Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 6, pp. 862-873

DOI: 10.4208/aamm.09-m09S07
December 2009

Improved Local Projection for the Generalized Stokes Problem

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Received 28 April 2009; Accepted (in revised version) 07 September 2009
Available online 18 November 2009

Abstract. We analyze pressure stabilized finite element methods for the solution of the generalized Stokes problem and investigate their stability and convergence properties. An important feature of the methods is that the pressure gradient unknowns can be eliminated locally thus leading to a decoupled system of equations. Although the stability of the method has been established, for the homogeneous Stokes equations, the proof given here is based on the existence of a special interpolant with additional orthogonal property with respect to the projection space. This makes it much simpler and more attractive. The resulting stabilized method is shown to lead to optimal rates of convergence for both velocity and pressure approximations.

AMS subject classifications: 65N12, 65N30, 65N15, 76D07

Key words: Generalized Stokes equations, stabilized finite elements, local projection, convergence, error estimates.

1 Introduction

Stabilized finite element methods that circumvent the restrictive inf − sup condition have been developed for Stokes-like problems (see, e.g., [4,14,16,19,20]). These residual-based methods represent one class of stabilized methods. They consist in modifying the standard Galerkin formulation by adding mesh-dependent terms, which are weighted residuals of the original differential equations. Although for properly chosen stabilization parameters, these methods are well posed for all velocity and pressure pairs. These methods are sensitive to the choice of the stabilization parameters. Another class of stabilized methods has been derived using Galerkin methods enriched with bubble functions (see, [1, 3]). Alternative stabilization techniques based on a continuous penalty method have also been proposed and analyzed in [10, 11].

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Recently, local projection methods that seem less sensitive to the choice of parameters and have better local conservation properties were proposed. The stabilization by projecting the pressure gradient has been analyzed in [12]. It was shown that the method is consistent in the sense that a smooth exact solution satisfies the discrete problem. Though the method may seem computationally expensive due to the nonlocal behaviour of the projection, iterative solvers were developed to make the method more efficient ([13]). Alternatively, a two-level approach with a projection onto a discontinuous finite element space of a lower degree defined on a coarser grid has been analyzed in [5,22,23]. In [6,7], low order approximations of the Oseen equations were analyzed.

A drawback of the two-level, from the implementation point of view, is that the added stabilizing term leads to a larger stencil which may not fit the data structure of an available programming code. In [21], stability of local projection methods is proved based on the existence of a special interpolant with additional properties with respect to the projection space. This general approach paves the way for introducing equal order stabilized methods by local projection onto a discontinuous space defined on the same mesh. In this case, the added stabilizing terms do not lead to a larger stencil like the two-level approach.

The main objective of this paper is to analyze the pressure gradient stabilization method for the generalized Stokes problem using the new approach. These kind of problems arise naturally in the time discretization of the unsteady Stokes problem, or the full Navier-Stokes equations by means of an operator splitting technique. Unlike the proof given by [22] and [23], where stability was shown using an inf-sup condition due to [16] and the equivalence of norms on finite dimensional spaces, here, the stability of the pressure-gradient method is proved for arbitrary Q^k-elements, by constructing a special interpolant with additional orthogonal property with respect to the projection space (see, e.g., [24,25]). As a result, optimal rates of convergence are found for the velocity and pressure approximations.

2 Variational formulation

Let Ω be a bounded two-dimensional polygonal region, $f \in L^2(\Omega)$, σ a positive real number, typically,

$$\sigma = \frac{1}{\Delta t},$$

where Δt is the time step in a time discretization procedure, and ν the kinematic viscosity coefficient. Then, the generalized homogeneous Stokes Problem reads: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ satisfying:

$$\begin{cases} \sigma\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$\mathbf{V} = (H_0^1(\Omega))^d, \quad \text{and} \quad \mathrm{Q} = \mathrm{L}_0^2(\blacksquare),$$

with $L_0^2(\Omega)$ denoting the set of square integrable functions with null average.

Define the forms

$$\begin{cases} A\big((\mathbf{u},p);(\mathbf{v},q)\big) = \sigma(\mathbf{u},\mathbf{v}) + \nu(\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla.\mathbf{v}) + (q,\nabla.\mathbf{u}), \\ F(\mathbf{v},q) = (\mathbf{f},\mathbf{v}), \end{cases} \tag{2.2}$$

for all $(\mathbf{v},q) \in \mathbf{V} \times Q$, with $(.,.)$, as usual, denoting the L^2-inner product on the region Ω. Then, the weak formulation of (2.1) reads in compact notation, as

$$A((\mathbf{u},p);(\mathbf{v},q)) = F(\mathbf{v},q) \quad , \quad \forall(\mathbf{v},q) \in \mathbf{V} \times Q. \tag{2.3}$$

Let $\mathbf{V}_h$ and Q_h be finite dimensional subspaces of $\mathbf{V}$ and Q, respectively. Then, the Galerkin discrete problem reads: Find $(\mathbf{u}_h,p_h) \in \mathbf{V}_h \times Q_h$ such that:

$$A((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h)) = F(\mathbf{v}_h,q_h) \quad , \quad \forall(\mathbf{v}_h,q_h) \in \mathbf{V}_h \times Q_h. \tag{2.4}$$

Note that the formulation (2.4) is stable only for velocity and pressure approximations satisfying the inf-sup condition (see, e.g., [18]).

3 Local projection stabilization

Let ζ_h be a shape regular partition of the region Ω into quadrilateral elements K (see, e.g., [8]). Denote by h_K the diameter of element K and by h the maximum diameter of the elements $K \in \zeta_h$.

We then define the equal order continuous finite element spaces

$$\begin{aligned} \mathbf{V}_h &= V_h^2 = \left\{\mathbf{v} \in \big(H_0^1(\Omega)\big)^2 : \ \mathbf{v}|_K \in \big(Q_h^k(K)\big)^2, \ \forall K \in \zeta_h\right\}, \\ Q_h &= \left\{q \in H^1(\Omega) : \ q\,|_K \in Q_h^k(K), \ \forall K \in \zeta_h\right\}, \end{aligned} \tag{3.1}$$

where Q_h^k denotes the standard continuous isoparametric finite element functions defined by means of a mapping from a reference element. On the reference quadrilateral, the approximation functions are polynomials of degree less than or equal to k in each variable. We shall also use P_h^k to denote the space of polynomials of degree less than or equal to k over ζ_h.

Additionally, we define the pressure-gradient finite element space by

$$\mathbf{Y}_h = Y_h^2 = \bigoplus_{K\in\zeta_h} (Q_h^{k-1,disc}(K))^2. \tag{3.2}$$

where $Q_h^{k-1,disc}$(respectively $P_h^{k,disc}$) denote the finite element spaces of discontinuous functions across elements of ζ_h.

Define the local projection operator $\pi_K : L^2(K) \to Q_h^{k-1}(K)$ by

$$(w - \pi_K w, \phi)_K = 0, \qquad \forall \phi \in Q_h^{k-1}(K), \tag{3.3}$$

which generates the global projection $\pi_h : L^2(\Omega) \to Y_h$ defined by

$$(\pi_h w)\,|_K = \pi_K(w|_K), \qquad \forall K \in \zeta_h\,, \quad \forall w \in L^2(\Omega). \tag{3.4}$$

The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ is given by

$$\kappa_h = id - \pi_h, \tag{3.5}$$

where id denotes the identity operator on $L^2(\Omega)$. For simplicity, we shall use the same notation id, π_M, π_h, and κ_h for vector-valued functions. Thus, $\kappa_h \nabla p$ is to be understood as acting on each component of ∇p separately.

Now, we are ready to introduce the stabilizing term

$$S(p_h; q_h) = \sum_{K \in \zeta_h} \alpha_K \left(\kappa_h \nabla p_h, \nabla q_h\right)_K = \sum_{K \in \zeta_h} \alpha_K \left(\kappa_h \nabla p_h, \kappa_h \nabla q_h\right)_K, \tag{3.6}$$

where α_K are element parameters that depend on the local mesh size.

Thus, our stabilized discrete problem reads as: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) + S(p_h; q_h) = F(\mathbf{v}_h, q_h)\,, \ \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \tag{3.7}$$

This can be written component-wise as: Find $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{V}_h \times Q_h \times \mathbf{Y}_h$ such that

$$\sigma(\mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla.\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.8a}$$

$$\sum_K \alpha_K(\nabla p_h, \nabla q_h) - \sum_K \alpha_K(\lambda_h, \nabla q_h) - (q_h, \nabla.\mathbf{u}_h) = 0, \quad \forall q_h \in Q_h, \tag{3.8b}$$

$$-\sum_K \alpha_K(\nabla p_h, \xi_h) + \sum_K \alpha_K(\lambda_h, \xi_h) = 0, \qquad \forall \xi_h \in \mathbf{Y}_h, \tag{3.8c}$$

where λ_h is the local L^2-projection of ∇p_h onto a discrete space $\mathbf{Y}_h$.

In order to investigate the properties of the bilinear form

$$A((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) + S(p_h; q_h),$$

on the product space $\mathbf{V}_h \times Q_h$, we introduce the mesh dependent norm

$$\|(\mathbf{v}_h, q_h)\|^2 = \sigma \,\|\mathbf{v}_h\|_{0,\Omega}^2 + \nu \,|\mathbf{v}_h|_{1,\Omega}^2 + (\sigma + \nu)\, \|q_h\|_{0,\Omega}^2 + S(q_h; q_h). \tag{3.9}$$

3.1 Stability

The main idea in the analysis of local projection methods is the construction of an interpolation operator $j_h : H^1(\Omega) \to Y_h$ with $j_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$, satisfying the usual approximation property

$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \leq C h_K^s \|v\|_{s,w(K)}, \quad \forall v \in H^s(w(K)),\ 1 \leqslant s \leqslant k+1, \tag{3.10}$$

where $w(K)$ denotes a certain local neighbourhood of K. We also need the following additional orthogonal property

$$(v - j_h v, \phi_h) = 0 \quad , \quad \forall \phi_h \in Y_h,\ \forall v \in H^1(\Omega). \tag{3.11}$$

Lemma 3.1. *Let $i_h : H^1(\Omega) \to V_h$ be an interpolation operator such that $i_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$ with the error estimate*

$$\|v - i_h v\|_{0,K} + h_K |v - i_h v|_{1,K} \leq C h_K^s \|v\|_{s,w(K)}, \quad \forall v \in H^s(\Omega),\ 1 \leqslant s \leqslant k+1. \tag{3.12}$$

Further, assume that the local inf-sup condition

$$\inf_{q_h \in Y_h(K)} \sup_{v_h \in V_h(K)} \frac{(v_h, q_h)_K}{\|v_h\|_{0,K} \|q_h\|_{0,K}} \geqslant \beta_1, \tag{3.13}$$

holds for all $K \in \zeta_h$, with a positive constant β_1 independent of the mesh size. Then, there exists an interpolation operator $j_h : H^1(\Omega) \to Y_h$ satisfying the properties (3.10) *and* (3.11).

Proof. For the construction of the interpolation operator j_h we refer to Theorem 2.2 in [21]. □

Remark 3.1. Note that condition (3.13) can be checked using Stenberg's technique on macro-elements $K \in \zeta_h$ which are equivalent to a reference element $\widehat{K}$. The inf − sup condition holds if the null space N_K is such that

$$N_K = \left\{ q_h \in Y_h(K) : (v_h, q_h)_K = 0, \quad \forall v_h \in V_h(K) \cap H_0^1(K) \right\} = \{0\}. \tag{3.14}$$

Note also that the fluctuation operator κ_h satisfies the approximation property

$$\|\kappa_h q\|_{0,K} \leq C h_K^l |q|_{l,K}, \quad \forall q \in H^l(K), \quad \forall K \in \zeta_h,\ 0 \leqslant l \leqslant k. \tag{3.15}$$

Since the L^2- local projection $\pi_K : L^2(K) \to Y_h(K)$ becomes the identity for the space $Q^{k-1}(K) \subset H^l(K)$, and the kernel of κ_h contains $P^{k-1}(K) \subset Q^{k-1}(K)$, We have that the Bramble-Hilbert Lemma gives the approximation properties stated in assumption (3.15).

Remark 3.2. The justification that the pair

$$V_h/Y_h = Q_h^k/Q_h^{k-1,disc}, \quad \text{for} \quad k \geqslant 1,$$

satisfies (3.13) follows from (3.14) using the one-to-one property of the mapping $F_K : \widehat{K} \to K$ combined with a positive bilinear function corresponding to the central node of $\widehat{K}$ (see, e.g., [17,21]). Further, using the same argument we can show that

$$V_h/Y_h = Q_h^k/P_h^{k-1,disc}, \qquad \text{for} \quad k \geq 1,$$

gives also a stable approximation.

The following theorem establishes the stability of the one-level local projection method in the sense of Babuska and Brezzi. (see, e.g., [24,25]).

Theorem 3.1. *Let properties* (3.10), (3.11), *and* (3.15) *hold and the parameters α_K be such that $\alpha_K = Ch_K^2$ for each element $K \in \zeta_h$.Then, the bilinear form of the pressure-gradient stabilized method satisfies*

$$\sup_{\substack{(\mathbf{w}_h,r_h)\in V_h\times Q_h \\ (\mathbf{w}_h,r_h)\neq 0}} \frac{A((\mathbf{v}_h,q_h);(\mathbf{w}_h,r_h)) + S(q_h;r_h)}{\|(\mathbf{w}_h,r_h)\|} \geq \beta \|(\mathbf{v}_h,q_h)\|,$$

for some positive constant β independent of the mesh parameter h.

Remark 3.3. For Stokes flow ($\sigma \to 0$), $\alpha_K = h^2$ has proven to be a good choice for the stabilization parameter (see, e.g., [5]). In addition, the analysis given in (see, e.g., [2]) reveals that for the current problem $\alpha_K = \sigma h^2/\nu$ is a reasonable choice because it takes into account the effect of the zero term.

Note that the above theorem guaranties unique solvability of the stabilized discrete problem (3.7). However, unlike the residual-based stabilization schemes (see, e.g., [16,19]), here, we do not have Galerkin orthogonality. As a consequence we need to estimate the consistency error (see, e.g., [17]).

Lemma 3.2. *Assume that the fluctuation operator κ_h satisfies the approximation property* (3.15)*. Let $(\mathbf{u},p) \in \mathbf{V} \times (Q \cap H^l(\Omega))$, $0 \leqslant l \leqslant k$, be the solution of the generalized Stokes problem (2.3), and $(\mathbf{u}_h,p_h) \in \mathbf{V}_h \times Q_h$ be the solution of the stabilized problem* (3.7)*. Then, the consistency error can be estimated by*

$$A((\mathbf{u}-\mathbf{u}_h,p-p_h);(\mathbf{v}_h,q_h)) \leqslant C\left(\sum_{K\in\zeta_h} \alpha_K h_K^{2l-2}|p|_{l,K}^2\right)^{\frac{1}{2}} \|(\mathbf{v}_h,q_h)\|,$$

for all $(\mathbf{v}_h,q_h) \in \mathbf{V}_h \times Q_h$.

3.2 Error analysis

As a consequence of the above stability and consistency results, we obtain the following error estimate.

Theorem 3.2. *Assume that the solution $(\mathbf{u}, p)$ of* (2.3) *belongs to* $\mathbf{V} \cap (H^{s+1}(\Omega))^2 \times (Q \cap H^l(\Omega))$, $1 \le s, l \le k$. *Then, the following error estimate holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \le C\left(h^s \|\mathbf{u}\|_{s+1,\Omega} + h^l \|p\|_{l,\Omega}\right). \tag{3.16}$$

Where, C is a positive constant independent of h.

Proof. Let $\widetilde{\mathbf{u}}_h = j_h\mathbf{u}$ and $\widetilde{p}_h = i_h p$ be the interpolants of the velocity and pressure, respectively. Then, Theorem 4 implies the existence of $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ such that

$$|\!|\!|(\mathbf{v}_h, q_h)|\!|\!| \leqslant C, \tag{3.17}$$

with

$$\|\widetilde{\mathbf{u}}_h - \mathbf{u}_h\|_{1,\Omega} + \|\widetilde{p}_h - p_h\|_{0,\Omega} \leqslant \frac{3}{\min\left\{\sigma^{\frac{1}{2}}, \nu^{\frac{1}{2}}\right\}} |\!|\!|(\widetilde{\mathbf{u}}_h - \mathbf{u}_h, \widetilde{p}_h - p_h)|\!|\!|,$$

with the right hand side satisfying

$$\begin{aligned}
&|\!|\!|(\widetilde{\mathbf{u}}_h - \mathbf{u}_h, \widetilde{p}_h - p_h)|\!|\!| \\
\leqslant&\frac{1}{\widetilde{\beta}\, |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|}\Big[A\big((\widetilde{\mathbf{u}}_h - \mathbf{u}_h, \widetilde{p}_h - p_h); (\mathbf{v}_h, q_h)\big) + S(\widetilde{p}_h - p_h; q_h)\Big] \\
\leqslant&\frac{1}{\widetilde{\beta}\, |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|}\Big[A\big((\widetilde{\mathbf{u}}_h - \mathbf{u}, \widetilde{p}_h - p); (\mathbf{v}_h, q_h)\big) + S(\widetilde{p}_h - p; q_h)\Big] \\
&+ \frac{1}{\widetilde{\beta}\, |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|}\Big[A\big((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)\big) + S(p - p_h; q_h)\Big].
\end{aligned} \tag{3.18}$$

Consequently, the consistency estimate of the method implies

$$\frac{1}{|\!|\!|(\mathbf{v}_h, q_h)|\!|\!|}\Big[A\big((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)\big) + S(p - p_h; q_h)\Big] \leqslant Ch^l \|p\|_{l,\Omega}. \tag{3.19}$$

The Galerkin terms of

$$A\big((\widetilde{\mathbf{u}}_h - \mathbf{u}, \widetilde{p}_h - p); (\mathbf{v}_h, q_h)\big) + S(\widetilde{p}_h - p; q_h)$$

can be estimated using the approximation properties of j_h and i_h. Hence, we get

$$\sigma(\widetilde{\mathbf{u}}_h - \mathbf{u}, \mathbf{v}_h) \leqslant \sigma \|\widetilde{\mathbf{u}}_h - \mathbf{u}\|_{0,\Omega} \|\mathbf{v}_h\|_{0,\Omega} \leqslant C\sigma h^{s+1} |\mathbf{u}|_{s+1,\Omega} |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|, \tag{3.20a}$$

$$\nu(\nabla(\widetilde{\mathbf{u}}_h - \mathbf{u}), \nabla\mathbf{v}_h) \leqslant \nu |\widetilde{\mathbf{u}}_h - \mathbf{u}|_{1,\Omega} |\mathbf{v}_h|_{1,\Omega} \leqslant C\nu h^s |\mathbf{u}|_{s+1,\Omega} |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|, \tag{3.20b}$$

$$|(p - \widetilde{p}_h, \nabla\cdot\mathbf{v}_h)| \leqslant C\|p - \widetilde{p}_h\|_{0,\Omega} |\mathbf{v}_h|_{1,\Omega} \leqslant Ch^l |p|_{l,\Omega} |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|. \tag{3.20c}$$

The fourth Galerkin term is estimated by applying the orthogonality property of j_h. Then, using $\alpha_K = Ch_K^2$ we get

$$\begin{aligned}
&|(\nabla\cdot(\widetilde{\mathbf{u}}_h-\mathbf{u}),q_h)| = |(\widetilde{\mathbf{u}}_h-\mathbf{u},\nabla q_h)| = |(\widetilde{\mathbf{u}}_h-\mathbf{u},\kappa_h\nabla q_h)|\\
\leqslant&\left(\sum_{K\in\zeta_h}\alpha_K^{-1}\|\widetilde{\mathbf{u}}_h-\mathbf{u}\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\zeta_h}\alpha_K\|\kappa_h\nabla q_h\|_{0,K}^2\right)^{\frac{1}{2}}\\
\leqslant&C\left(\sum_{K\in\zeta_h}\frac{h_K^2}{\alpha_K}h_K^{2s}\|\mathbf{u}\|_{s+1,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\zeta_h}\alpha_K\|\kappa_h\nabla q_h\|_{0,K}^2\right)^{\frac{1}{2}},
\end{aligned}$$

which gives

$$|(\nabla\cdot(\widetilde{\mathbf{u}}_h-\mathbf{u}),q_h)| \leqslant Ch_K^s\|\mathbf{u}\|_{s+1,K}\|(\mathbf{v}_h,q_h)\|. \tag{3.21}$$

The stability term is estimated using the L_2-stability of the fluctuation operator κ_h, the approximation properties of i_h and $\alpha_K = Ch_K^2$, hence we obtain

$$\begin{aligned}
S(\widetilde{p}_h-p;q_h) =& \sum_{K\in\zeta_h}\alpha_K\big(\kappa_h\nabla(\widetilde{p}_h-p),\kappa_h\nabla q_h\big)\\
\leqslant&\left(\sum_{K\in\zeta_h}\alpha_K\|\kappa_h\nabla(\widetilde{p}_h-p)\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\zeta_h}\alpha_K\|\kappa_h\nabla q_h\|_{0,K}^2\right)^{\frac{1}{2}}\\
\leqslant& C_1\left(\sum_{K\in\zeta_h}C_2h_K^2h_K^{2l-2}\|p\|_{l,w(K)}^2\right)^{\frac{1}{2}}\|(\mathbf{v}_h,q_h)\|,
\end{aligned}$$

which yields

$$S(\widetilde{p}_h-p;q_h) \leqslant Ch_K^l\|p\|_{l,\Omega}\|(\mathbf{v}_h,q_h)\|. \tag{3.22}$$

Thus, using (3.19)-(3.22) we obtain the desired estimate (3.16). □

Remark 3.4. We note that because of the compatibility of the $Q_h^k/P_h^{k-1,disc}$ approximation (see, e.g., [9]) the stability of (3.7) and the above optimal error estimates hold also for such approximation.

4 Numerical results

In this section, numerical results for two-dimensional generalized Stokes flows are presented. The performance of the $Q_h^1-Q_h^1$ velocity-pressure approximation is assessed for $\alpha_K=\sigma h^2/\nu$.

4.1 Test 1 problem

The first problem consists in solving generalized Stokes problem in the unit square $[0,1]\times[0,1]$, with exact solution:

$$\mathbf{u}(x,y)=(u_1(x,y),u_2(x,y))^T, \qquad \mathrm{p}(x,y)=(x-0.5)(y-0.5),$$

with

$$u_1=2x^2(1-x)^2y(1-y)(1-2y), \qquad u_2=-2x(1-x)(1-2x)y^2(1-y)^2.$$

Numerical results obtained for

$$\sigma=1, \quad \nu=1, \quad \text{and} \quad \nu=0.1,$$

respectively, are displayed in Fig. 1. These results indicate that the error norms $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ and $\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$ converge at the predicted rates, while $\|\mathrm{p}-\mathrm{p}_h\|_{0,\Omega}$ seems to converge one degree higher than predicted. Further, it appears that for $\nu\le 0.1$, the velocity converges at a higher rate than expected for both L_2 and H_1 norms. This behaviour is beleived to be due to the symmetry of the problem.

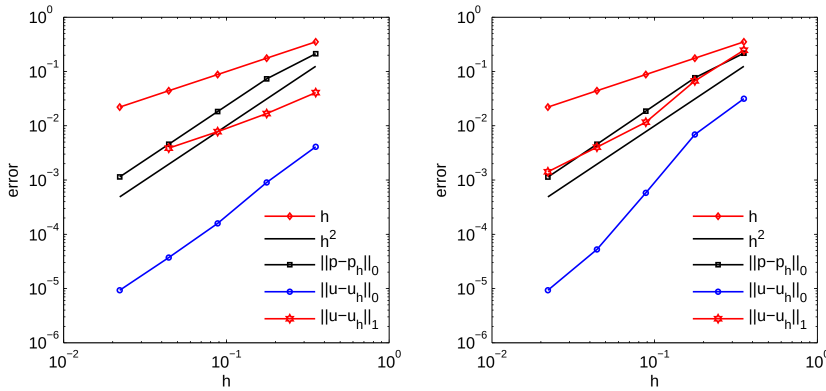

Figure 1: Rates of convergence for $\sigma=1$, $\nu=1$ (left) and $\nu=0.1$ (right) for Test 1.

4.2 Lid-driven cavity flow

Next, we address the lid-driven cavity problem, with domain Ω as before, $\mathbf{f}=\mathbf{0}$. Our aim here is to assess the performance of the method using a graded mesh near $x=0$, $x=1$, $y=0$, and $y=1$. We impose a leaky boundary condition, that is

$$u_1(0,y)=u_1(1,y)=u_1(x,0)=0, \quad \text{and} \quad u_1(x,1)=1, \quad \text{for} \quad 0\le x\le 1.$$

Numerical results are obtained for $\nu=1$ and $\nu=10^{-4}$, both using $\sigma=1$. Elevations for the pressure field and the horizontal velocity are displayed in Figs. 2 and 3. We

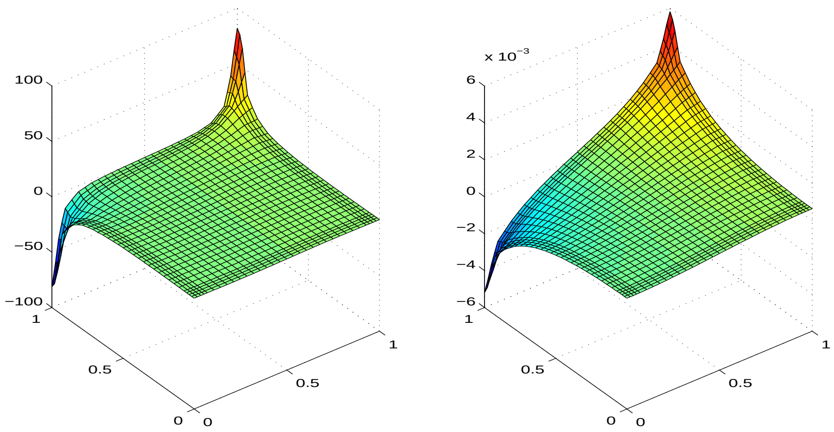

Figure 2: Elevation of the pressure field for $\sigma = 1$, $\nu = 1$ (left) and $\nu = 10^{-4}$ (right) for the lid-driven cavity problem.

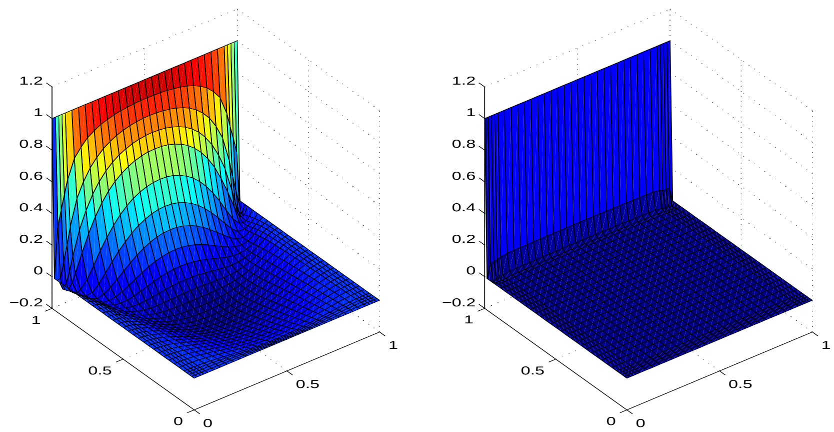

Figure 3: Elevation of the horizontal velocity for $\sigma = 1$, $\nu = 1$ (left) and $\nu = 10^{-4}$ (right) for the lid-driven cavity problem.

observe that there are no oscillations for the pressure for both cases, which shows that the method treats well the inf-sup condition and the boundary layer for the reaction dominated regime. Further, the horizontal velocity solution is comparable to the one reported in [3].

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