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On Polynomial Maximum Entropy Method for Classical Moment Problem

Jiu Ding[1,*], Noah H. Rhee[2] and Chenhua Zhang[1]

[1] *Department of Mathematics, The University of Southern Mississippi, Hattiesburg, MS 39406-5045, USA*
[2] *Department of Mathematics and Statistics, University of Missouri–Kansas City, Kansas City, MO 64110-2499, USA*

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Abstract. The maximum entropy method for the Hausdorff moment problem suffers from ill conditioning as it uses monomial basis $\{1,x,x^2,\cdots,x^n\}$. The maximum entropy method for the Chebyshev moment probelm was studied to overcome this drawback in [4]. In this paper we review and modify the maximum entropy method for the Hausdorff and Chebyshev moment problems studied in [4] and present the maximum entropy method for the Legendre moment problem. We also give the algorithms of converting the Hausdorff moments into the Chebyshev and Lengendre moments, respectively, and utilizing the corresponding maximum entropy method.

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1 Introduction

The *Hausdorff moment problem* is to find an unknown density f^* such that

$$\int_0^1 f^*(x)x^i dx = \mu_i, \quad i=0,1,2,\cdots.$$

It is well known [11] that the above problem has a solution if and only if the moment sequence $\{\mu_i\}$ is *positive definite*, i.e., $\Delta^m \mu_i \geq 0$ for all m and i, where Δ^m is the m-th forward difference.

*Corresponding author.
Email: jiu.ding@usm.edu (J. Ding), rheen@umkc.edu (N. H. Rhee), chenhua.zhang@usm.edu (C. H. Zhang)

Many mathematical physics problems are equivalent to the moment problem. Practically, it is often the case that only a finitely many moments are available. Thus, it is desired to determine a density f that satisfies all the known moment conditions

$$\int_0^1 f(x)x^i dx = \mu_i, \quad i=0,1,2,\cdots,n.$$

Mathematically there are infinitely many candidates for the required function, so how to locate the *best one* among them is important in applications.

The maximum entropy principle provides the most unbiased criterion for choosing the best candidate with the given moments. In other words, the determined density function gives the maximum entropy among all the densities with the given moments. The realization of this principle is the solution of the following optimization problem: maximize

$$H(f) = -\int_0^1 f(x)\ln f(x)dx$$

among all the density functions subject to

$$\int_0^1 f(x)x^i dx = \mu_i, \quad i=0,1,2,\cdots,n.$$

Here the objective function H is called the *Boltzmann* entropy.

This principle was first proposed by Jayne in 1957 to numerically recover an unknown density function in mathematical physics [8]. The resulting numerical scheme is well known now as the maximum entropy method [10], and its idea has been extended to solving, for example, Frobenius-Perron operator equations [1,2,4–6] for the computation of a stationary density of an interval mapping $S:[0,1]\to[0,1]$.

Although it is widely useful in physical science and engineering [3, 10], the classical maximum entropy method using Hausdorff moments has an intrinsic drawback of sensitivity issue. Namely, the resulting system of nonlinear equations from the above constrained maximization problem is ill-conditioned due to the involvement of the standard monomial basis $\{1,x,x^2,\cdots,x^n\}$. So in [4] to overcome this drawback Chebyshev polynomial basis was used.

In this paper we review and modify the Hausdorff and Chebyshev maximum entropy methods and study Legendre maximum entropy method. We also consider the algorithms of converting the Hausdorff moments into the Chebyshev and Legendre moments, respectively, and solving the corresponding moment problems with the maximum entropy method.

We briefly review the basic properties of the Chebyshev and Legendre polynomials in the next section. Then we review the general maximum entropy method in Section 3. In Section 4 we develop a polynomial maximum entropy method. Then in Section 5 we consider the algorithms of converting the Hausdorff moments into the Chebyshev or Legendre moments. Numerical experiments of all the algorithms discussed in the paper are performed and compared in Section 6. We conclude in Section 7.