Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **8**, No. 1, pp. 19-36

DOI: 10.4208/aamm.2013.m322
February 2016

Existence and Asymptotic Behavior of Positive Solutions for Variable Exponent Elliptic Systems

Honghui Yin[1] and Zuodong Yang[2,3,*]

[1] *School of Mathematical Sciences, Huaiyin Normal University, Jiangsu 223001, China*
[2] *School of Teacher Education, Nanjing Normal University, Nanjing 210097, China*
[3] *Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China*

Received 11 August 2013; Accepted (in revised version) 27 November 2014

Abstract. In this paper, our main purpose is to establish the existence of positive solution of the following system

$$\begin{cases} -\Delta_{p(x)}u=F(x,u,v), & x\in\Omega, \\ -\Delta_{q(x)}v=H(x,u,v), & x\in\Omega, \\ u=v=0, & x\in\partial\Omega, \end{cases}$$

where $\Omega=B(0,r)\subset\mathbf{R}^N$ or $\Omega=B(0,r_2)\setminus\overline{B(0,r_1)}\subset\mathbf{R}^N$, $0<r$, $0<r_1<r_2$ are constants. $F(x,u,v)=\lambda^{p(x)}[g(x)a(u)+f(v)]$, $H(x,u,v)=\theta^{q(x)}[g_1(x)b(v)+h(u)]$, $\lambda,\theta>0$ are parameters, $p(x),q(x)$ are radial symmetric functions, $-\Delta_{p(x)}=-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. We give the existence results and consider the asymptotic behavior of the solutions. In particular, we do not assume any symmetric condition, and we do not assume any sign condition on $F(x,0,0)$ and $H(x,0,0)$ either.

AMS subject classifications: 35J60, 35J62

Key words: Positive solution, $p(x)$-Laplacian, asymptotic behavior, sub-supersolution.

1 Introduction

In this paper, our main purpose is to establish the existence of positive solution of the following system

$$\begin{cases} -\Delta_{p(x)}u=F(x,u,v), & x\in\Omega, \\ -\Delta_{q(x)}v=H(x,u,v), & x\in\Omega, \\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

*Corresponding author.
Email: yinhh@hytc.edu.cn (H. H. Yin), zdyang_jin@263.net (Z. D. Yang)

where $\Omega=B(0,r)\subset\mathbf{R}^N$ or $\Omega=B(0,r_2)\setminus\overline{B(0,r_1)}\subset\mathbf{R}^N$, r and $r_1<r_2$ are positive constants, $F(x,u,v)=\lambda^{p(x)}[g(x)a(u)+f(v)]$, $H(x,u,v)=\theta^{q(x)}[g_1(x)b(v)+h(u)]$ and $p(x),q(x)\in C^1(\overline{\Omega})$ are radial symmetric positive functions, i.e., $p(x)=p(|x|)$, $q(x)=q(|x|)$, the operator $-\Delta_{p(x)}=-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian and the corresponding equation is called a variable exponent equation.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It aries from nonlinear elasticity theory, electro-rheological fluids, etc. (see [17, 27]). Many results have been obtained on this kind of problems, for example [1–3, 5–7, 9, 13]. On the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [1, 3, 5]. For the existence results for the elliptic problems with variable exponents, we refer to [7, 13, 21–24].

For the special case, $p(x)\equiv p$ (a constant), (1.1) becomes the well known p-Laplacian system. There have been many papers on this class of problems, see [4, 12, 19] and the reference therein. We point out that elliptic equations involving the $p(x)$-Laplacian are not trivial generalizations of similar problems studied in the constant case, since the $p(x)$-Laplacian operator is nonhomogeneity. Thus, some techniques which can be applied in the case of the p-Laplacian operators will fail in that new station, such as the Lagrange Multiplier Theorem. Another example is that, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)}=\inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}}\frac{\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx}{\int_\Omega\frac{1}{p(x)}|u|^{p(x)}dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)}>0$ (see [11]). But the facts that the first eigenvalue $\lambda_p>0$ and the existence of the first eigenfunction are very important in the study of p-Laplacian problems. There are more difficulties in discussing the existence and asymptotic behavior of solutions of variable exponent problems.

In [12], the authors studied the existence of positive weak solutions for the following problem:

$$\begin{cases}-\Delta_p u=\lambda f(v), & x\in\Omega,\\ -\Delta_p v=\lambda g(u), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega.\end{cases}\tag{1.2}$$

Under the condition of

$$\lim_{s\to\infty}\frac{f(M[g(s)]^{\frac{1}{p-1}})}{s^{p-1}}=0,\quad \forall M>0,\tag{1.3}$$

the authors gave the existence of positive solutions for problem (1.2).

In [4], the author considered the existence and nonexistence of positive weak solutions to the following p-Laplacian problem:

$$\begin{cases} -\Delta_p u = \lambda u^\alpha v^\gamma, & x \in \Omega, \\ -\Delta_q v = \lambda u^\delta v^\beta, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Recently, in [20], the authors considered the existence of positive solutions to the following quasilinear elliptic system in a bounded domain $\Omega \subset R^N$:

$$\begin{cases} -\Delta_p u = \lambda[g(x)a(u) + f(v)], & x \in \Omega, \\ -\Delta_q v = \theta[g_1(x)b(v) + h(u)], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $\lambda, \theta > 0$ are parameters and $g(x)$, $g_1(x)$ may be negative near the boundary $\partial\Omega$.

We note that in order to obtain the existence results, the first eigenfunction of $-\Delta_p$ is used to construct the sub-solution for problems (1.2), (1.4) and (1.5). But for the variable exponent problems, maybe the first eigenvalue and the first eigenfunction of the operator $-\Delta_{p(x)}$ do not exist. Even if the first eigenfunction of $-\Delta_{p(x)}$ exists, because of the nonhomogeneity of $-\Delta_{p(x)}$, we still cannot to construct the sub-solution of variable exponent problems with the first eigenfunction. In many cases, the radial symmetric conditions are affective to deal with variable exponent problems, see [7,8,22,24] and reference therein. In [21, 22, 26], with a condition similar to (1.3), the author discussed the existence of positive solutions of the following problems:

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(v), & x \in \Omega, \\ -\Delta_{p(x)} v = \lambda g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

and

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)} f(v), & x \in \Omega, \\ -\Delta_{p(x)} v = \lambda^{p(x)} g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

We call (1.1) is $(p(x), q(x))$-type and call (1.6), (1.7) are $(p(x), p(x))$-type. Since both $-\Delta_{p(x)}$ operator and $-\Delta_{q(x)}$ operator are contained, the study of $(p(x), q(x))$-type is more complicated than that of $(p(x), p(x))$-type.

Motivated by the above results, we study problem (1.1) in this paper. Our aim is to give the existence and asymptotic behavior of positive weak solutions for problem (1.1). The paper gives the existence of positive weak solutions via sub-supersolution method. Our results partially generalized the results of [12, 20–22, 26].

The paper is organized as follows. In Section 2, we recall some facts that will be needed in the paper. In Section 3, we consider the existence of positive solutions of (1.1). We will show the asymptotic behavior of the positive solutions of problem (1.1) in the fourth section. In Section 5, we give an example.

2 Notations and preliminaries

In order to deal with $p(x)$-Laplacian problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [6, 14, 17, 18]). For any $f(x)\in C(\overline{\Omega})$, we write

$$f^+=\max_{x\in\overline{\Omega}} f(x), \quad f^-=\min_{x\in\overline{\Omega}} f(x).$$

Denote

$$L^{p(x)}(\Omega)=\Big\{u\Big|u \text{ is a measurable real-valued funcion, } \int_\Omega |u(x)|^{p(x)}dx<\infty\Big\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)}=\inf\Big\{\lambda>0\Big|\int_\Omega\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}dx\le 1\Big\},$$

and $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ becomes a Banach space, and we call it variable exponent Lebesgue space.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega)||\nabla u|\in L^{p(x)}(\Omega)\},$$

and it can be equipped with the norm

$$\|u\|=|u|_{p(x)}+|\nabla u|_{p(x)}, \quad \forall u\in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we call it variable exponent Sobolev space. From [6], we know that spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces.

We define

$$(L(u),v)=\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx, \quad \forall u,v\in W_0^{1,p(x)}(\Omega),$$

then $L:W_0^{1,p(x)}(\Omega)\to(W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded and strictly monotone operator, and it is a homeomorphism (see [9, Theorem 3.1]).

Definition 2.1. (1) $(u,v)\in(W_0^{1,q(x)}(\Omega),W_0^{1,q(x)}(\Omega))$ is called a (weak) solution of problem (1.1) if it satisfies

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx=\int_\Omega F(x,u,v)\varphi dx, \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx=\int_\Omega H(x,u,v)\psi dx, \end{cases}$$

for any $(\varphi,\psi)\in(W_0^{1,q(x)}(\Omega),W_0^{1,q(x)}(\Omega))$.

(2) $(u,v)\in(W^{1,p(x)}(\Omega),W^{1,p(x)}(\Omega))$ is called a sub-solution (super-solution) of problem (1.1) if $(u,v)\leq(\geq)(0,0)$ on $\partial\Omega$ and

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx\leq(\geq)\int_\Omega F(x,u,v)\varphi dx, \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx\leq(\geq)\int_\Omega H(x,u,v)\psi dx, \end{cases}$$

for any $(\varphi,\psi)\in(W_0^{1,p(x)}(\Omega),W_0^{1,q(x)}(\Omega))$ with $\varphi,\psi\geq 0$.

Define $A:W^{1,p(x)}(\Omega)\to(W_0^{1,p(x)}(\Omega))^*$ as

$$\langle Au,\varphi\rangle=\int_\Omega(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi+m(x,u)\varphi)dx,\quad \forall u\in W^{1,p(x)}(\Omega),\quad \forall\varphi\in W_0^{1,p(x)}(\Omega),$$

where $m(x,u)$ is continuous on $\overline{\Omega}\times R$, $m(x,\cdot)$ is increasing and satisfies

$$|m(x,t)|\leq C_1+C_2|t|^{p^*(x)-1},$$

where

$$p^*(x)=\frac{Np(x)}{N-p(x)},$$

if $p(x)<N$ and $p^*(x)=\infty$ if $p(x)\geq N$, here and hereafter, we use C_i to denote positive constants. It is easy to check that A is a continuous bounded mapping. From [25], we have the following lemma.

Lemma 2.1 (Comparison Principle). *Let $u,v\in W^{1,p(x)}(\Omega)$. If $Au-Av\leq 0$ in $(W_0^{1,p(x)}(\Omega))^*$ and $u\leq v$ on $\partial\Omega$ (i.e., $(u-v)^+\in W_0^{1,p(x)}(\Omega)$), then $u\leq v$ a.e. in Ω.*

The following conditions will be required in our results:

(D1) $\Omega=B(0,r)\subset\mathbf{R}^N$ is an open ball with center 0 and radius $r>0$;

(D2) $p(x),q(x)\in C^1(\overline{\Omega})$ are radial symmetric functions and $1<p^-\leq p^+$, $1<q^-\leq q^+$;

(D3) $g,g_1\in C(\overline{\Omega})$ are positive functions;

(D4) $f,h\in C^1([0,\infty))$ are nondecreasing, $\lim_{s\to\infty}f(s)=\infty$, $\lim_{s\to\infty}h(s)=\infty$ and

$$\lim_{s\to\infty}\frac{f\big(M[h(s)]^{\frac{1}{q^--1}}\big)}{s^{p^--1}}=0,\quad \forall M>0,$$

(a combined sub-linear effect at ∞).

(D5) $a,b\in C^1([0,\infty))$ are nondecreasing, $\lim_{s\to\infty}a(s)=\infty$, $\lim_{s\to\infty}b(s)=\infty$ and

$$\lim_{s\to\infty}\frac{a(s)}{s^{p^--1}}=0,\quad \lim_{s\to\infty}\frac{b(s)}{s^{q^--1}}=0.$$

3 Existence of positive solutions

In the present paper, we use $(\lambda,\theta)>(\lambda^*,\theta^*)$ to denote $\lambda>\lambda^*$, $\theta>\theta^*$ and the same meaning for other cases, and denote by $\rho(x)=|x|$, then we have the following result:

Theorem 3.1. *If (D1)-(D5) hold, then there exist* $(\lambda_*,\theta_*)>(0,0)$ *such that for any* $(\lambda,\theta)>(\lambda_*,\theta_*)$, *problem* (1.1) *has at least one positive solution.*

Proof. According to the sub-super solution method for $p(x)$-Laplacian equations (see [10]), we only need to construct a positive sub-solution (ϕ_1,ϕ_2) and a super-solution (z_1,z_2) of (1.1) such that $(\phi_1,\phi_2)\le(z_1,z_2)$, then there exists a positive solution (u,v) of (1.1) satisfies $(\phi_1,\phi_2)\le(u,v)\le(z_1,z_2)$. That's complete the proof.

By (D3)-(D5), we see that there exists a $M>2$, such that

$$a(s)g(x)+f(0)\ge 1,\quad b(s)g_1(x)+h(0)\ge 1,\quad \text{when } s\ge M-1,\quad x\in\Omega. \tag{3.1}$$

Let

$$\sigma=\frac{\ln M}{k},\quad \tau=\frac{\ln M}{l},$$

then there exists $k_1=l_1>1$ such that for any $k>k_1$, $l>l_1$, we have $\sigma,\tau\in(0,r)$, we denote

$$\phi_1(x)=\phi_1(\rho)=\begin{cases} e^{k(r-\rho)}-1, & r-\sigma<\rho\le r,\\ e^{k\sigma}-1+\displaystyle\int_\rho^{r-\sigma} ke^{k\sigma}\Big(\frac{t}{r-\sigma}\Big)^{\frac{1}{p(t)-1}}dt, & 0\le\rho\le r-\sigma,\end{cases}$$

and

$$\phi_2(x)=\phi_2(\rho)=\begin{cases} e^{l(r-\rho)}-1, & r-\tau<\rho\le r,\\ e^{l\tau}-1+\displaystyle\int_\rho^{r-\tau} le^{l\tau}\Big(\frac{t}{r-\tau}\Big)^{\frac{1}{q(t)-1}}dt, & 0\le\rho\le r-\tau.\end{cases}$$

It is easy to see that $\phi_1,\phi_2\in C^1(\overline{\Omega})$. By computation, we have

$$-\Delta_{p(x)}\phi_1=\begin{cases} -(ke^{k(r-\rho)})^{p(\rho)-1}\Big[k(p(\rho)-1)-p'(\rho)\ln k-kp'(\rho)(r-\rho)-\dfrac{N-1}{\rho}\Big],\\ \qquad r-\sigma<\rho<r,\\ -(ke^{k\sigma})^{p(\rho)-1}\Big[p'(\rho)(\ln k+k\sigma)\dfrac{\rho}{r-\sigma}-\dfrac{1}{r-\sigma}+\dfrac{N-1}{\rho}\dfrac{\rho}{r-\sigma}\Big],\\ \qquad 0<\rho<r-\sigma.\end{cases} \tag{3.2}$$

Denote

$$\alpha=\min\Big\{\frac{\inf p(x)-1}{4(\sup|\nabla p(x)|+1)},1\Big\},\quad \alpha_1=\min\Big\{\frac{\inf q(x)-1}{4(\sup|\nabla q(x)|+1)},1\Big\},$$

and
$$\beta=|f(0)|+a(M-1)\max_{x\in\overline{\Omega}} g(x), \quad \beta_1=|h(0)|+b(M-1)\max_{x\in\overline{\Omega}} g_1(x).$$
From (3.2), there exists $k_2>0$ such that when $k>k_2$, we have
$$-\Delta_{p(x)}\phi_1\leq -k^{p(\rho)}\alpha, \quad r-\sigma<\rho<r. \tag{3.3}$$
Let $\lambda=\alpha/\beta k$, we have $k^{p(x)}\alpha\geq\lambda^{p(x)}\beta$, then
$$-\Delta_{p(x)}\phi_1\leq -\lambda^{p(x)}\beta\leq\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)], \quad r-\sigma<\rho<r. \tag{3.4}$$
When $0<\rho<r-\sigma$, there exists $C_1>0$ such that
$$-\Delta_{p(x)}\phi_1\leq C_1(ke^{k\sigma})^{p(\rho)-1}\ln k. \tag{3.5}$$
Then there exists $k_3>0$ such that when $k>k_3$, $\lambda=\alpha/\beta k$, we have
$$C_1(ke^{k\sigma})^{p(x)-1}\ln k\leq\lambda^{p(x)}. \tag{3.6}$$
From (3.1), (3.5) and (3.6), we have
$$-\Delta_{p(x)}\phi_1\leq\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)], \quad 0<\rho<r-\sigma. \tag{3.7}$$
Let $k_*=\max\{k_1,k_2,k_3\}$. Similarly, we obtain l_2, l_3 and denote $l_*=\max\{l_1,l_2,l_3\}$. Denote
$$\lambda_*=\frac{\alpha}{\beta}k_*, \quad \theta_*=\frac{\alpha_1}{\beta_1}l_*.$$
Then for any $(\lambda,\theta)>(\lambda_*,\theta_*)$, we let
$$\sigma=\frac{\alpha\ln M}{\beta\lambda}, \quad \tau=\frac{\alpha_1\ln M}{\beta_1\theta},$$
and (3.3), (3.7) still hold, that is
$$-\Delta_{p(x)}\phi_1\leq\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)] \text{ a.e. on } \Omega. \tag{3.8}$$
Similarly, we have
$$-\Delta_{q(x)}\phi_2\leq\theta^{q(x)}[b(\phi_2)g_1(x)+h(\phi_1)] \text{ a.e. on } \Omega. \tag{3.9}$$

From (3.8) and (3.9), we can see that (ϕ_1,ϕ_2) is a sub-solution of (1.1). For any $(\lambda,\theta)>(\lambda_*,\theta_*)$, we consider the following problem
$$\begin{cases} -\Delta_{p(x)}z_1=\lambda^{p^+}\eta, & x\in\Omega,\\ -\Delta_{q(x)}z_2=2\theta^{q^+}h(\omega_1), & x\in\Omega,\\ z_1=z_2=0, & x\in\partial\Omega, \end{cases} \tag{3.10}$$

here $\omega_1=\max_{x\in\overline{\Omega}} z_1(x)$ and η is a positive constant. We will show that (z_1,z_2) is a super-solution of (1.1).

By directly computation, we can see

$$z_1=\int_\rho^r\Big(\frac{\lambda^{p^+}\eta}{N}t\Big)^{\frac{1}{p(t)-1}}dt,\quad z_2=\int_\rho^r\Big(\frac{2\theta^{q^+}h(\omega_1)}{N}t\Big)^{\frac{1}{q(t)-1}}dt,$$

is a positive solution of problem (3.10). Obviously, there exists a $\zeta\in[0,r]$ such that

$$\omega_1=\max_{x\in\overline{\Omega}} z_1=\int_0^r\Big(\frac{\lambda^{p^+}\eta}{N}t\Big)^{\frac{1}{p(t)-1}}dt=(\lambda^{p^+}\eta)^{\frac{1}{p(\zeta)-1}}\int_0^r\Big(\frac{t}{N}\Big)^{\frac{1}{p(t)-1}}dt,$$

when η is large, we obtain

$$C_2(\lambda^{p^+}\eta)^{\frac{1}{p^+-1}}\le\omega_1\le C_2(\lambda^{p^+}\eta)^{\frac{1}{p^--1}},\tag{3.11}$$

where

$$C_2=\int_0^r\Big(\frac{t}{N}\Big)^{\frac{1}{p(t)-1}}dt$$

is a positive constant. Similarly, we have

$$C_3(2\theta^{q^+}h(\omega_1))^{\frac{1}{q^+-1}}\le\omega_2\le C_3(2\theta^{q^+}h(\omega_1))^{\frac{1}{q^--1}}.$$

For any $\varphi\in W^{1,p(x)}(\Omega)$ with $\varphi\ge0$, we have

$$\int_\Omega|\nabla z_1|^{p(x)-2}\nabla z_1\nabla\varphi dx=\int_\Omega\lambda^{p^+}\eta\varphi dx,\tag{3.12a}$$

$$\int_\Omega|\nabla z_2|^{q(x)-2}\nabla z_2\nabla\varphi dx=\int_\Omega 2\lambda^{q^+}h(\omega_1)\varphi dx.\tag{3.12b}$$

From (3.11), we know that ω_1 is large when η is large, by (D3)-(D5), we have

$$\lim_{s\to\infty}\frac{f[C_3(2\theta^{q^+}h(s))^{\frac{1}{q^--1}}]+a(s)\max_{x\in\overline{\Omega}}g(x)}{s^{p^--1}}=0.$$

Then when η is large enough, combining (3.11), we obtain

$$\lambda^{p^+}\eta\ge\Big(\frac{1}{C_2}\omega_1\Big)^{p^--1}\ge\lambda^{p^+}\Big\{f\big[C_3(2\theta^{q^+}h(\omega_1))^{\frac{1}{q^--1}}\big]+a(\omega_1)\max_{x\in\overline{\Omega}}g(x)\Big\}.\tag{3.13}$$

Since f, a are nondecreasing functions, from (3.12a) and (3.13), and use (3.11) again, we have

$$\begin{aligned}&\int_\Omega|\nabla z_1|^{p(x)-2}\nabla z_1\nabla\varphi dx\\ \ge&\int_\Omega\lambda^{p^+}\Big\{f\big[C_3(2\theta^{q^+}h(\omega_1))^{\frac{1}{q^--1}}\big]+a(\omega_1)\max_{x\in\overline{\Omega}}g(x)\Big\}\varphi dx\\ \ge&\int_\Omega\lambda^{p(x)}[a(z_1)g(x)+f(z_2)]\varphi dx.\end{aligned}$$

Since h is nondecreasing, we have

$$\int_{\Omega}\theta^{q^+}h(\omega_1)\varphi dx \geq \int_{\Omega}\theta^{q^+}h(z_1)\varphi dx. \tag{3.14}$$

From (D4) and (D5), when η large enough, then

$$b\big[C_3(2\theta^{q^+}h(\omega_1))^{\frac{1}{q^- -1}}\big]\max_{x\in\overline{\Omega}}g(x) \leq h(\omega_1). \tag{3.15}$$

From (3.12b), (3.14) and (3.15), we obtain

$$\begin{aligned}
&\int_{\Omega}|\nabla z_2|^{q(x)-2}\nabla z_2\nabla\varphi dx\\
\geq &\int_{\Omega}\theta^{q^+}\Big\{b\big[C_3(2\lambda^{q^+}h(\omega_1))^{\frac{1}{q^- -1}}\big]\max_{x\in\overline{\Omega}}g(x)+\theta^{q^+}h(z_1)\Big\}\varphi dx\\
\geq &\int_{\Omega}\theta^{q(x)}[b(z_2)g(x)+h(z_1)]\varphi dx.
\end{aligned}$$

Thus, we obtain that (z_1,z_2) is a super-solution of (1.1).

Now, we only need to show that $(\phi_1,\phi_2)\leq(z_1,z_2)$ in Ω. When η is large enough, we have

$$\lim_{\rho\to r^-}\frac{\phi_1(\rho)}{z_1(\rho)}=\frac{k}{\big(\frac{\lambda^{p^+}\eta}{N}r\big)^{\frac{1}{p(r)-1}}}<1.$$

By the continuity of $\phi_1(x)$ and $z_1(x)$, there exists $\varepsilon>0$ such that

$$\phi_1(x)\leq z_1(x), \quad r-\varepsilon<\rho\leq r.$$

When $0\leq\rho\leq r-\varepsilon$, we can see that $\phi_1(x)$ is bounded and

$$z_1(x)=\int_{\rho}^{r}\Big(\frac{\lambda^{p^+}\eta}{N}t\Big)^{\frac{1}{p(t)-1}}dt \geq \int_{r-\varepsilon}^{r}\Big(\frac{\lambda^{p^+}\eta}{N}t\Big)^{\frac{1}{p(t)-1}}dt\to\infty \quad \text{as} \quad \eta\to\infty.$$

Then

$$\phi_1(x)\leq z_1(x), \quad x\in\Omega,$$

when η is large enough.

By (3.11), we can see that ω_1 is large enough when η is large enough, and so $h(\omega_1)$ is large enough. Similarly as above argument, when η is large enough, we have

$$\phi_2(x)\leq z_2(x), \quad x\in\Omega.$$

Thus, we complete the proof of Theorem 3.1. $\square$

Remark 3.1. We note that if we replace (D3) with

(D3') $g,g_1 \in C(\overline{\Omega})$, they are positive far away from $\partial\Omega$, i.e., there exists $\varepsilon > 0$ small enough such that g, g_1 are positive on $\Omega \backslash \partial\Omega_\varepsilon$, where $\partial\Omega_\varepsilon = \{x \in \Omega | d(x) < \varepsilon\}$ and $d(x)$ denotes the distance of $x \in \Omega$ to the boundary of Ω.

Then (3.1) is satisfied on $\Omega \backslash \partial\Omega_\varepsilon$. If we take

$$\beta = |f(0)| + a(M-1)\max_{x\in\overline{\Omega}} |g(x)|,$$
$$\beta_1 = |h(0)| + b(M-1)\max_{x\in\overline{\Omega}} |g_1(x)|,$$

in the proof of Theorem 3.1, then Theorem 3.1 still hold. Since we do not assume any sign-changing conditions on $f(0)$ or $h(0)$. Hence in our system (1.1), $F(x,0,0)$ or $H(x,0,0)$ could be negative for some $x \in \Omega$. In fact, we usually assume $F(x,u,v)$, $H(x,u,v)$ non-negative (see [3, 21, 23]) and it is well known that the study of positive solutions with sign-changing weight is mathematically challenging (see [15, 16, 20]).

Remark 3.2. From Corollary 5 in [20], we note that when $p(x) = q(x) \equiv p$ (a constant), then problem (1.5) has at least one positive solution when $\lambda = \theta$ is large enough. Thus, our results in the present paper is a complement and generalization partly to the results in [20].

If we replace the condition (D1) with

(D1') $\Omega = B(0,r_2) \backslash \overline{B(0,r_1)} \subset R^N$, where $0 < r_1 < r_2$ are constants.

Then we have

Theorem 3.2. *If (D1') and (D2)-(D5) hold, then there exist* $(\lambda_*,\theta_*) > (0,0)$ *such that for any* $(\lambda,\theta) > (\lambda_*,\theta_*)$, *problem* (1.1) *has at least one positive solution.*

Proof. We denote

$$\phi_1(x) = \phi_1(\rho) = \begin{cases} e^{k(r_2-\rho)} - 1, & r_2 - \sigma < \rho \le r_2, \\ e^{k\sigma} - 1 + \displaystyle\int_\rho^{r_2-\sigma} k e^{k\sigma} \Big(\frac{r_2 - \varepsilon_2 - t}{\sigma - \varepsilon_2}\Big)^{\frac{1}{p(t)-1}} dt, & r_2 - \varepsilon_2 < \rho \le r_2 - \sigma, \\ e^{k\sigma} - 1 + \displaystyle\int_{r_1+\sigma}^{r_1+\varepsilon_1} k e^{k\sigma} \Big(\frac{r_1 + \varepsilon_1 - t}{\varepsilon_1 - \sigma}\Big)^{\frac{1}{p(t)-1}} dt, & r_1 + \varepsilon_1 < \rho \le r_2 - \varepsilon_2, \\ e^{k\sigma} - 1 + \displaystyle\int_{r_1+\sigma}^{\rho} k e^{k\sigma} \Big(\frac{r_1 + \varepsilon_1 - t}{\varepsilon_1 - \sigma}\Big)^{\frac{1}{p(t)-1}} dt, & r_1 + \sigma < \rho \le r_1 + \varepsilon_1, \\ e^{k(\rho - r_1)} - 1, & r_1 \le \rho \le r_1 + \sigma, \end{cases}$$

and

$$\phi_2(x)=\phi_2(\rho)=\begin{cases} e^{l(r_2-\rho)}-1, & r_2-\tau<\rho\le r_2,\\ e^{l\tau}-1+\displaystyle\int_{\rho}^{r_2-\tau} le^{l\tau}\Big(\frac{r_2-\epsilon_2-t}{\tau-\epsilon_2}\Big)^{\frac{1}{q(t)-1}}dt, & r_2-\epsilon_2<\rho\le r_2-\tau,\\ e^{l\tau}-1+\displaystyle\int_{r_1+\tau}^{r_1+\epsilon_1} le^{l\tau}\Big(\frac{r_1+\epsilon_1-t}{\epsilon_1-\tau}\Big)^{\frac{1}{q(t)-1}}dt, & r_1+\epsilon_1<\rho\le r_2-\epsilon_2,\\ e^{l\tau}-1+\displaystyle\int_{r_1+\tau}^{\rho} le^{l\tau}\Big(\frac{r_1+\epsilon_1-t}{\epsilon_1-\tau}\Big)^{\frac{1}{q(t)-1}}dt, & r_1+\tau<\rho\le r_1+\epsilon_1,\\ e^{l(\rho-r_1)}-1, & r_1\le\rho\le r_1+\tau,\end{cases}$$

where we assume

$$\sigma=\frac{\ln M}{k},\quad \tau=\frac{\ln M}{l},$$

M is a positive constant such that (3.1) hold. Then there exists $k_1=l_1>1$ such that for any $k>k_1$, $-l>l_1$, we have $\sigma,\tau\in(0,(r_2-r_1)/4)$, and ε_1, ε_2, ϵ_1, ϵ_2 are positive constants satisfying

$$r_1+\sigma<r_1+\varepsilon_1<r_2-\varepsilon_2<r_2-\sigma \tag{3.16}$$

and

$$r_1+\tau<r_1+\epsilon_1<r_2-\epsilon_2<r_2-\tau. \tag{3.17}$$

It is easy to see that we can take ε_1, ε_2 and ϵ_1, ϵ_2 such that

$$\int_{r_2-\varepsilon_2}^{r_2-\sigma}\Big(\frac{r_2-\varepsilon_2-t}{\sigma-\varepsilon_2}\Big)^{\frac{2}{p^--1}}dt=\int_{r_1+\sigma}^{r_1+\varepsilon_1}\Big(\frac{r_1+\varepsilon_1-t}{\varepsilon_1-\sigma}\Big)^{\frac{2}{p^--1}}dt, \tag{3.18a}$$

$$\int_{r_2-\varepsilon_2}^{r_2-\tau}\Big(\frac{r_2-\varepsilon_2-t}{\tau-\varepsilon_2}\Big)^{\frac{2}{p^--1}}dt=\int_{r_1+\tau}^{r_1+\varepsilon_1}\Big(\frac{r_1+\varepsilon_1-t}{\varepsilon_1-\tau}\Big)^{\frac{2}{p^--1}}dt, \tag{3.18b}$$

hold, then $\phi_1,\phi_2\in C^1(\Omega)$. Obviously, we have that $\varepsilon_1\to\sigma^+$ when $\varepsilon_2\to\sigma^+$, then we can choose $\varepsilon_1,\varepsilon_2>0$ such that (3.16) and (3.18a) hold simultaneously. Similarly, there exist $\epsilon_1,\epsilon_2>0$ such that (3.17) and (3.18b) hold simultaneously.

By computation

$$-\Delta_{p(x)}\phi_1(x)=\begin{cases} -(ke^{k(r_2-\rho)})^{p(\rho)-1}\Big[k(p(\rho)-1)-p'(\rho)\ln k-kp'(\rho)(r_2-\rho)-\dfrac{N-1}{\rho}\Big], \\ \qquad r_2-\sigma<\rho<r_2, \\ -(ke^{k\sigma})^{p(\rho)-1}\Big[p'(\rho)(\ln k+k\sigma)\dfrac{r_2-\varepsilon_2-\rho}{\varepsilon_2-\sigma}-\dfrac{1}{\varepsilon_2-\sigma}+\frac{N-1}{\rho}\dfrac{r_2-\varepsilon_2-\rho}{\varepsilon_2-\sigma}\Big], \\ \qquad r_2-\varepsilon_2<\rho<r_2-\sigma, \\ 0, \qquad r_1+\varepsilon_1<\rho<r_2-\varepsilon_2, \\ -(ke^{k\sigma})^{p(\rho)-1}\Big[p'(\rho)(\ln k+k\sigma)\dfrac{r_1+\varepsilon_1-\rho}{\varepsilon_1-\sigma}-\dfrac{1}{\varepsilon_1-\sigma}+\dfrac{N-1}{\rho}\dfrac{r_1+\varepsilon_1-\rho}{\varepsilon_1-\sigma}\Big], \\ \qquad r_1+\sigma<\rho<r_1+\varepsilon_1, \\ -(ke^{k(\rho-r_1)})^{p(\rho)-1}\Big[k(p(\rho)-1)+p'(\rho)\ln k+kp'(\rho)(\rho-r_1)+\dfrac{N-1}{\rho}\Big], \\ \qquad r_1<\rho<r_1+\sigma. \end{cases}$$

Then there exists $k_2>0$ such that when $k>k_2$, we have

$$-\Delta_{p(x)}\phi_1\le -k^{p(\rho)}\alpha, \quad r_2-\sigma<\rho<r_2 \quad \text{or} \quad r_1<\rho<r_1+\sigma, \tag{3.19}$$

where α, α_1, β, β_1 are defined as in Theorem 3.1. Let $\lambda=\alpha k/\beta$, we have $k^{p(x)}\alpha\ge\lambda^{p(x)}\beta$, then when $r_2-\sigma<\rho<r_2$ or $r_1<\rho<r_1+\sigma$, we have

$$-\Delta_{p(x)}\phi_1\le -\lambda^{p(x)}\beta\le\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)]. \tag{3.20}$$

When $r_2-\varepsilon_2<\rho<r_2-\sigma$ or $r_1+\sigma<\rho<r_1+\varepsilon_1$, there exists $C_4>0$ such that

$$-\Delta_{p(x)}\phi_1\le C_4(ke^{k\sigma})^{p(x)-1}\ln k. \tag{3.21}$$

Then there exists $k_3>0$ such that when $k>k_3$ and $\lambda=\alpha k/\beta$, we have

$$C_4(ke^{k\sigma})^{p(x)-1}\ln k\le\lambda^{p(x)}. \tag{3.22}$$

From (3.1), (3.21) and (3.22), when $r_2-\varepsilon_2<\rho<r_2-\sigma$ or $r_1+\sigma<\rho<r_1+\varepsilon_1$, we have

$$-\Delta_{p(x)}\phi_1\le\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)]. \tag{3.23}$$

Obviously, when $r_1+\varepsilon_1<\rho<r_2-\varepsilon_2$, we have

$$-\Delta_{p(x)}\phi_1=0\le\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)]. \tag{3.24}$$

Let $k_*=\max\{k_1,k_2,k_3\}$. Similarly, we obtain l_2, l_3 and denote $l_*=\max\{l_1,l_2,l_3\}$. Denote

$$\lambda_*=\frac{\alpha}{\beta}k_*, \quad \theta_*=\frac{\alpha_1}{\beta_1}l_*.$$

Then for any $\lambda>\lambda_*$, $\theta>\theta_*$, we let $\sigma=\alpha\ln M/\beta\lambda$, $\tau=\alpha_1\ln M/\beta_1\theta$ and (3.20), (3.23)-(3.24) still hold, that is

$$-\Delta_{p(x)}\phi_1\le\lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)] \quad \text{a.e. on } \Omega. \tag{3.25}$$

Similarly, we have

$$-\Delta_{q(x)}\phi_2\le\theta^{q(x)}[b(\phi_2)g_1(x)+h(\phi_1)] \quad \text{a.e. on } \Omega. \tag{3.26}$$

From (3.25) and (3.26), we can see that (ϕ_1,ϕ_2) is a sub-solution of (1.1). Let z be a radial solution of

$$-\Delta_{p(x)}z(x)=\mu \quad \text{in } \Omega, \quad z=0 \quad \text{on } \partial\Omega,$$

then

$$z=\int_{r_1}^{\rho} t^{\frac{1-N}{p(t)-1}}\mu^{\frac{1}{p(t)-1}}\Big|\frac{C}{\mu}-\frac{t^N}{N}\Big|^{\frac{1}{p(t)-1}-1}\Big(\frac{C}{\mu}-\frac{t^N}{N}\Big)dt,$$

where C is some positive constant such that

$$z(r_1)=z(r_2)=0.$$

Then

$$\frac{r_1^N}{N}<\frac{C}{\mu}<\frac{r_2^N}{N}.$$

Assume $\omega=\max_{x\in\overline{\Omega}}z=z(\rho_0)$, then $C=\mu\rho_0^N/N$. From the argument of Theorem 2.2 in [22], we know that

$$C_5\mu^{\frac{1}{p^+-1}}\le\omega\le C_6\mu^{\frac{1}{p^--1}}, \tag{3.27}$$

where C_5, C_6 are positive constants independent on μ.

Similarly to the proof of Theorem 3.1, we can see that the solution (z_1,z_2) of (3.10) is still a supersolution for (1.1) when η is large enough.

Now we denote

$$\zeta_1=1+\max\Big\{\max_{x\in\overline{\Omega}}\phi_1,\max_{x\in\overline{\Omega}}|\nabla\phi_1|\Big\},$$

and

$$\zeta_2=1+\max\Big\{\max_{x\in\overline{\Omega}}\phi_2,\max_{x\in\overline{\Omega}}|\nabla\phi_2|\Big\}.$$

Similarly to the argument in [22], we obtain that there exist positive constants σ_1, σ_2 such that

$$z_1'(\rho)\ge\zeta_1, \qquad r_1\le\rho\le r_1+\sigma_1, \tag{3.28a}$$

$$z_1'(\rho)\le-\zeta_1, \qquad r_2-\sigma_2\le\rho\le r_2, \tag{3.28b}$$

and

$$z_1(\rho)\geq\zeta_1, \quad r_1+\sigma_1\leq\rho\leq r_2-\sigma_2. \tag{3.29}$$

By (3.28a)-(3.29) and $z_1(x)=\phi_1(x)=0$, $x\in\partial\Omega$, we obtain that

$$\phi_1\leq z_1, \quad x\in\Omega.$$

Similarly, we obtain that

$$\phi_2\leq z_2, \quad x\in\Omega.$$

That's completes the proof. □

4 Asymptotic behavior of positive solutions

In this section, when parameters $(\lambda,\theta)>(\lambda_*,\theta_*)$, we will discuss the asymptotic behavior of maximum of solutions about parameters λ, θ, and the asymptotic behavior of solutions near the boundary of Ω.

Theorem 4.1. *If (D1)-(D5) hold and (u,v) is a solution of* (1.1) *which has been obtained in Theorem 3.1, then*

(i) There exist positive constants C_7 and C_8 such that

$$C_7\lambda\leq\max_{x\in\overline{\Omega}}u(x)\leq C_6(\lambda^{p^+}\eta)^{\frac{1}{p^--1}}, \tag{4.1a}$$

$$C_8\theta\leq\max_{x\in\overline{\Omega}}v(x)\leq C_6(2\theta^{q^+}h(\omega))^{\frac{1}{q^--1}}. \tag{4.1b}$$

(ii) When $d(x)\to 0$, we have

$$u(x)=\mathcal{O}(d(x)), \quad v(x)=\mathcal{O}(d(x)).$$

Proof. (i) By the definition of ϕ_1, we have

$$\begin{aligned}\max_{x\in\overline{\Omega}}u(x)\geq\max_{x\in\overline{\Omega}}\phi_1(x)=&e^{k\sigma}-1+\int_0^{r-\sigma}ke^{k\sigma}\Big(\frac{t}{r-\sigma}\Big)^{\frac{1}{p(t)-1}}dt\\ \geq&\lambda\frac{\alpha}{\beta}M\int_0^{r-\sigma}\Big(\frac{t}{r-\sigma}\Big)^{\frac{1}{p(t)-1}}dt\\ =&C_7\lambda.\end{aligned}$$

By (3.11), we have

$$\max_{x\in\overline{\Omega}}u(x)\leq\max_{x\in\overline{\Omega}}z_1(x)\leq C_6(\lambda^{p^+}\eta)^{\frac{1}{p^--1}}.$$

Thus, we obtain (4.1a). Similarly, (4.1b) is valid too.

(ii) Since $\Omega=B(0,r)$, we have $d(x)=r-\rho$, when $d(x)\to 0$, we have

$$u(x)\geq \phi_1(x)=e^{kd(x)}-1\geq C_9\lambda d(x)$$

and

$$v(x)\geq \phi_2(x)=e^{ld(x)}-1\geq C_{10}\theta d(x),$$

where C_9, C_{10} are positive constants. On the other hand, we have

$$u(x)\leq z_1(x)=\int_\rho^r \Big(\frac{\lambda^{p^+}\eta}{N}t\Big)^{\frac{1}{p(t)-1}}dt=\int_{r-d(x)}^r \Big(\frac{\lambda^{p^+}\eta}{N}t\Big)^{\frac{1}{p(t)-1}}dt\leq \Big(\frac{\lambda^{p^+}\eta}{N}r\Big)^{\frac{1}{p^--1}}d(x),$$

when η is large enough. Thus, we obtain

$$u(x)=\mathcal{O}(d(x)) \quad \text{as } \ d(x)\to 0.$$

Similarly, when η is large enough, we have

$$v(x)\leq z_2(x)\leq \Big(\frac{2\theta^{q^+}h(\omega)}{N}r\Big)^{\frac{1}{p^--1}}d(x),$$

and obtain

$$v(x)=\mathcal{O}(d(x)) \quad \text{as } \ d(x)\to 0.$$

This completes the proof. □

When $\Omega=B(0,r_2)\setminus\overline{B(0,r_1)}$, we have almost the same results as Theorem 4.1, that is

Theorem 4.2. *If (D1'), (D2)-(D5) hold and (u,v) is a solution of* (1.1) *which has been obtained in Theorem 3.2, then*

(i) There exist positive constants C_{11} and C_{12} such that

$$C_{11}\lambda\leq \max_{x\in\overline{\Omega}}u(x)\leq C_6(\lambda^{p^+}\eta)^{\frac{1}{p^--1}}, \tag{4.2a}$$

$$C_{12}\theta\leq \max_{x\in\overline{\Omega}}v(x)\leq C_6(2\theta^{q^+}h(\omega))^{\frac{1}{p^--1}}. \tag{4.2b}$$

(ii) When $d(x)\to 0$, we have

$$u(x)=\mathcal{O}(d(x)), \quad v(x)=\mathcal{O}(d(x)).$$

Proof. (i) By the definition of ϕ_1, we have

$$\begin{aligned}\max_{x\in\overline{\Omega}}u(x)\geq \max_{x\in\overline{\Omega}}\phi_1(x)&=e^{k\sigma}-1+\int_{r_1+\sigma}^{r_1+\varepsilon_1}ke^{k\sigma}\Big(\frac{r_1+\varepsilon_1-t}{\varepsilon_1-\sigma}\Big)^{\frac{1}{p(t)-1}}dt\\ &\geq\lambda\frac{\alpha}{\beta}M\int_{r_1+\sigma}^{r_1+\varepsilon_1}\Big(\frac{r_1+\varepsilon_1-t}{\varepsilon_1-\sigma}\Big)^{\frac{1}{p(t)-1}}dt\\ &=C_{11}\lambda.\end{aligned}$$

By (3.27), we have

$$\max_{x\in\overline{\Omega}} u(x) \le \max_{x\in\overline{\Omega}} z_1(x) \le C_6(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}.$$

Thus, we obtain (4.2a). Similarly, (4.2b) is valid too.

(ii) Since $\Omega = B(0,r_2)\setminus\overline{B(0,r_1)}$, for any $x\in\Omega$, we have $d(x)=\min\{r_2-\rho,\rho-r_1\}$, when $d(x)\to 0$, we have $r_1\le\rho\le r_1+\sigma$ or $r_2-\sigma<\rho\le r_2$. Then

$$u(x)\ge\phi_1(x)=e^{kd(x)}-1\ge C_9\lambda d(x). \tag{4.3}$$

Similarly, we have

$$v(x)\ge C_{10}\theta d(x). \tag{4.4}$$

For

$$z_1=\int_{r_1}^{\rho} t^{\frac{1-N}{p(t)-1}}(\lambda^{p^+}\eta)^{\frac{1}{p(t)-1}}\Big|\frac{\rho_1}{N}-\frac{t^N}{N}\Big|^{\frac{1}{p(t)-1}-1}\Big(\frac{\rho_1}{N}-\frac{t^N}{N}\Big)dt,$$

where $r_1<\rho_1<r_2$ satisfies $\max_{x\in\overline{\Omega}} z_1 = z_1(\rho_1)$. For $z_1(r_1)=z_1(r_2)=0$ and z_1 is continuous, it is easy to obtain that

$$z_1(x)=\mathcal{O}(d(x)) \quad \text{as } \ d(x)\to 0. \tag{4.5}$$

Thus, from (4.3) and (4.5), we have

$$u(x)=\mathcal{O}(d(x)) \quad \text{as } \ d(x)\to 0.$$

Similarly, we have

$$v(x)=\mathcal{O}(d(x)) \quad \text{as } \ d(x)\to 0.$$

This completes the proof. $\square$

5 An example

We consider the following problem

$$\begin{cases} -\Delta_{p(x)}u=\lambda^{p(x)}\Big[\dfrac{1}{e^{|x|}}u^s+v^m\Big], & x\in\Omega,\\ -\Delta_{q(x)}v=\lambda^{q(x)}\Big[\dfrac{1}{e^{|x|}}v^t+u^n\Big], & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases} \tag{5.1}$$

We assume:

(D6) $0\le s<p^- -1$, $0\le t<q^- -1$, $0<m,n$ and $mn<(p^- -1)(q^- -1)$.

If we set $g(x)=g_1(x)=e^{-|x|}$, $a(u)=u^s$, $b(v)=v^t$, $f(v)=v^m$ and $h(u)=u^n$, then (D3)-(D5) are satisfied. Then we have the following result:

Theorem 5.1. *If (D1) (or (D1′)), (D2) and (D6) hold, then there exist* $(\lambda_*,\theta_*)>(0,0)$ *such that for any* $(\lambda,\theta)>(\lambda_*,\theta_*)$, *problem* (5.1) *has at least one positive solution* (u,v), *and* (u,v) *satisfying*

(i) There exist positive constants C_{13} *and* C_{14} *such that*

$$C_{13}\lambda\le\max_{x\in\overline{\Omega}}u(x)\le C_6(\lambda^{p^+}\eta)^{\frac{1}{p^--1}},$$
$$C_{14}\theta\le\max_{x\in\overline{\Omega}}v(x)\le C_6(2\theta^{q^+}h(\omega))^{\frac{1}{p^--1}}.$$

(ii) When $d(x)\to 0$, *we have*

$$u(x)=\mathcal{O}(d(x)),\quad v(x)=\mathcal{O}(d(x)).$$

Acknowledgments

The authors would like to thank the referees for the helpful suggestions. The authors are supported by the National Natural Science Foundation of China (No. 11171092 and No. 11471164) and the Natural Science Foundation of Jiangsu Education Office (No. 12KJB110002).

References

[1] E. ACERBI AND G. MINGIONE, *Regularity results for a class of functionals with nonstandard growth*, Arch. Ration. Mech. Anal., 156 (2001), pp. 121–140.

[2] M. BENDAHMANEA AND P. WITTBOLD, *Renormalized solutions for nonlinear elliptic equations with variable exponents and* L^1 *data*, Nonlinear Anal., 70 (2009), pp. 567–583.

[3] E. ACERBI AND G. MINGIONE, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal., 164 (2002), pp. 213–259.

[4] C. H. CHEN, *On positive weak solutions for a class of quasilinear elliptic systems*, Nonlinear Anal., 62 (2005), pp. 751–756.

[5] X. L. FAN, *Global* $C^{1,\alpha}$ *regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations, 235 (2007), pp. 397–417.

[6] X. L. FAN AND D. ZHAO, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263 (2001), pp. 424–446.

[7] X. L. FAN, H. Q. WU AND F. Z. WANG, *Hartman-type results for* $p(t)$*-Laplacian systems*, Nonlinear Anal., 52 (2003), pp. 585–594.

[8] X. L. FAN, Y. Z. ZHAO AND D. ZHAO, *Compact imbedding theorems with symmetry of Strauss-Lions type for the spaces* $W^{1,p(x)}$, J. Math. Anal. Appl., 255 (2001), pp. 333–348.

[9] X. L. FAN AND Q. H. ZHANG, *Existence of solutions for* $p(x)$*-Laplacian Dirichlet problem*, Nonlinear Anal., 52 (2003), pp. 1843–1852.

[10] X. L. FAN, *On the sub-supersolution method for* $p(x)$*-Laplacian equations*, J. Math. Anal. Appl., 330 (2007), pp. 665–682.

[11] X. L. FAN, Q. H. ZHANG AND D. ZHAO, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem,* J. Math. Anal. Appl., 302 (2005), pp. 306–317.

[12] D. D. HAI AND R. SHIVAJI, *An existence result on positive solutions of p-Laplacian systems,* Nonlinear Anal., 56 (2004), pp. 1007–1010.

[13] A. EI HAMIDI, *Existence results to elliptic systems with nonstandard growth con-ditions*, J. Math. Anal. Appl., 300 (2004), pp. 30–42.

[14] O. KOVÁČIK AND J. RÁKOSNÍK, *On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$*, Czechoslovak Math. J., 41 (1991), pp. 592–618.

[15] P. L. LIONS, *On the existence of positive solutions of semilinear elliptic equations,* SIAM Rev., 24 (1982), pp. 441–467.

[16] S. ORUGANTI, J. SHI AND R. SHIVAJI, *Diffusive equations with constant yield harvesting, I: steady states,* Trans. Amer. Math. Soc., 354 (2002), pp. 3601–3619.

[17] M. RÚZICKA, Electro-Rheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math.,Vol. 1784, Springer-Verlag, Berlin, 2000.

[18] S. G. SANKO, *Denseness of $C_0^\infty(R^N)$ in the generalized Sobolev spaces $W^{m,p(x)}(R^N)$*, Dokl. Ross. Akad. Nauk., 369(4) (1999), pp. 451–454.

[19] H. H. YIN AND Z. D. YANG, *Existence and nonexistence of entire positive solutions for quasilinear systems with singular and super-linear terms,* Differential Equations Appl., 2(2) (2010), pp. 241–249.

[20] H. H. YIN, *Existence results for classes of quasilinear elliptic systems with sign-changing weight,* Int. J. Nonlinear Sci., 10 (2010), pp. 53–60.

[21] Q. H. ZHANG, *Existence of positive solutions for a class of $p(x)$-Laplacian systems,* J. Math. Anal. Appl., 333 (2007), pp. 591–603.

[22] Q. H. ZHANG, *Existence of positive solutions for elliptic systems with nonstandard $p(x)$-growth conditions via sub-supersolution method,* Nonlinear Anal., 67 (2007), pp. 1055–1067.

[23] Q. H. ZHANG, *Existence and asymptotic behavior of positive solutions for a variable exponent elliptic system without variational structure,* Nonlinear Anal., 72 (2010), pp. 354–363.

[24] Q. H. ZHANG, *Existence of radial solutions for $p(x)$-Laplacian equations in R^N*, J. Math. Anal. Appl., 315(2) (2006), pp. 506–516.

[25] Q. H. ZHANG, *A strong maximum principle for differential equations with nonstand $p(x)$-growth conditions,* J. Math. Anal. Appl., 312(1) (2005), pp. 24–32.

[26] Q. H. ZHANG, *Existence and asymptotic behavior of positive solutions for variable exponent elliptic systems,* Nonlinear Anal., 70 (2009), 305–316.

[27] V. V. ZHIKOV, *Averaging of functionals of the calculus of variations and elasticity theory,* Math. USSR Izv., 29 (1987), pp. 33–36.