases} \varphi_0 \leq g_0, \\ \varphi_n \leq g_0 + \displaystyle\sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s\varphi_s, \quad n \geq 1. \end{cases} \tag{2.10}$$

Then φ_n satisfies

$$\begin{cases} \varphi_1 \leq g_0(1+k_0) + p_0, \\ \varphi_n \leq g_0 \displaystyle\prod_{s=0}^{n-1}(1+k_s) + \sum_{s=0}^{n-2} p_s \prod_{r=s+1}^{n-1}(1+k_r) + p_{n-1}, \quad n \geq 2. \end{cases} \tag{2.11}$$

Moreover, if $g_0 \geq 0$ and $p_n \geq 0$ for $n \geq 0$, it follows

$$\varphi_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right) \exp\left(\sum_{s=0}^{n-1} k_s\right), \quad n \geq 1. \tag{2.12}$$

3 Expanded mixed finite element discretization

First, we set three variables: the pressure p, the gradient $\Upsilon=-\nabla p$, and the flux $\Psi=K(p)\Upsilon$.

We suppose that the following assumption is satisfied.

(**A4**) The reaction term f is a sufficiently smooth function with bounded derivatives through the second order on Ω. For all $(\hat{p},\hat{\Upsilon}) \in W \times \boldsymbol{V}$ there exists $F > 0$ such that

$$f_p(\hat{p},\hat{\Upsilon}) \leq F, \qquad f_\Upsilon(\hat{p},\hat{\Upsilon}) \leq F.$$

Now, we define the weak form of the nonlinear parabolic equations (1.1)-(1.3) as follows: Find $(p,\Upsilon,\Psi) \in W \times \boldsymbol{V} \times \tilde{\boldsymbol{V}}$ such that

$$\begin{aligned}
&\Big(\frac{\partial p}{\partial t}, w\Big) + (\nabla\cdot\Psi, w) = \big(f(p,-\Upsilon), w\big), && w \in W, && (3.1)\\
&(\Upsilon,\boldsymbol{v}) = (p,\nabla\cdot\boldsymbol{v}), && \boldsymbol{v}\in\tilde{\boldsymbol{V}}, && (3.2)\\
&(\Psi,\boldsymbol{v}) = \big(K(p)\Upsilon,\boldsymbol{v}\big), && \boldsymbol{v}\in\boldsymbol{V}. && (3.3)
\end{aligned}$$

The discrete time mixed finite element approximation to (3.1)-(3.3) can be defined as follows: Find $(p_h^n,\Upsilon_h^n,\Psi_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ such that

$$\begin{aligned}
&\Big(\frac{p_h^n - p_h^{n-1}}{\triangle t}, w_h\Big) + (\nabla\cdot\Psi_h^n, w_h) = \big(f(p_h^n,-\Upsilon_h^n), w_h\big), && w_h \in W_h, && (3.4)\\
&(\Upsilon_h^n,\boldsymbol{v}_h) = (p_h^n,\nabla\cdot\boldsymbol{v}_h), && \boldsymbol{v}_h\in\tilde{\boldsymbol{V}}_h, && (3.5)\\
&(\Psi_h^n,\boldsymbol{v}_h) = \big(K(p_h^n)\Upsilon_h^n,\boldsymbol{v}_h\big), && \boldsymbol{v}_h\in\boldsymbol{V}_h. && (3.6)
\end{aligned}$$

At the time of $t=t^n$, we rewrite (3.1)-(3.3) in the following form

$$\begin{aligned}
&\Big(\frac{\partial p^n}{\partial t}, w_h\Big) + (\nabla\cdot\Psi^n, w_h) = \big(f(p^n,-\Upsilon^n), w_h\big), && w_h \in W_h, && (3.7)\\
&(\Upsilon^n,\boldsymbol{v}_h) = (p^n,\nabla\cdot\boldsymbol{v}_h), && \boldsymbol{v}_h\in\tilde{\boldsymbol{V}}_h, && (3.8)\\
&(\Psi^n,\boldsymbol{v}_h) = \big(K(p^n)\Upsilon^n,\boldsymbol{v}_h\big), && \boldsymbol{v}_h\in\boldsymbol{V}_h. && (3.9)
\end{aligned}$$

Using the definition of Π_h and Q_h, and the assumption (**A3**) that $\nabla\cdot\boldsymbol{V}_h \subset W_h$, we can obtain

$$\begin{aligned}
&\Big(\frac{Q_h p^n - Q_h p^{n-1}}{\triangle t}, w_h\Big) + (\nabla\cdot\Pi_h\Psi^n, w_h)\\
&\qquad\qquad = \big(f(p^n,-\Upsilon^n), w_h\big) + (\partial_t p^n - p_t^n, w_h), && (3.10)\\
&(Q_h\Upsilon^n,\boldsymbol{v}_h) = (Q_h p^n,\nabla\cdot\boldsymbol{v}_h), && (3.11)\\
&(\Pi_h\Psi^n,\boldsymbol{v}_h) = \big(K(p^n)\Upsilon^n,\boldsymbol{v}_h\big) + (\Pi_h\Psi^n - \Psi^n,\boldsymbol{v}_h). && (3.12)
\end{aligned}$$

Set

$$\mu^n = Q_h p^n - p_h^n, \quad \xi^n = Q_h\Upsilon^n - \Upsilon_h^n, \quad \lambda^n = \Pi_h\Psi^n - \Psi_h^n.$$

Subtracting (3.10)-(3.12) from (3.4)-(3.6), we derive the error equations:

$$\begin{aligned}&\left(\frac{\mu^n-\mu^{n-1}}{\triangle t},w_h\right)+(\nabla\cdot\lambda^n,w_h)\\&\qquad\qquad=\big(f(p^n,-Y^n)-f(p_h^n,-Y_h^n),w_h\big)+(\partial_t p^n-p_t^n,w_h),\qquad(3.13)\\&(\xi^n,\boldsymbol{v}_h)=(\mu^n,\nabla\cdot\boldsymbol{v}_h),\qquad(3.14)\\&(\lambda^n,\boldsymbol{v}_h)=\big(K(p^n)Y^n-K(p_h^n)Y_h^n,\boldsymbol{v}_h\big)+(\Pi_h\Psi^n-\Psi^n,\boldsymbol{v}_h).\qquad(3.15)\end{aligned}$$

Letting $w_h=\mu^n,\boldsymbol{v}_h=\lambda^n,\boldsymbol{v}_h=\xi^n$ in Eqs. (3.13)-(3.15) respectively, we can get

$$\begin{aligned}&\left(\frac{\mu^n-\mu^{n-1}}{\triangle t},\mu^n\right)+(\nabla\cdot\lambda^n,\mu^n)\\&\qquad\qquad=\big(f(p^n,-Y^n)-f(p_h^n,-Y_h^n),\mu^n\big)+(\partial_t p^n-p_t^n,\mu^n),\qquad(3.16)\\&(\xi^n,\lambda^n)=(\mu^n,\nabla\cdot\lambda^n),\qquad(3.17)\\&(\lambda^n,\xi^n)=\big(K(p_h^n)(Y^n-Q_hY^n),\xi^n\big)-\big((K(p_h^n)-K(p^n))Y^n,\xi^n\big)\\&\qquad\qquad+\big(K(p_h^n)\xi^n,\xi^n\big)+(\Pi_h\Psi^n-\Psi^n,\xi^n).\qquad(3.18)\end{aligned}$$

Noticing that

$$\begin{aligned}-\frac{1}{2}\left(\|\mu^{n-1}\|^2+\|\mu^n\|^2\right)&\le-(\mu^{n-1},\mu^n),\qquad(3.19)\\\big(K(p_h^n)\xi^n,\xi^n\big)&=\|K(p_h^n)^{\frac{1}{2}}\xi^n\|^2.\qquad(3.20)\end{aligned}$$

We have

$$(\partial_t\mu^n,\mu^n)+\big(K(p_h^n)\xi^n,\xi^n\big)\ge\frac{1}{2\triangle t}\left(\|\mu^n\|^2-\|\mu^{n-1}\|^2\right)+\|K(p_h^n)^{\frac{1}{2}}\xi^n\|^2.\qquad(3.21)$$

Applying the Taylor expansions, we have

$$f(p^n,-Y^n)-f(p_h^n,-Y_h^n)=f_p(\bar{p},\bar{Y})(p^n-p_h^n)-f_Y(\bar{p},\bar{Y})(Y^n-Y_h^n).$$

Then,

$$\begin{aligned}&\big(f(p^n,-Y^n)-f(p_h^n,-Y_h^n),\mu^n\big)\\\le&\big(F(p^n-Q_hp^n),\mu^n\big)+\big(F\mu^n,\mu^n\big)-\big(F(Y^n-Q_hY^n),\mu^n\big)-(F\xi^n,\mu^n).\qquad(3.22)\end{aligned}$$

Moreover,

$$\begin{aligned}&|(\partial_t p^n-p_t^n,\mu^n)|=\left|\left(\int(t-t^{n-1})p_{tt}(\cdot,s)ds,\mu^n\right)\right|\\\le&\triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}(\cdot,s)\|ds\|\mu^n\|\le(\triangle t)^2\left(\int_{t^{n-1}}^{t^n}\left\|\frac{\partial^2 p}{\partial t^2}(\cdot,s)\right\|\right)^2+\|\mu^n\|^2,\qquad(3.23)\end{aligned}$$

and

$$\begin{aligned}
&\big((K(p_h^n)-K(p^n))Y^n,\xi^n\big)\le K^*\parallel Y^n\parallel_\infty\parallel p_h^n-p^n\parallel\cdot\parallel\xi^n\parallel\\
\le&CK^*\big(\parallel p_h^n-Q_hp^n\parallel+\parallel Q_hp^n-p^n\parallel\big)\parallel\xi^n\parallel\\
=&CK^*(\parallel\mu^n\parallel+\parallel Q_hp^n-p^n\parallel)\parallel\xi^n\parallel\\
\le&C\big(\parallel\mu^n\parallel^2+\parallel Q_hp^n-p^n\parallel^2\big)+\delta\parallel\xi^n\parallel^2,
\end{aligned}\tag{3.24}$$

where we have used the assumptions (**A1**)-(**A2**). From (3.16)-(3.24), applying the Hölder's inequality gives

$$\begin{aligned}
&\frac{1}{2\triangle t}(\|\mu^n\|^2-\|\mu^{n-1}\|^2)+\|K(p_h^n)^{\frac12}\xi^n\|^2\\
\le&\left(\frac{\mu^n-\mu^{n-1}}{\triangle t},\mu^n\right)+\|K(p_h^n)^{\frac12}\xi^n\|^2\\
\le&(\partial_tp^n-p_t^n,\mu^n)+(\Pi_h\Psi^n-\Psi^n,\xi^n)+(F(p^n-Q_hp^n),\mu^n)\\
&+(F\mu^n,\mu^n)-(F(Y^n-Q_hY^n),\mu^n)-(F\xi^n,\mu^n)\\
&+\big(K(p_h^n)(Y^n-Q_hY^n),\xi^n\big)+\big((K(p_h^n)-K(p^n))Y^n,\xi^n\big)\\
\le&C\|\mu^n\|^2+C\triangle t^2+\delta\|\xi^n\|^2+C[\|\Pi_h\Psi^n-\Psi^n\|^2\\
&+\|Y^n-Q_hY^n\|^2+\|p^n-Q_hp^n\|^2].
\end{aligned}\tag{3.25}$$

We next multiply both sides of (3.25) by $2\triangle t$ and sum over n, $n=1,\cdots,N$, applying the discrete Gronwall's Lemma, to obtain

$$\begin{aligned}
&\|\mu^N\|^2+\sum_{n=1}^N\triangle t\|K(p_h^n)^{\frac12}\xi^n\|^2\\
\le&C\sum_{n=1}^N\triangle t\Big[\|\Pi_h\Psi^n-\Psi^n\|^2+\|Y^n-Q_hY^n\|^2\\
&\qquad+\|p^n-Q_hp^n\|^2\Big]+\|\mu^0\|^2+C(\triangle t)^2.
\end{aligned}$$

With a proper choice of the initial function $p_h^0=Q_hp^0$, we have $\mu^0=0$, combining (2.4) and (2.6), we get

$$\|\mu^N\|^2+\sum_{n=1}^N\triangle t\|K(p_h^n)^{\frac12}\xi^n\|^2\le C\left((\triangle t)^2+h^{2k+2}\right).$$

Then, we have

$$\|\mu^N\|+\left(\sum_{n=1}^N\triangle t\|K(p_h^n)^{\frac12}\xi^n\|^2\right)^{\frac12}\le C\left(\triangle t+h^{k+1}\right).\tag{3.26}$$

Applying the triangle inequality, we have the following result.

Theorem 3.1. *Suppose that assumptions* $(\mathbf{A1})$-$(\mathbf{A4})$ *are valid. Let* $(p_h^n, \Upsilon_h^n, \Psi_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ *be the solution of the mixed finite element equations* (3.4)-(3.6) *for* $n \geq 1$. *If we choose the initial function*

$$p_h^0 = Q_h p_0,$$

for $1 \leq M \leq N$ *and* $k \geq 1$, *then there exists a positive constant* C, *independent of* h *such that*

$$\begin{aligned}&\|p^M - p_h^M\| + \left(\sum_{n=1}^{M} \triangle t \|K(p_h^n)^{\frac{1}{2}}(\Upsilon^n - \Upsilon_h^n)\|^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{M} \triangle t \|\Psi^n - \Psi_h^n\|^2\right)^{\frac{1}{2}} \\ \leq & C\left(\triangle t + h^{k+1}\right).\end{aligned} \tag{3.27}$$

4 A two-grid method and error estimates

Note that the two-grid algorithms used in Wu and Allen [15] involve only one Newton iteration on the fine grid. Nevertheless, some more interesting results can be derived if one more Newton iteration is carried out on the fine grid. The idea is from Xu's work [16]. Thus, based on the works of Dawson [7], we can construct an algorithm of the two-grid method for Eqs. (1.1)-(1.3) discretized by expanded mixed finite element methods.

To iteratively solve the nonlinear system (3.4)-(3.6), we find $(\tilde{\tilde{p}}_h^n, \tilde{\tilde{\Upsilon}}_h^n, \tilde{\tilde{\Psi}}_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ in three steps as follows:

Step 1: On the coarse grid Γ_H, compute $(p_H^n, \Upsilon_H^n, \Psi_H^n) \in W_H \times \boldsymbol{V}_H \times \tilde{\boldsymbol{V}}_H$ to satisfy the following original nonlinear system:

$$\left(\frac{p_H^n - p_H^{n-1}}{\triangle t}, w_H\right) + (\nabla \cdot \Psi_H^n, w_H) = (f^n, w_H), \qquad w_H \in W_H, \tag{4.1}$$

$$(\Upsilon_H^n, \boldsymbol{v}_H) - (p_H^n, \nabla \cdot \boldsymbol{v}_H) = 0, \qquad \boldsymbol{v}_H \in \tilde{\boldsymbol{V}}_H, \tag{4.2}$$

$$(\Psi_H^n, \boldsymbol{v}_H) = (K(p_H^n)\Upsilon_H^n, \boldsymbol{v}_H), \qquad \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{4.3}$$

Step 2: On the fine grid Γ_h, compute $(\tilde{p}_h^n, \tilde{\Upsilon}_h^n, \tilde{\Psi}_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ to satisfy the following linear system:

$$\left(\frac{\tilde{p}_h^n - \tilde{p}_h^{n-1}}{\triangle t}, w_h\right) + (\nabla \cdot \tilde{\Psi}_h^n, w_h) = (f^n, w_h), \qquad w_h \in W_h, \tag{4.4}$$

$$(\tilde{\Upsilon}_h^n, \boldsymbol{v}_h) - (\tilde{p}_h^n, \nabla \cdot \boldsymbol{v}_h) = 0, \qquad \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h, \tag{4.5}$$

$$(\tilde{\Psi}_h^n, \boldsymbol{v}_h) = (K(p_H^n)\tilde{\Upsilon}_h^n, \boldsymbol{v}_h) + (K'(p_H^n)\Upsilon_H^n(\tilde{p}_h^n - p_H^n), \boldsymbol{v}_h), \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.6}$$

Step 3: On the fine grid Γ_h, compute $(\tilde{\tilde{p}}_h^n, \tilde{\tilde{Y}}_h^n, \tilde{\tilde{\Psi}}_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ to satisfy the following linear system:

$$\left(\frac{\tilde{\tilde{p}}_h^n - \tilde{\tilde{p}}_h^{n-1}}{\triangle t}, w_h\right) + (\nabla \cdot \tilde{\tilde{\Psi}}_h^n, w_h) = (f^n, w_h), \qquad w_h \in W_h, \tag{4.7}$$

$$(\tilde{\tilde{Y}}_h^n, \boldsymbol{v}_h) - (\tilde{\tilde{p}}_h^n, \nabla \cdot \boldsymbol{v}_h) = 0, \qquad \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h, \tag{4.8}$$

$$(\tilde{\tilde{\Psi}}_h^n, \boldsymbol{v}_h) = (K(\tilde{p}_h^n)\tilde{\tilde{Y}}_h^n, \boldsymbol{v}_h) + (K'(\tilde{p}_h^n)\tilde{Y}_h^n(\tilde{\tilde{p}}_h^n - \tilde{p}_h^n), \boldsymbol{v}_h), \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.9}$$

Remark 4.1. Eq. (4.9) is motivated by the Taylor expansion

$$K(\tilde{\tilde{p}}_h^n)\tilde{\tilde{Y}}_h^n = K(\tilde{p}_h^n)\tilde{\tilde{Y}}_h^n + K'(\tilde{p}_h^n)\tilde{\tilde{Y}}_h^n(\tilde{\tilde{p}}_h^n - \tilde{p}_h^n) + \frac{K''(x^n)}{2}\tilde{\tilde{Y}}_h^n(\tilde{\tilde{p}}_h^n - \tilde{p}_h^n)^2, \tag{4.10}$$

for some x^n between $\tilde{\tilde{p}}_h^n$ and $\tilde{p}_h^n$.

Lemma 4.1. *Suppose that assumptions* (**A1**)-(**A4**) *are fulfilled. Let* $(\tilde{p}_h^n, \tilde{Y}_h^n, \tilde{\Psi}_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ *be the solution of the mixed finite element equations* (4.1)-(4.6) *for* $n \geq 1$. *If we choose the initial function*

$$\tilde{p}_h^0 = Q_h p_0,$$

for $1 \leq M \leq N$ *and* $k \geq 1$, *then there exists a positive constant* C, *independent of* h *such that*

$$\|p^M - \tilde{p}_h^M\| + \left(\sum_{n=1}^{M} \triangle t \|K(p_H^n)^{\frac{1}{2}}(Y^n - \tilde{Y}_h^n)\|^2\right)^{\frac{1}{2}} \leq C(\triangle t + h^{k+1} + H^{2k+1}).$$

The proof can be found in [7].

Moreover, by an argument similar to that in Lemma 4.1, we can also prove the following result:

Remark 4.2. If we choose the initial function $\tilde{p}_h^0 = Q_h p_0$, then for $1 \leq n \leq N$, $2 \leq q < \infty$, and $k \geq 1$, we have

$$\|p^n - \tilde{p}_h^n\|_{0,q} \leq C(\triangle t + h^{k+1} + H^{2k+1}). \tag{4.11}$$

Now, we can prove the following main theorem.

Theorem 4.1. *Suppose that assumptions* (**A1**)-(**A4**) *are fulfilled. Let* $(\tilde{\tilde{p}}_h^n, \tilde{\tilde{Y}}_h^n, \tilde{\tilde{\Psi}}_h^n) \in W_h \times \boldsymbol{V}_h \times \tilde{\boldsymbol{V}}_h$ *be the solution of the mixed finite element equations* (4.1)-(4.9) *for* $n \geq 1$. *If we choose the initial function*

$$\tilde{\tilde{p}}_h^0 = Q_h p_0, \tag{4.12}$$

then for $1 \leq M \leq N$ *and* $k \geq 1$, *we have*

$$\|p^M - \tilde{\tilde{p}}_h^M\| + \left(\sum_{n=1}^{M} \triangle t \|K(\tilde{p}_h^n)^{\frac{1}{2}}(Y^n - \tilde{\tilde{Y}}_h^n)\|^2\right)^{\frac{1}{2}} \leq C(\triangle t + h^{k+1} + H^{4k+1}). \tag{4.13}$$

Proof. We now derive an estimate for this two-grid scheme. We set

$$\alpha^n = Q_h p^n - \tilde{\tilde{p}}_h^n, \quad \beta^n = Q_h Y^n - \tilde{\tilde{Y}}_h^n, \quad \gamma^n = \Pi_h \Psi^n - \tilde{\tilde{\Psi}}_h^n.$$

Using (3.10)-(3.12), (4.7)-(4.9), and the definition of projection Q_h, Π_h, we get the error equations:

$$\Big(\frac{\alpha^n - \alpha^{n-1}}{\triangle t}, w_h\Big) + (\nabla \cdot (\Pi_h \Psi^n - \tilde{\tilde{\Psi}}_h^n), w_h) = (E_p, w_h), \qquad w_h \in W_h, \tag{4.14}$$

$$(Q_h Y^n - \tilde{\tilde{Y}}_h^n, \boldsymbol{v}_h) - (Q_h p^n - \tilde{\tilde{p}}_h^n, \nabla \cdot \boldsymbol{v}_h) = 0, \qquad \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h, \tag{4.15}$$

and

$$\begin{aligned}
&(\Pi_h \Psi^n - \tilde{\tilde{\Psi}}_h^n, \boldsymbol{v}_h) \\
=&(\Pi_h \Psi^n - \Psi^n, \boldsymbol{v}_h) - (K(p^n)(Q_h Y^n - Y^n), \boldsymbol{v}_h) + \Big(K(\tilde{p}_h^n)(Q_h Y^n - \tilde{\tilde{Y}}_h^n), \boldsymbol{v}_h\Big) \\
&+ \Big((\alpha^n + (p^n - Q_h p^n))K'(\tilde{p}_h^n)\tilde{Y}_h^n, \boldsymbol{v}_h\Big) + \Big(K'(\tilde{p}_h^n)(p^n - \tilde{p}_h^n)(Q_h Y^n - \tilde{Y}_h^n), \boldsymbol{v}_h\Big) \\
&+ \frac{1}{2}\Big((p^n - \tilde{p}_h^n)^2 K''(x^n) Q_h Y^n, \boldsymbol{v}_h\Big), \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h,
\end{aligned} \tag{4.16}$$

for some x^n between $\tilde{p}_h^n$ and p_h^n and $E_p = \partial_t p^n - p_t^n$.

Choosing $w_h = \alpha^n$ in (4.14), $\boldsymbol{v}_h = \gamma^n$ in (4.15), $\boldsymbol{v}_h = \beta^n$ in (4.16), adding (4.14) and (4.15), and subtracting (4.16), yield

$$\begin{aligned}
&\Big(\frac{\alpha^n - \alpha^{n-1}}{\triangle t}, \alpha^n\Big) + \Big(K(\tilde{p}_h^n)\beta^n, \beta^n\Big) \\
=& (E_p, \alpha^n) - (\Pi_h \Psi^n - \Psi^n, \beta^n) + \Big(K(p^n)(Q_h Y^n - Y^n), \beta^n\Big) \\
&- \Big((\alpha^n + (p^n - Q_h p^n))K'(\tilde{p}_h^n)\tilde{Y}_h^n, \beta^n\Big) - \Big(K'(\tilde{p}_h^n)(p^n - \tilde{p}_h^n)(Q_h Y^n - \tilde{Y}_h^n), \beta^n\Big) \\
&- \frac{1}{2}\Big((p^n - \tilde{p}_h^n)^2 K''(x^n) Q_h Y^n, \beta^n\Big),
\end{aligned} \tag{4.17}$$

for some x^n between $\tilde{p}_h^n$ and p_h^n. Note that

$$\begin{aligned}
&\Big((p^n - \tilde{p}_h^n)^2 K''(x^n) Q_h Y^n, \beta^n\Big) \\
=&\Big(\big[(\tilde{p}_h^n - p_H^n)^2 - 2(\tilde{p}_h^n - p^n)(p^n - p_H^n) - (p^n - p_H^n)^2\big] \cdot K''(x^n)(Q_h Y^n - Y^n \\
&+ Y^n - Y_H^n), \beta^n\Big) + \Big((p^n - \tilde{p}_h^n)^2 K''(x^n)(Y_H^n - Y^n + Y^n), \beta^n\Big),
\end{aligned} \tag{4.18}$$

and using the Hölder's inequality, we get

$$\Big((p^n - \tilde{p}_h^n)^2 K''(x^n) Y^n, \beta^n\Big) \leq C\|p^n - \tilde{p}_h^n\|_{0,4}^2 \|\beta^n\|. \tag{4.19}$$

Using the Hölder's inequality and (3.23), multiplying by $\triangle t$ and summing $n = 1$ to M, $M \leq N$, using the approximation properties, we see that

$$
\begin{aligned}
&\|\alpha^M\|^2 - \|\alpha^0\|^2 + \sum_{n=1}^{M} \triangle t \|K(\tilde{p}_h^n)^{\frac{1}{2}} \beta^n\|^2 \\
\leq& C(\triangle t)^2 + \delta \sum_{n=1}^{M} \triangle t \|\beta^n\|^2 + C\left[\sum_{n=1}^{M} \triangle t \|\alpha^n\|^2 + h^{2k+2}\right] \\
&+ C\sum_{n=1}^{M} \triangle t \Big[\|(p^n - Q_h p^n)\tilde{Y}_h^n\|^2 + \|(p^n - \tilde{p}_h^n)(Q_h Y^n - \tilde{Y}_h^n)\|^2 + T_1\Big] \\
\equiv& C(\triangle t)^2 + \delta \sum_{n=1}^{M} \triangle t \|\beta^n\|^2 + C\left[\sum_{n=1}^{M} \triangle t \|\alpha^n\|^2 + h^{2k+2} + T_2\right],
\end{aligned}
\tag{4.20}
$$

where

$$
\begin{aligned}
T_1 =& \|p_H^n - \tilde{p}_h^n\|_\infty^2 \|p_H^n - \tilde{p}_h^n\|^2 \|Q_h Y^n - Y^n\|_\infty^2 \\
&+ \|p^n - \tilde{p}_h^n\|_\infty^2 \|p^n - p_H^n\|^2 \|Q_h Y^n - Y^n\|_\infty^2 \\
&+ \|p^n - \tilde{p}_H^n\|_\infty^2 \|p^n - \tilde{p}_H^n\|^2 \|Q_h Y^n - Y^n\|_\infty^2 \\
&+ \|p^n - \tilde{p}_h^n\|_\infty^2 \|p^n - \tilde{p}_h^n\|^2 \|Y_H^n - Y^n\|_\infty^2 + \|p^n - \tilde{p}_h^n\|_{0,4}^4,
\end{aligned}
\tag{4.21}
$$

$$
T_2 = C\sum_{n=1}^{M} \triangle t \Big[\|(p^n - Q_h p^n)\tilde{Y}_h^n\|^2 + \|(p^n - \tilde{p}_h^n)(Q_h Y^n - \tilde{Y}_h^n)\|^2 + T_1\Big]. \tag{4.22}
$$

Using (4.11) and the Hölder's inequality, we have

$$
\begin{aligned}
T_1 =& \|p_H^n - \tilde{p}_h^n\|_\infty^2 \|p_H^n - \tilde{p}_h^n\|^2 \|Q_h Y^n - Y^n\|_\infty^2 \\
&+ \|p^n - \tilde{p}_h^n\|_\infty^2 \|p^n - p_H^n\|^2 \|Q_h Y^n - Y^n\|_\infty^2 \\
&+ \|p^n - \tilde{p}_H^n\|_\infty^2 \|p^n - \tilde{p}_H^n\|^2 \|Q_h Y^n - Y^n\|_\infty^2 \\
&+ \|p^n - \tilde{p}_h^n\|_\infty^2 \|p^n - \tilde{p}_h^n\|^2 \|Y_H^n - Y^n\|_\infty^2 + \|p^n - \tilde{p}_h^n\|_{0,4}^4 \\
\leq& h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^4 h^{2k+2} \\
&+ h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^2 (H^{k+1} + \triangle t)^2 h^{2k+2} \\
&+ h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^4 \|Y^n - Y_H^n\|_\infty^2 + (h^{k+1} + H^{2k+1} + \triangle t)^4.
\end{aligned}
\tag{4.23}
$$

Combining (4.22)-(4.23), Lemma 4.1 and Theorem 3.1, we can obtain

$$
\begin{aligned}
T_2 =& C\sum_{n=1}^{M} \triangle t \Big[\|(p^n - Q_h p^n)\tilde{Y}_h^n\|^2 + \|(p^n - \tilde{p}_h^n)(Q_h Y^n - \tilde{Y}_h^n)\|^2 + T_1\Big] \\
\leq& C\sum_{n=1}^{M} \triangle t \Big[h^{2k+2} + h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^4 h^{2k+2} \\
&+ h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^2 (H^{k+1} + \triangle t)^2 h^{2k+2} + (h^{k+1} + H^{2k+1} + \triangle t)^4\Big]
\end{aligned}
$$

$$+ h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^4 \sum_{n=1}^{M} \triangle t \|Y^n - Y_H^n\|_\infty^2.$$

From (2.4), (2.8)-(2.9), and Theorem 3.1, we have

$$\begin{aligned}
&\sum_{n=1}^{M} \triangle t \|Y^n - Y_H^n\|_\infty^2 \\
\le& C \sum_{n=1}^{M} \triangle t (\|Y^n - Q_H Y^n\|_\infty^2 + \|Q_H Y^n - Y_H^n\|_\infty^2) \\
\le& C \sum_{n=1}^{M} \triangle t H^{2k+2} + C \sum_{n=1}^{M} \triangle t (H^{-1} \|Q_H Y^n - Y_H^n\|)^2 \\
\le& C H^{-2} (H^{k+1} + \triangle t)^2,
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
\|Y^n - Y_H^n\|_\infty \le& \|Y^n - Q_H Y^n\|_\infty + \|Q_H Y^n - Y_H^n\|_\infty \\
\le& H^{-1}(H^{k+1} + \triangle t).
\end{aligned} \tag{4.25}$$

Then, we obatin

$$T_2 \le C \left(h^{2k+2} + H^{8k+2} + H^{10k+4} h^{-2} + (\triangle t)^2 \right), \tag{4.26}$$

where $h \le H^{k+1}$, we have

$$T_2 \le C \left(h^{2k+2} + H^{8k+2} + (\triangle t)^2 \right). \tag{4.27}$$

Noticing that $\alpha^0 = 0$, and applying the discrete Gronwall's Lemma, we deduce that

$$\|\alpha^M\| + \left(\sum_{n=1}^{M} \triangle t \|K(\tilde{p}_h^n)^{\frac{1}{2}} \beta^n\|^2 \right)^{\frac{1}{2}} \le C(h^{k+1} + H^{4k+1} + \triangle t). \tag{4.28}$$

Applying the triangle inequality and the approximation properties, we can prove the results of the Theorem.

5 Conclusion and future works

In this paper, we have presented a new two-grid scheme for expanded mixed finite element solution of nonlinear parabolic equations. The theory demonstrates a remarkable fact about the two-grid scheme: we can iterate on a very coarse grid Γ_H and still get good approximations by taking one iteration on the fine grid Γ_h. We have derived an error estimate for the algorithms of the two-grid method. It is shown that the algorithm achieve asymptotically optimal approximation as long as the mesh sizes satisfy

$$h = \mathcal{O}\Big(H^{\frac{4k+1}{k+1}}\Big).$$

The next step of our work is to carry out some the numerical experiment with the proposed two-grid scheme. Furthermore, we shall consider the two-grid schemes for expanded mixed finite element solution of nonlinear hyperbolic equations.

Acknowledgements

This work is supported by Guangdong Province Universities and Colleges Pearl River Scholar Funded Scheme (2008), National Science Foundation of China 10971074, the National Basic Research Program under the Grant 2005CB321703 and Hunan Provincial Innovation Foundation For Postgraduate CX2009B119

References

[1] F. Brezzi, J. Douglas Jr. and L. D. Marini, *Two families of mixed finite elemetns for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[2] Y. Chen, Y. Huang and D. Yu, *A two-grid method for expanded mixed finite element solution of semilinear reaction-diffusion equations*, Internat. J. Numer. Methods in Engineering, 57 (2003), pp. 193–209.

[3] Y. Chen and L. Li, *L^p error estimates of two-grid schemes of expanded mixed finite element methods*, Appl. Math. Comp., 209 (2009), pp. 197–205.

[4] Y. Chen, D. Yu and S. Liu, *Analysis of two-grid method for reaction-diffusion equations by expanded mixed finite element methods*, Internat. J. Numer. Methods in Engineering, 69 (2006), pp. 408–422.

[5] Z. Chen, *Expanded mixed element methods for quasilinear second-order elliptic problems(I)*, RAIRO: Modèl. Math. Anal. Numer., 32 (1998), pp. 479–499.

[6] Z. Chen, *Expanded mixed element methods for quasilinear second-order elliptic problems(II)*, RAIRO: Modèl. Math. Anal. Numer., 32 (1998), pp. 501–520.

[7] C. N. Dawson and M. F. Wheeler, *Two-grid methods for mixed finite element approximations of non-linear parabolic equations*, Contemporary Math., 180 (1994), pp. 191–203.

[8] C. N. Dawson, M. F. Wheeler and C. S. Woodward, *A two-grid finite difference scheme for nonlinear parabolic equations*, SIAM J. Numer. Anal., 35 (1998), pp. 435–452.

[9] J. Douglas, Jr., R. E. Ewing and M. F. Wheeler, *The approximation of the pressure by a mixed method in the simulation of miscible displacement*, RAIRO Anal. Numér., 17 (1983), pp. 17–33.

[10] L. Evans, *Partial differential equations*, Amer. Math. Soc., Providence, RI, 1998.

[11] Z. Lu and H. Zhang, *V cycle multi-grid-cycle method of viscoelastic fluid flow obeying an Oldroyd B type constitutive law*, Numer. Anal. Appl., 11 (2008), pp. 83–94.

[12] F. Pretorius and M. W. Choptuik, *Adaptive mesh refinement for coupled elliptic-hyperbolic systems*, J. Comp. Phys., 218 (2006), pp. 246–274.

[13] P. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems, Math. Aspects of the Finite Element Method*, Lecture Notes in Math., 606, (Springer, Berlin, 1977), pp. 292–315.

[14] V. THOMÉE, Galerkin Finite Element Methods for Parabolic Problems, Springer-Verlag, 1997.

[15] L. WU AND M. B. ALLEN, *A two-grid method for mixed finite-element solution of reaction-diffusion equations*, Numer. Methods for Partial Diffenrential Equations, 15 (1999), pp. 317–332.

[16] J. XU, *Two-grid discretization techniques for linear and non-linear PDEs*, SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777.

[17] J. XU, *A novel two-grid method for semilinear equations*, SIAM J. Sci. Comp., 15 (1994), pp. 231–237.

[18] J. XU AND A. H. ZHOU, *Some multiscale methods for partial differential equations*, Contemporary Math., 306 (2002), pp. 1–27.