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A Comparative Study of Finite Element and Finite Difference Methods for Two-Dimensional Space-Fractional Advection-Dispersion Equation

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Abstract. The paper makes a comparative study of the finite element method (FEM) and the finite difference method (FDM) for two-dimensional fractional advection-dispersion equation (FADE) which has recently been considered a promising tool in modeling non-Fickian solute transport in groundwater. Due to the non-local property of integro-differential operator of the space-fractional derivative, numerical solution of FADE is very challenging and little has been reported in literature, especially for high-dimensional case. In order to effectively apply the FEM and the FDM to the FADE on a rectangular domain, a backward-distance algorithm is presented to extend the triangular elements to generic polygon elements in the finite element analysis, and a variable-step vector Grünwald formula is proposed to improve the solution accuracy of the conventional finite difference scheme. Numerical investigation shows that the FEM compares favorably with the FDM in terms of accuracy and convergence rate whereas the latter enjoys less computational effort.

AMS subject classifications: 26A33, 65D32, 65N06, 65N30

Key words: Space-fractional derivative, advection-dispersion, finite element, finite difference.

1 Introduction

Fractional derivative models have been extensively investigated in recent decades for describing the anomalous diffusion or dispersion [1, 2], energy dissipation of vibration and wave [3, 4], and dynamic system [5], with fewer parameters than the classical models of integer-order derivative. A state-of-the-art review of applications of fractional calculus on solid and fluid mechanics can be found in monographs [6, 7]. Due to the

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integro-differential and convolution expression of the fractional derivative, the analytical solution of fractional derivative equations are not always obtainable, especially for high-dimensional irregular computing domain and complex boundary conditions.

Recent decade has witnessed fast growing developments of numerical methods for fractional derivative equations. In comparison with scalar time-fractional derivative, space-fractional derivative, particularly the fractional Laplacian [8, 9], is of more difficulty for discretization due to its vector integral expression. It is stressed that the differentiation directions of the space-fractional derivative are usually confined to coordinate axes in most of the existing literature [10–12]. However, for the FADE model of our interest in describing the multidimensional non-Fickian solution transport little has been reported on the consideration of the non-coordinate differentiation directions of the space-fractional derivative. It is significant to consider the non-coordinate derivative in the FADE in order to generate a full family of multivariable Lévy stable laws that underlie particle random walks with occasional large jumps [13, 14]. The motivation of this study is to seek effective numerical methods for discretizing the space-fractional derivative having non-coordinate differentiation directions.

FEM and FDM have long been considered well-known mesh-based approximation methods for solving a tremendous amount of engineering and scientific problems. Great effort has been made to apply these two methods to fractional models with coordinate-directed space-fractional derivative [10–12, 15–18]. Nevertheless, special care should be taken to the FADE with non-coordinate-directed derivative due to the convolution characteristic and the direction dependence of the space-fractional derivative. Roop [19] has presented for the FADE a finite element scheme based on the variational statements derived in [20], but the scheme is confined to the use of triangular elements and will be difficult to extend to rectangular elements because of the complicated mathematical analysis arising from the non-coordinate derivative. Note that rectangular elements are usually preferred for rectangular computing domains. Meerschaert et al. [21] proposed a vector Grünwald formula (VGF), a type of finite difference scheme, for discretizing the space-fractional derivative on an infinite domain. But due to taking fixed spatial steps, this formula will become inaccurate for a finite computing domain.

In this study, we make a comparative study of the FEM and the FDM for two-dimensional FADE. A backward-distance algorithm is presented to diversify the types of elements that can be used in the finite element analysis, and a variable-step VGF is proposed to improve the solution accuracy of the conventional VGF. Unless otherwise stated, we call the backward-distance-algorithm based FEM the FEM and the variable-step VGF the FDM for simplicity. In the FEM, the backward-distance algorithm is intended for all types of polygon elements, such as triangle, rectangle, and parallelogram. The algorithm can automatically derive the desired distance irrespective of the relative locations of the quadrature point and the finite element. The FDM adopts variable spatial steps to guarantee more grid points are selected in arbitrary differentiation directions of the space-fractional derivative. This avoids the accuracy decrease due to very few computing points taken along certain differentiation direction.