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Convergence Analysis of Legendre-Collocation Methods for Nonlinear Volterra Type Integro Equations

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Abstract. A Legendre-collocation method is proposed to solve the nonlinear Volterra integral equations of the second kind. We provide a rigorous error analysis for the proposed method, which indicate that the numerical errors in L^2-norm and L^∞-norm will decay exponentially provided that the kernel function is sufficiently smooth. Numerical results are presented, which confirm the theoretical prediction of the exponential rate of convergence.

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1 Introduction

The integro-differential equations (IDEs) arise from the mathematical modeling of many scientific phenomena. Nonlinear phenomena, that appear in many applications in scientific fields can be modeled by nonlinear integro-differential equations. This paper is concerned with the nonlinear Volterra integral equations (VIEs) of the second kind

$$y(t)=\int_0^t \widehat{K}(t,\tau,y(\tau))d\tau+\widehat{g}(t), \quad t\in[0,T], \tag{1.1}$$

where kernel function $\widehat{K}:S\times\mathbb{R}\to\mathbb{R}$ (where $S:=\{(t,\tau):0\le\tau\le t\le T\}$) and $\widehat{g}(t):[0,T]\to\mathbb{R}$ are known, function $y(t)$ is the unknown function to be determined. It will always

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be assumed that problem (1.1) possesses a unique solution, $\widehat{K}$ is continuous for all S and Lipschitz continuous with its third argument. We will consider the case that the solutions are sufficiently smooth. Consequently it is natural to implement very high-order numerical methods such as spectral methods for the solutions.

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations, such as the linearization method [1], differential transform method [2], RF-pair method [4], product integration method [30], Hermite-type collocation method [5], semi-analytical-numerical techniques, such as the Adomian decomposition method [36], Taylor expansion approach [3], collocation methods [9] and references therein. Nevertheless, few works touched the spectral approximations to integral-differential equations. Then Chebyshev spectral methods were investigated in [19] for the first kind Fredholm integral equations under multiple-precision arithmetic. However, no theoretical results were provided to justify the high accuracy numerically obtained. Recently, Tang and Xu [32] developed a novel spectral Legendre-collocation method to solve linear Volterra integral equations of (1.1). Xie, Li and Tang [38] developed a spectral Petrov-Galerkin methods for linear Volterra type integral equations. Chen, Li and Tang [16–18] proposed a spectral Jacobi-collocation approximation for linear Volterra integral equations with weakly singular kernels. For nonlinear case, polynomial spline collocation methods for the nonlinear Basset equation is discussed in [12]. Legendre spectral Galerkin method has been proposed to nonlinear Volterra integral equations in [35]. In this paper, the main purpose of this work is to provide Legendre-collocation methods for nonlinear Volterra integral equations and provide a rigorous error analysis which theoretically justifies the spectral rate of convergence. The linear kind of (1.1) was provided in [32], but they pointed out that the rate of convergence seems not optimal, in this paper, the optimal order of convergence $\mathcal{O}(N^{-m})$ is obtained.

The paper is organized as follows. In Section 2, we outline the spectral approaches for (1.1). Some lemmas useful for establishing the convergence results will be provided in Section 3. The convergence analysis will be carried out in Section 4, and Section 5 contains numerical results, which will be used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with conclusions and future work.

2 Legendre-collocation method

For a given $N \geq 0$, we denote by $\{\theta_k\}_{k=0}^N$ the Legendre points, and by $\{\omega_k\}_{k=0}^N$ the corresponding Legendre weights. Then, the Legendre-Gauss integration formula is

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{N} f(\theta_k)\omega_k. \tag{2.1}$$

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t=\frac{1}{2}T(1+x), \qquad x=\frac{2t}{T}-1,$$
$$\tau=\frac{1}{2}T(1+s), \qquad s=\frac{2\tau}{T}-1,$$

and let

$$u(x)=y\Big(\frac{1}{2}T(1+x)\Big), \quad g(x)=\widehat{g}\Big(\frac{1}{2}T(1+x)\Big),$$
$$K(x,s,u)=\frac{T}{2}\widehat{K}\Big(\frac{1}{2}T(1+x),\frac{1}{2}T(1+s),y\Big(\frac{1}{2}T(1+s)\Big)\Big),$$

we obtain form (1.1) that

$$u(x)=\int_{-1}^{x}K(x,s,u(s))ds+g(x), \quad x\in I=[-1,1]. \tag{2.2}$$

Set the collocation points as the set of $(N+1)$ Legendre-Gauss points, $\{x_i\}_{i=0}^{N}$ associated with ω_k. Assume that (2.2) holds at x_i:

$$u(x_i)=\int_{-1}^{x_i}K(x_i,s,u(s))ds+g(x_i). \tag{2.3}$$

The main difficulty in obtaining high order of accuracy is to compute the integral term in (2.3). In particular, for small values of x_i, there is little information available for $u(s)$. To overcome this difficulty, we will transfer the integral interval $[-1,x_i]$ to a fixed interval $[-1,1]$ and then make use some appropriate quadrature rule. More precisely, we first make a simple linear transformation:

$$s(x,\theta)=\frac{1+x}{2}\theta+\frac{x-1}{2}, \quad -1\le\theta\le 1. \tag{2.4}$$

Then (2.3) becomes

$$u(x_i)=\int_{-1}^{1}\tilde{K}\big(x_i,s(x_i,\theta),u(s(x_i,\theta))\big)d\theta+g(x_i), \tag{2.5}$$

where

$$\tilde{K}(x_i,s(x_i,\theta),u(s(x_i,\theta)))=\frac{1+x_i}{2}K(x_i,s(x_i,\theta),u(s(x_i,\theta))).$$

Next, using Legendre-Gauss integration formula, the integration term in (2.5) can be approximated by

$$\int_{-1}^{1}\tilde{K}(x_i,s(x_i,\theta),u(s(x_i,\theta)))d\theta\approx\sum_{k=0}^{N}\tilde{K}\big(x_i,s(x_i,\theta_k),u(s(x_i,\theta_k))\big)\omega_k, \tag{2.6}$$

where the set $\{\theta_i\}_{i=0}^N$ coincide with the Legendre-Gauss collocation points $\{x_i\}_{i=0}^N$.

We use u_i to approximate the function value $u(x_i)$, $0 \le i \le N$, and use

$$U(x)=\sum_{j=0}^{N} u_j F_j(x) \tag{2.7}$$

to approximate the function $u(x)$, namely, $u(x_i) \approx u_i$, $u(x) \approx U(x)$, and

$$U(s(x_i,\theta_k))=\sum_{j=0}^{N} u_j F_j(s(x_i,\theta_k)), \tag{2.8}$$

where $F_j(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^N$ which is the set of $(N+1)$ Legendre-Gauss points

$$F_j(x)=\prod_{i=0,i\neq j}^{N} \frac{x-x_i}{x_j-x_i}. \tag{2.9}$$

Then, the Legendre collocation method is to seek $U(x)$ such that $\{u_i\}_{i=0}^N$ satisfies the following collocation equations

$$u_i=\sum_{k=0}^{N} \tilde{K}\Big(x_i,s(x_i,\theta_k),\sum_{j=0}^{N} u_j F_j(s(x_i,\theta_k))\Big)\omega_k+g(x_i). \tag{2.10}$$

The numerical scheme (2.10) leads to a nonlinear system for $\{u_i\}_{i=0}^N$, we can get the values of $\{u_i\}_{i=0}^N$ by solving the system of nonlinear equations using a proper solver (e.g., Newton method).

There are some recent studies for using the collocation methods and the product integration methods for solving multi-dimensional Volterra integral equations. In two dimensions, we have

$$u(x,y)=\int_{-1}^{x}\int_{-1}^{y} K(x,y,s,\zeta,u(s,\zeta))dsd\zeta+g(x,y), \quad (x,y)\in I=[-1,1], \tag{2.11}$$

we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $N+1$ Legendre-Gauss, or Legendre-Gauss-Radau, or Legendre-Gauss-Lobatto points, and by $\{\omega_i^{(1)}\}_{i=0}^N$ the corresponding weights. Similarly, we denote the collocation points by $\{y_j\}_{j=0}^N$, which is the set of $N+1$ Legendre-Gauss, or Legendre-Gauss-Radau, or Legendre-Gauss-Lobatto points, and by $\{\omega_j^{(2)}\}_{j=0}^N$ the corresponding weights. Assume that (2.11) holds at the Legendre-collocation point-pairs (x_i,y_j), we have

$$u(x_i,y_j)=\int_{-1}^{x_i}\int_{-1}^{y_j} K(x_i,y_j,s,\zeta,u(s,\zeta))dsd\zeta+g(x_i,y_j). \tag{2.12}$$

We use $u(x_i,y_j)\approx u_{i,j}, u(x,y)\approx U(x,y)=\sum_{i=0}^N\sum_{j=0}^N u_{i,j}F_i(x)F_j(y)$, then using the linear transformation, the Legendre-collocation method is to seek $U(x,y)$ such that $u_{i,j}$ satisfies the following collocation equations:

$$\begin{aligned}u_{i,j}=&\frac{1+x_i}{2}\frac{1+y_j}{2}\sum_{p=0}^N\sum_{l=0}^N K\Big(x_i,y_j,s(x_i,\theta_p),\zeta(y_j,\theta_l),\sum_{i=0}^N\sum_{j=0}^N u_{i,j}F_i(s(x_i,\theta_p))F_j(\zeta(y_j,\theta_l))\Big)\omega_i^{(1)}\omega_j^{(2)}\\&+g(x_i,y_j).\end{aligned}\tag{2.13}$$

3 Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

Lemma 3.1 (see [15], Integration Error from Gauss Quadrature). *Assume that a $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight ω_j is used to integrate the product $u\varphi$, where $u\in H^m(I)$ for some $m\geq 1$. Let $\mathcal{P}_N$ denote the space of all polynomials of degree not exceeding N, $\varphi\in\mathcal{P}_N$. Then there exists a constant C independent of N such that*

$$\Big|\int_{-1}^1 u(x)\varphi(x)dx-(u,\varphi)_N\Big|\leq CN^{-m}|u|_{H^{m,N}(I)}\|\varphi\|_{L^2(I)},\tag{3.1}$$

where

$$|u|_{H^{m,N}(I)}=\Big(\sum_{j=\min(m,N+1)}^m\|u^{(j)}\|^2_{L^2(I)}\Big)^{1/2},\quad (u,\varphi)_N=\sum_{j=0}^N u(x_j)\varphi(x_j)\omega_j.\tag{3.2}$$

Lemma 3.2 (see [34], Lemma 3.2). *Assume that $u\in H^m(I)$ and denote I_Nu its interpolation polynomial associated with the $(N+1)$ Legendre-Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{x_j\}_{j=0}^N$, namely,*

$$I_Nu=\sum_{i=0}^N u(x_i)F(x_i).$$

Then the following estimates hold

$$\|u-I_Nu\|_{L^2(I)}\leq CN^{-m}|u|_{H^{m,N}(I)},\tag{3.3a}$$

$$\|u-I_Nu\|_{L^\infty(I)}\leq CN^{\frac{3}{4}-m}|u|_{H^{m,N}(I)}.\tag{3.3b}$$

Using Theorem 1 in [24], we have the following estimate for the Lagrange interpolation associated with the Jacobi Gaussian collocation points.

Lemma 3.3. *For every bounded function v, there exists a constant C, independent of v such that*

$$\sup_N\Big\|\sum_{j=0}^N v(x_j)F_j(x)\Big\|_{L^2(I)}\leq C\max_{x\in[-1,1]}|v(x)|,$$

where $F_j(x)$ is the Lagrange interpolation basis function associated with the Legendre collocation points $\{x_j\}_{j=0}^N$.

From [23], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials.

Lemma 3.4. *Assume that $\{F_j(x)\}_{j=0}^N$ is the N-th Lagrange polynomials associated with the Gauss, or Gauss-Radau, or Gauss-Lobatto points of the Legendre polynomials, then*

$$\max_{x\in[-1,1]}\sum_{j=0}^{N}|F_j(x)|=\mathcal{O}\big(N^{\frac{1}{2}}\big). \tag{3.4}$$

Lemma 3.5 (see [21], Gronwall Inequality)**.** *Suppose $L\geq 0$ and $G(x)$ is a non-negative, locally integrable function defined on $[-1,1]$ satisfying*

$$E(x)\leq G(x)+L\int_{-1}^{x}E(\tau)d\tau.$$

Then there exists a constant C such that

$$\|E\|_{L^p(I)}\leq L\|G\|_{L^p(I)},\quad p\geq 1.$$

4 Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed approximations. Firstly, we will carry our convergence analysis in L^2 space.

Theorem 4.1. *Let $u(x)$ be the exact solution of the nonlinear Volterra integro equation* (2.2)*, which is assumed to be sufficiently smooth, $K(x,t,u)$ is satisfies m times Lipschitz continuous with its third argument. $U(x)$ is obtained by using the spectral collocation scheme* (2.10)*. If $u\in H^m(I)$, then for $m\geq 1$,*

$$\|U(x)-u(x)\|_{L^2(I)}\leq CN^{-m}\Big(\max_{x\in[-1,1]}|K(x,s,u(s))|_{H^{m,N}(I)}+|u|_{H^{m,N}(I)}\Big) \tag{4.1}$$

provided that N is sufficiently large, where C is a constant independent of N.

Proof. First, form (2.10), we have as

$$u_i=\int_{-1}^{1}\tilde{K}(x_i,s(x_i,\theta),U(s(x_i,\theta)))d\theta+g(x_i)+I_{i,1}, \tag{4.2}$$

where

$$I_{i,1}=\sum_{k=0}^{N}\tilde{K}\big(x_i,s(x_i,\theta_k),U(s(x_i,\theta_k))\big)\omega_k-\int_{-1}^{1}\tilde{K}(x_i,s(x_i,\theta),U(s(x_i,\theta)))d\theta. \tag{4.3}$$

Using the integration error estimates from Legendre-Gauss polynomials quadrature in Lemma 3.1, we have

$$|I_{i,1}(x)|\leq CN^{-m}|K(x,s(x,\theta),U(s(x,\theta)))|_{H^{m,N}(I)},$$

note that

$$|K(x,s(x,\theta),U(s(x,\theta)))|\leq|K(x,s,u(s))|+|K(x,s,U(s))-K(x,s,u(s))|.$$

Then we get

$$|I_{i,1}(x)|\leq CN^{-m}\big(|K(x,s,u(s))|_{H^{m,N}(I)}+|K(x,s,U(s))-K(x,s,u(s))|_{H^{m,N}(I)}\big). \tag{4.4}$$

We assume that $K(x,t,u)$ is m times Lipschitz continuous with its third argument,

$$|K(x,s,u_1)-K(x,s,u_2)|\leq L_0|u_1-u_2|, \tag{4.5a}$$
$$\Big|\frac{\partial K(x,s,u_1)}{\partial u}-\frac{\partial K(x,s,u_2)}{\partial u}\Big|\leq L_1|u_1-u_2|,\cdots, \tag{4.5b}$$
$$\Big|\frac{\partial^m K(x,s,u_1)}{\partial u^m}-\frac{\partial^m K(x,s,u_2)}{\partial u^m}\Big|\leq L_m|u_1-u_2|, \tag{4.5c}$$

and using the definition of $|\cdot|_{H^{m,N}(I)}$ in (3.2), we have

$$\begin{aligned}
&|K(x,s,U(s))-K(x,s,u(s))|_{H^{m,N}(I)}\\
=&\Big(\sum_{j=\min(m,N+1)}^{m}\Big\|\frac{\partial^j K(x,s,U)}{\partial U^j}-\frac{\partial^j K(x,s,u)}{\partial u^j}\Big\|^2_{L^2(I)}\Big)^{\frac{1}{2}}\\
\leq&\sum_{j=\min(m,N+1)}^{m}\Big\|\frac{\partial^j K(x,s,U)}{\partial U^j}-\frac{\partial^j K(x,s,u)}{\partial u^j}\Big\|_{L^2(I)}\\
\leq&\sum_{j=\min(m,N+1)}^{m}L_i\|U-u\|_{L^2(I)}\\
\leq&C\|U-u\|_{L^2(I)}.
\end{aligned} \tag{4.6}$$

Then (4.4) can be rewritten as

$$\begin{aligned}
|I_{i,1}(x)|\leq&CN^{-m}|K(x,s(x,\theta),U(s(x,\theta)))|_{H^{m,N}(I)}\\
\leq&CN^{-m}\big(|K(x,s,u(s))|_{H^{m,N}(I)}+\|U-u\|_{L^2(I)}\big).
\end{aligned} \tag{4.7}$$

Multiplying $F_i(x)$ on both sides of (4.2) and summing up from 0 to N yield

$$U(x)=I_N\int_{-1}^{x}K(x,s,U(s))ds+I_Ng(x)+J_1(x), \tag{4.8}$$

where

$$J_1(x)=\sum_{i=0}^{N}I_{i,1}F_i(x). \tag{4.9}$$

It follows from (2.2) that

$$\begin{aligned}U(x)=&I_N\int_{-1}^{x}K(x,s,U(s))ds+I_N\Big(u(x)-\int_{-1}^{x}K(x,s,u(s))ds\Big)+J_1(x)\\=&I_Nu(x)+I_N\int_{-1}^{x}\big(K(x,s,U(s))-K(x,s,u(s))\big)ds+J_1(x).\end{aligned} \tag{4.10}$$

Let $e(x)$ denote the error function,

$$e(x)=U(x)-u(x).$$

Then, (4.10) can be written as

$$e(x)=I_N\int_{-1}^{x}\big(K(x,s,U(s))-K(x,s,u(s))\big)ds+J_1(x)+J_2(x), \tag{4.11}$$

where

$$J_2(x)=I_Nu(x)-u(x). \tag{4.12}$$

Consequently,

$$|e(x)|\le I_N\int_{-1}^{x}\big|K(x,s,U(s))-K(x,s,u(s))\big|ds+|J_1(x)|+|J_2(x)|. \tag{4.13}$$

It follows the Lipschitz conditions (4.5) that

$$\begin{aligned}|e(x)|\le&L_0I_N\int_{-1}^{x}|U(s)-u(s)|ds+|J_1(x)|+|J_2(x)|\\\le&L_0\int_{-1}^{x}|e(s)|ds+|J_1(x)|+|J_2(x)|+J_3(x),\end{aligned} \tag{4.14}$$

where

$$J_3(x)=L_0I_N\int_{-1}^{x}|e(s)|ds-L_0\int_{-1}^{x}|e(s)|ds. \tag{4.15}$$

It follows from the Gronwall inequality of Lemma 3.5

$$\|e(x)\|_{L^2(I)}\le C\big(\|J_1(x)\|_{L^2(I)}+\|J_2(x)\|_{L^2(I)}+\|J_3(x)\|_{L^2(I)}\big). \tag{4.16}$$

Using Lemma 3.3 and the estimates (4.7), we have

$$\|J_1\|_{L^2(I)} \leq C \max_{0\leq i\leq N} |I_{i,1}| \leq CN^{-m}\Big(\max_{x\in[-1,1]} |K(x,s,u(s))|_{H^{m,N}(I)} + \|e(x)\|_{L^2(I)}\Big). \tag{4.17}$$

Due to Lemma 3.2,

$$\|J_2\|_{L^2(I)} = \|I_N u(x) - u(x)\|_{L^2(I)} \leq CN^{-m}|u|_{H^{m,N}(I)}. \tag{4.18}$$

By virtue of Lemma 3.2 with $m=1$,

$$\begin{aligned}\|J_3\|_{L^2(I)} =&\Big\|L_0 I_N \int_{-1}^{x} |e(s)|ds - L_0 \int_{-1}^{x} |e(s)|ds\Big\|_{L^2(I)} \\ \leq& CN^{-1}\Big|\int_{-1}^{x}|e(s)|ds\Big|_{H^1(I)} \\ \leq& CN^{-1}\|e\|_{L^2(I)}.\end{aligned} \tag{4.19}$$

Combining (4.17), (4.18), (4.19), gives

$$\begin{aligned}\|e\|_{L^2(I)} \leq& CN^{-m}\Big(\max_{x\in[-1,1]} |K(x,s,u(s))|_{H^{m,N}(I)} + \|e(x)\|_{L^2(I)}\Big) \\ &+ CN^{-m}|u|_{H^{m,N}(I)} + CN^{-1}\|e\|_{L^2(I)}.\end{aligned} \tag{4.20}$$

Provided that N is sufficiently large, we have the desired estimate (4.1). □

Theorem 1 in [32] have convergence analysis with the order of convergence $\mathcal{O}(N^{1/2-m})$ which seems not optimal, in our result, the optimal order of convergence $\mathcal{O}(N^{-m})$ is obtained.

Next, we will give the error estimates in L^∞ space.

Theorem 4.2. *If the hypotheses given in Theorem 4.1 hold, then*

$$\|U(x)-u(x)\|_{L^\infty(I)} \leq CN^{\frac{1}{2}-m} \max_{x\in[-1,1]} |K(x,s,u(s))|_{H^{m,N}(I)} + CN^{\frac{3}{4}-m}|u|_{H^{m,N}(I)}, \tag{4.21}$$

provided that N is sufficiently large and C is a constant independent of N.

Proof. Following the same procedure as in the proof of Theorem 4.1, we have

$$|e(x)| \leq L\int_{-1}^{x} |e(s)|ds + |J_1(x)| + |J_2(x)| + J_3(x), \tag{4.22}$$

where J_1, J_2 and J_3 are defined by (4.9), (4.12) and (4.15), respectively. It follows from the Gronwall inequality (see Lemma 3.5) that

$$\|e(x)\|_{L^\infty(I)} \leq C\big(\|J_1(x)\|_{L^\infty(I)} + \|J_2(x)\|_{L^\infty(I)} + \|J_3(x)\|_{L^\infty(I)}\big). \tag{4.23}$$

Using Lemma 3.4 and the estimates (4.7), we have

$$\|J_1\|_{L^\infty(I)} \le CN^{\frac{1}{2}} \max_{0\le i\le N} |I_{i,1}| \le CN^{\frac{1}{2}-m}\Big(\max_{x\in[-1,1]} |K(x,s,u(s))|_{H^{m,N}(I)} + \|e(x)\|_{L^2(I)}\Big). \tag{4.24}$$

From Lemma 3.2, we have

$$\|J_2\|_{L^\infty(I)} = \|I_N u(x) - u(x)\|_{L^\infty(I)} \le CN^{\frac{3}{4}-m} |u|_{H^{m,N}(I)}. \tag{4.25}$$

It follows again from Lemma 3.2 with $m=1$ that

$$\begin{aligned}\|J_3\|_{L^\infty(I)} =& \Big\| L_0 I_N \int_{-1}^{x} |e(s)| ds - L_0 \int_{-1}^{x} |e(s)| ds \Big\|_{L^\infty(I)} \le CN^{-\frac{1}{4}} \Big| \int_{-1}^{x} |e(s)| ds \Big|_{H^1(I)} \\ \le& CN^{-\frac{1}{4}} \|e\|_{L^2(I)} \le CN^{-\frac{1}{4}} \|e\|_{L^\infty(I)}. \end{aligned} \tag{4.26}$$

The desired estimate (4.21) is obtained by combining (4.23), (4.24), (4.25) and (4.26). □

5 Numerical experiments

We will provide some numerical examples below using the spectral technique proposed in this work.

All the experiments are implemented on Matlab 7.1, the resulting nonlinear algebraic system (2.10) and (2.13) is solved by the Matlab build-in function fsolve with the initial value 0 and tolerance $1.0e-15$. The computing environment is: Thinkpad Laptop (Intel i5-3230M CPU 2.50GHz, Memory 4.0GB), and the operator system is Windows XP.

Example 5.1. Consider the following nonlinear Volterra equation (2.2) with

$$K(x,s,u(s)) = \frac{1}{2+2u^2(s)}, \tag{5.1a}$$

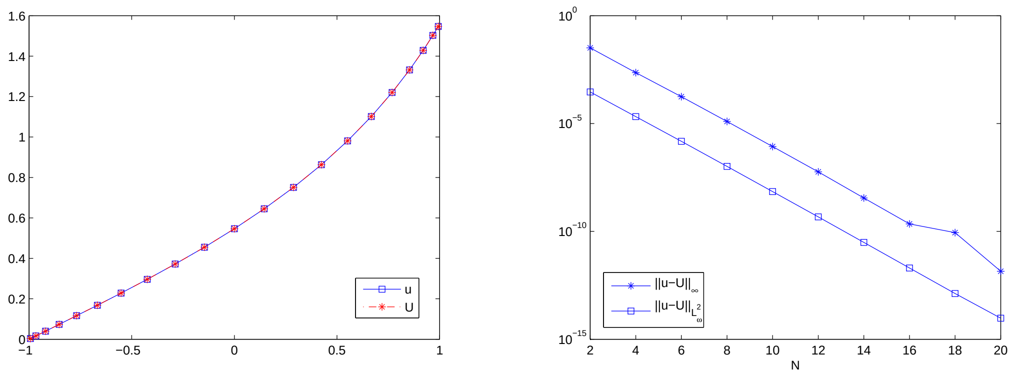
$$g(x) = \tan((1+x)/2) - 0.25\sin(1+x) - 0.25(1+x), \tag{5.1b}$$

so that the exact solution is $u(x) = \tan((1+x)/2)$.

Table 1 shows the errors $\|U-u\|_{L^\infty(-1,1)}$ and $\|U-u\|_{L^2(-1,1)}$ for $2\le N\le 20$ obtained by using the spectral collocation methods described above. Fig. 1 presents the approximate

Table 1: The errors $\|U-u\|_{L^\infty(-1,1)}$, $\|U-u\|_{L^2(-1,1)}$ and CPU time used for Example 5.1.

N	L^∞-error	L^2-error	CPU time (s)
4	0.0023	2.0988e-005	0.0156
8	1.2486e-005	1.0368e-007	0.0469
12	5.7446e-008	4.6983e-010	0.1250
16	2.2265e-010	2.0085e-012	0.3750
20	1.4155e-012	9.4950e-015	1.0625



(a) (b)

Figure 1: Example 5.1: Comparison between approximate solution $U(x)$ and exact solution $u(x)$ (a). The errors $U-u$ versus the number of collocation points in L^2 and L^∞ norms (b).

and exact solution on left side, which are found in excellent agreement, on right side, the numerical errors $U-u$ is plotted for $2\le N\le 24$ in both L^2 and L^∞ norms. As expected, the exponential rate of convergence is observed for the nonlinear problem, which confirmed our theoretical predictions.

Example 5.2. Consider the following nonlinear Volterra equation (2.2) with

$$K(x,s,u(s))=e^{x-3s}u^2(s), \tag{5.2a}$$

$$\begin{aligned} g(x)=&\frac{1}{2(1+36\pi^2)}\big(e^{-x}+36\pi^2e^{-x}-e^{-x}\cos 6\pi x\\ &+6\pi e^{-x}\sin 6\pi x-36e\pi^2\big)e^x+e^x\sin 3\pi x. \end{aligned} \tag{5.2b}$$

The exact solution is $u(x)=e^x\sin 3\pi x$.

Table 2: The errors $\|U-u\|_{L^\infty(-1,1)}$, $\|U-u\|_{L^2(-1,1)}$ and CPU time used for Example 5.2.

N	L^∞-error	L^2-error	CPU time (s)
12	0.0478	1.7039e-004	0.2031
14	0.0160	4.6958e-005	0.4063
16	5.4537e-004	7.2824e-006	0.6250
18	8.4306e-005	7.5959e-007	0.9219
20	2.5527e-006	5.7623e-008	1.4063
22	3.7190e-007	3.3416e-009	2.0938
24	1.0254e-008	1.5301e-010	3.3438
26	6.3190e-010	5.6591e-012	4.6250
28	1.8130e-011	1.7091e-013	6.3125
30	4.0181e-012	3.6829e-015	8.3594

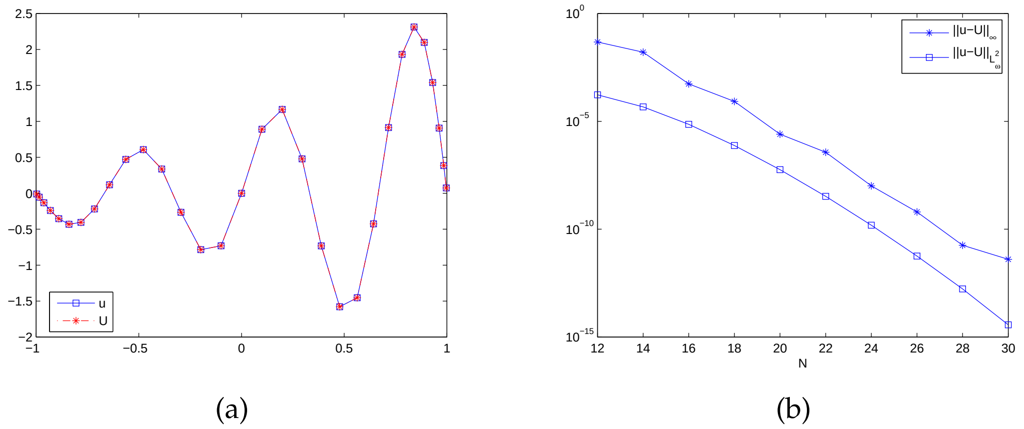


Figure 2: Example 5.2: Comparison between approximate solution $U(x)$ and exact solution $u(x)$ (a). The errors $U-u$ versus the number of collocation points in L^2 and L^∞ norms (b).

Numerical errors with several values of N are displayed in Table 2 and Fig. 2. Again the exponential rate of convergence is observed for the nonlinear problem.

Example 5.3. The third example is concerned with a 2D nonlinear Volterra equation with second kind. Consider the Eq. (2.11) with

$$K(x,y,s,\zeta,u(s,\zeta)) = e^{x+y}\cot(2s+\zeta)u^2(s,\zeta), \tag{5.3a}$$

$$g(x) = \frac{1}{16}e^{x+y}\big(\sin(4x+2y)-\sin(2y-4)-\sin(4x-2)-\sin 6\big)+\sin(2x+y). \tag{5.3b}$$

This problem has a unique solution $u(x,y)=\sin(2x+y)$.

Numerical errors with several values of N are displayed in Table 3 and Fig. 3. Although our convergence theory does not cover multi-dimensional case, the exponential rate of convergence is observed for the 2D nonlinear problem. It is expected that the analysis techniques proposed in this work can be used to extend Theorem 4.1 and Theorem 4.2 to obtain a spectral convergence rate for (2.13).

Table 3: The errors $\|U-u\|_{L^\infty(-1,1)}$, $\|U-u\|_{L^2(-1,1)}$ and CPU time used for Example 5.3.

N	L^∞-error	L^2-error	CPU time (s)
2	0.4253	0.0151	0.1563
4	0.0292	0.2835	2.2969
6	7.5784e-04	4.2484e-04	18.7969
8	1.4431e-05	5.6910e-06	81.2031
10	1.2489e-07	5.1461e-08	259.4688
12	9.4263e-10	3.1804e-10	1.3286e+003
14	4.8134e-012	1.5284e-012	8.2548e+003
16	2.0539e-014	5.7171e-015	1.6597e+004

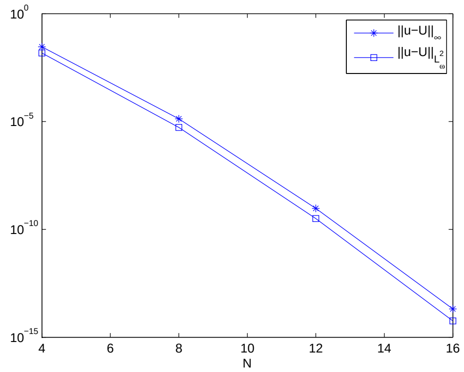


Figure 3: Example 5.3: The errors $U-u$ versus the number of collocation points in L^2 and L^∞ norms.

6 Conclusions and future work

This paper proposes a numerical method for the nonlinear Volterra integral equations based on Legendre spectral approach. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of approximate solutions decay exponentially in L^2-norm and L^∞-norm, which is a desired feature for a spectral method.

In our future work, the spectral collocation methods will be studied for the nonlinear Volterra integral-differential equations with smooth kernels and weakly singular kernels, and extend this method to high dimension.

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